Total variation in image processing

Let \( O \) denote a region in the plane occupied by a noisy image and let \( u = u(x,y) \) denote a (possibly in colors) component intensity of a pixel at point \( (x,y) \). A denoising method was proposed in [BJ92, CL97] involving minimization of a Bidirectional-Filtering (BF) functional

\[
\int (\nabla u | v) + \frac{m}{\varepsilon} (u - v)^2 \ dx \ dy
\]

with a given \( \lambda > 0 \).

The total variation flow has following desirable properties:

- for small \( \varepsilon \) it quickly smooths out grain-like noise;
- \( \lambda > 0 \) does not smooth out well defined contours;
- it acts as (weighted) mean curvature flow on level sets.

Mathematics of ROF functional

The minimizers of the \((\lambda > 1)\) power \((\ell^\lambda_2)^2\) functional \((\lambda)\) typically belong only to \(BV(\Omega)\). This means that they can be understood as a vector measure and is a causal jump discontinuities. However, certain preservation of regularity properties were obtained for \((\lambda)\) with \( \lambda \in (1,2) \):

- \( u \) admits a modulus of continuity and \( \Omega \) is bounded, then the minimizer of \((\lambda)\) admits the same modulus of continuity.
- \( u \) is \( BV\) continuous in a ball, then \( u \) is \( BV\) continuous in a smaller ball.
- the set of jump discontinuities of \( u \) is contained in that of \( v \) and jump sizes of \( u \) are not bigger than those of \( v \).

In \([A+07]\) the following characterization of the subdifferential \( \partial (\lambda) \) was obtained:

\[
\partial (\lambda) (f) = \{ x \in L^2(\Omega) \mid \exists v \in L^2(\Omega) \text{ s.t. } f(x) = \lambda \}, \quad \forall \lambda > 1
\]

In the case \( \lambda = 1 \) one can, using (3), try to identify the class of sets \( A \) whose shape is preserved by minimizing (1), i.e. such that \( f(\lambda = 1) = 1 \).

In the case of convex sets a characterization in terms of Choquard condition

\[
\int_{\Omega} f(x) \ dx = \int_{\Omega} \int_{\Omega} |x-y|^{-\lambda} \ dx \ dy
\]

was obtained in \([BC05]\). Here, \( f \) denotes perimeter, equal to \( H^1 \) measure of boundary for Lipschitz sets. Using similar arguments, a significant class of explicit solutions was proposed in \([BC07]\).

Total variation denoising in \( \delta \) anisotropy

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Total variation denoising in \( \delta \) anisotropy

In 2004, minimizing anisotropic analogues of (1), in which the Euclidean norm of \( \nabla u \) is replaced with any other norm on the plane, was proposed [EO04] for denoising images, whose main features are expected to be arranged in certain directions. Minimizing the kind of functional may actually amplify contours in characteristic directions of the chosen norm \( \delta \) (jump sizes can increase and new jumps can appear). A natural choice of \( \delta \) turns out to be \( \ell^2 \) functional to \( \delta \)-anisotropic version of (1):

\[
\int (\nabla u | v) + \frac{m}{2\widetilde{\varepsilon}} (u - v)^2 \ dx \ dy
\]

Variants of (1) were successfully applied for many purposes.

Preservation of PCR functions

Let \( O \) be a rectangle or \( O = \mathbb{R}^2 \). We say that \( u \) is piecewise constant on \( \mathbb{R}^2 \) and \( u \in \ell^2(\mathbb{R}^2) \) if there is a decomposition of the support of \( u \) into a finite number of rectangles with sides parallel to coordinate axes, such that \( u \) is constant on it. We have:

Theorem 1. \( \ell^2 \) Moll-Mucha [M+08], \( u \in \ell^2(\mathbb{R}^2) \) and \( \delta \) the minimizer of (5), then \( u \in \ell^2(\mathbb{R}^2) \).

Note that PCR functions are dense in \( BV(\mathbb{R}^2) \). In fact it is even strictly dense in \( BV(\mathbb{R}^2) \) with wrt norm \( \| \cdot \|_{\ell^2} \). Hence, we can use it to approximate any reasonable initial datum.

Rectilinear geometry

The proof of Theorem 1 follows by arguing validity of an explicit algorithm for computing the solution. In order to state the algorithm, we need to recall the following notions from rectilinear geometry:

- we say that a polygon \( P \) is rectilinear if all of its sides are parallel to the coordinate axes;
- we call any finite set of line parallel to the coordinate axes a grid;
- we say that grid \( G \) is spanned by rectilinear polygon \( F \) if it consists of all line extending sides of \( F \);
- we say that grid \( G \) is spanned by a function \( v \in \ell^2(\mathbb{R}^2) \) if it consists of all lines extending sides of level sets of \( v \).

Algorithm for finding minimizer

Given a rectilinear polygon \( D \), we introduce functional \( J_{\ell^2} \) as the \( \ell^2 \)-anisotropic version of the standard functional stated in [Sci].

\[
J_{\ell^2}(u) = \int_D \left( | \nabla u | + \frac{m}{2\widetilde{\varepsilon}} (u - v)^2 \right) \ dx \ dy
\]

where \( \nabla_{\ell^2} \) is the \( \ell^2 \)-anisotropic perimeter of \( u \) in \( D \) equal to \( \int_{\partial u \cap \partial D} |n| \Omega_{\ell^2} \ dx \). For Lipschitz \( u \), first we denote by \( E \) the largest minimizer of \( J_{\ell^2}(u) \). Then, in the \( x \) or \( y \) step we denote by \( R \) the \( \delta \)-anisotropic version of E.

Preservation of continuity

Approximating functions with PCR functions, we can use explicit description of their evolution to obtain

Theorem 2. \( u \) is piecewise constant on \( \mathbb{R}^2 \) and \( \delta = \ell^2 \) in continuous, then the minimizer of (5) is \( \ell^2 \) continuous. Furthermore, if \( u \) satisfies

\[
|u(x_1,y_1) - u(x_2,y_2)| \leq \omega(\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}) \]

with \( \omega \), \( \omega \rightarrow 0 \) as \( \| \cdot \|_{\ell^2} \rightarrow 0 \), \( \| \cdot \|_{\ell^2} \)-norm decreasing, then \( u \) satisfies the same inequality.

This follows from a lemma saying that even though minimizing (5) may increase jump-size, the maximal jump does not increase: Note that if \( u \) is not convex, Theorem 2 does not hold in general (see example below).

Generalizations

- Using similar methods, we can prove analogous results for the \( \ell^p \)-anisotropic total variation flow \( u = \delta_{\ell^p}(\cdot) |\nabla u| \).
- We expect that similar results hold in the case that \( \delta \) is \( n \)-dimensional, \( n > 2 \). However, our current approximation lemma works only in \( \mathbb{R}^2 \).
- We expect our results could be generalized to the vector-valued case.