Uniform a-priori bounds for solutions of fourth-order semilinear problems in critical dimension

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Abstract

We establish uniform a-priori bounds for the solutions of the semilinear problem $\Delta^4 u = h(x, u)$ in the unit ball $B \subset \mathbb{R}^N$ endowed with Dirichlet boundary conditions, where $h : B \times \mathbb{R} \to \mathbb{R}$ is a superlinear and subcritical nonlinearity in the sense of Trudinger-Moser inequality. Uniform a-priori estimates are indeed the main ingredient in the proof of the existence of positive solutions via degree theory. The same analysis can be performed also in the general polyharmonic framework and in the case of Navier boundary conditions. This work has been carried out under the supervision of François Hamel, Enea Parini and Bernhard Ruf.

1. The problem, the state of art and the main assumptions

Find a constant $C > 0$ depending on the data of the problem $\Omega, h$ such that

$$||u||_{L^\infty(\Omega)} \leq C$$

for all solutions of the problem

$$\begin{align*}
\Delta^4 u &= h(x,u) & \text{in } \Omega, \\
u &= u_{x_1} = 0 & \text{on } \partial \Omega.
\end{align*} \tag{1}$$

2nd order ($m = 1, - \Delta$) 4th order ($m = 2, \Delta^2$)

| $N > 2m$ | by the Trudinger-Moser inequality, we consider exponential nonlinearities up to $t \to e^{ct}$. |
| $N = 2m$ | by Adams inequality, we can consider exponential nonlinearities up to $t \to e^{ct^2}$. |

Brezis-Turner, 77; Gidas-Spruck, 81; de Figueiredo-Lions-Nussbaum, 82

Oswald, 85; Soranzo, 94; Reichel-Weth, 09

The sequence $\{u_k\}$ is bounded in $C^{2m}(\Omega)$ for any $p > 1$, so is compact in $C^{2m}(\Omega)$ and the limit profile $u$ is a strong solution of

$$\Delta^4 u = \alpha \frac{u}{(\sqrt{\alpha} u + 1)^{3/2}} \quad \text{in } \mathbb{R}^N,$$

where we recall $\alpha = \lim_{x \to x_k} u(x) \in [0, \infty[$.

We have to distinguish two cases:

3.1 $f$ is subcritical: $\beta \in (0, \infty)$

It is rather easy to infer a contradiction:

$$+\infty \to \int_{\Omega} \rho(x)\|u\|_4^4 \leq \int_{\Omega} \rho(x)\|h(x, u_\rho(x))\|_4^4 \leq C \rho(x)$$

Theorem 1. [Subcritical case] Let $B \subset \mathbb{R}^N$ be the unit ball and $h$ be a subcritical nonlinearly satisfying assumptions (H4)-(H2) and the ones on $\alpha$ of Proposition 1, according to the growth of $f$. Then there exists $C > 0$ such that $||u||_{L^\infty(\Omega)} \leq C$ for all solutions $u$ of (1).

3.2 $f$ is critical: $\beta = \infty$

• By a result of [WX99], we characterize explicitly the limit profile: $u(x) = -4k_0|\Omega|^{1/4}$. Moreover, suppose that $\Omega$ is a ball and $h(\cdot, u_{\rho}(x)) \in L^4(\Omega) \cap L^\infty(\Omega)$, and that

$$\sigma(x) \geq \rho(x) \|h(x, u_\rho(x))\|_4 \geq \Theta > 0.$$

• The set $S$ of blow-up points is finite and $u_{\rho}(x)$ is uniformly bounded in $W^{2m, \infty}(S \setminus \{x\})$.

• Applying a Pohozaev identity in a neighborhood of a blow-up point $x_\rho$, we infer that

$$\sigma(x) \geq \rho(x) \|h(x, u_\rho(x))\|_4 \geq \Theta > 0.$$ 

which contradicts (2). We can thus state the following result.

Theorem 2. [Critical case] Let $B \subset \mathbb{R}^N$ be the unit ball and $h$ be a critical nonlinearity satisfying assumptions (H4)-(H2) and the ones on $\alpha$ of Proposition 1 according to the growth of $f$. Moreover, suppose that $\Omega$ is a ball and $h(\cdot, u_{\rho}(x)) \in L^4(\Omega) \cap L^\infty(\Omega)$, such that the problem (1) with $h(t, u) = A(t)f(u)$ and $\Omega = B(0)$ admits a solution $u \in L^\infty$. It suffices to consider $w(x) = w(|x|) = \sigma(x) = |x|^{n/2}|\log|x|^{1/2}|^{1/2}$, with $n < \frac{4}{3}$ and $\lambda \in C^2(\Omega \setminus \{0\})$, radial and decreasing, with values in $[0, 1]$ and $0$ in $\{1\}$. In this case, $\rho$ is chosen so that $\lim_{x \to 0^+} \rho(x) = 0$ and $\lambda(x) = c e^{-\lambda \Delta^4 u}$. We infer the existence of a positive solution for (1):

$$\sigma(x) = \rho(x) \|h(x, u_\rho(x))\|_4 \geq \Theta > 0.$$ 

4. Sharpness of the result: a counterexample

• The whole analysis can be performed also in the polyharmonic context ($m=\alpha > 0$) and with Navier boundary conditions in smooth bounded and convex domains.

Finally, suppose moreover that...

$$\lim_{x \to x_k} \frac{\|u\|_4}{\rho(x)} < \frac{\lambda}{4} \text{ uniformly in } B,$$

and applying the Krasnoselski"i degree theory we infer the existence of a positive solution for (1):

$$\sigma(x) \geq \rho(x) \|h(x, u_\rho(x))\|_4 \geq \Theta > 0$$

and there exists $\lambda > 0$ such that $\|h\|_{L^\infty(\Omega \setminus \{x\})}$ and we can extend Theorems 1 and 2 in the case of a bounded domain of class $C^\infty$.

5. Consequences and generalizations

• The whole analysis can be performed also in the polyharmonic context ($m=\alpha > 0$) and with Navier boundary conditions in smooth bounded and convex domains.

6. An open problem

We obtained uniform a-priori bounds only in the case of the ball (see Section 2).

References


