QUASICONCAVE SOLUTIONS TO ELLIPTIC PROBLEMS IN CONVEX RINGS

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Abstract. We investigate the convexity of level sets of solutions to general elliptic equations in a convex ring Ω. In particular, if u is a classical solution which has constant (distinct) values on the two connected components of ∂Ω, we consider its quasi-concave envelope $u^*$ (i.e. the function whose superlevel sets are the convex envelopes of those of u) and we find suitable assumptions which force $u^*$ to be a subsolution of the equation. If a comparison principle holds, this yields $u = u^*$ and then u is quasi-concave.

1. Introduction

Let $Ω_{t_0}$ and $Ω_{t_1}$ be two bounded open subsets of $\mathbb{R}^n$ such that $\overline{Ω}_{t_1} \subset Ω_{t_0}$ and let $Ω = Ω_{t_0} \setminus \overline{Ω}_{t_1}$. Let $u \in C^2(Ω) \cap C(Ω)$ be a classical solution of the following Dirichlet problem

$$
\begin{cases}
F(x, u, Du, D^2u) = 0 & \text{in } Ω \\
u = t_0 & \text{on } ∂Ω_{t_0} \\
u = t_1 & \text{on } ∂Ω_{t_1},
\end{cases}
$$

where $t_0 < t_1$ and $F(x, t, p, A)$ is a proper and (degenerate) elliptic operator defined on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S_n$ (here and throughout $S_n$ denotes the set of real symmetric $n \times n$ matrices).

If both $Ω_{t_0}$ and $Ω_{t_1}$ are convex (and $t_0 \leq u \leq t_1$ in $Ω$), it is quite natural to ask whether all the superlevel sets of u are convex or, in other words, to ask whether u is a quasi-concave function (note that, throughout the paper, we systematically extend $u = t_1$ in $Ω_{t_1}$). Without suitable assumptions, the answer can be negative (see [14], for instance), then we look for conditions on the operator $F$ which guarantee an affirmative answer. This problem has been extensively studied in literature: we refer to the classic monograph by Kawohl [7] for a thorough picture of the state of the art up to 1985 and to [5] and [6] for more recent references.

The strategy we will use here is to consider the so called quasi-concave envelope $u^*$ of u, i.e. the smallest quasi-concave function greater than or equal to u (or, equivalently, the continuous function whose superlevel sets

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are the convex envelopes of the corresponding superlevel sets of \( u \) and to find assumptions which force \( u \) to coincide with \( u^* \). In particular, since \( u^* \geq u \) by definition, it is sufficient to care only about the reverse inequality, which can be obtained via a (viscosity) comparison principle when \( u^* \) is a (viscosity) subsolution of (1.1). Then our aim is to find suitable assumption on the operator \( F \) which force \( u^* \) to be a subsolution.

This technique, suggested in [7], was firstly successfully employed in [3] and then further developed in [5]; moreover, it was also adapted to study the starshapedness of level sets in [15]. In [3], there were proved some relations between the double normal derivatives and the tangential part of the Laplacian of \( u^* \) and the corresponding quantities of \( u \), calculated at suitably related points; hence, it was possible to treat operators which are easily decomposable in a normal (with respect to the level sets of the involved function) and a tangential part. In [5] this restriction about the structure of the operator has been avoided thanks to a limit process: indeed, the quasi concave envelope \( u^* \) can be seen as the limit as \( p \) tends to \(-\infty\) of the \( p\)-concave envelope \( u_p \) of \( u \), that is the smallest \( p\)-concave function greater than or equal to \( u \) (we recall that \( p\)-concave, when \( p<0 \), means that \( u_p \) is convex). This allows to treat more general operators, but doesn’t permit to obtain sharp results, neither to retrieve already known results in some basic cases.

In the present paper, thanks to some new results from [13] about the Minkowski addition of functions (see §2.3 for definition and properties), we exploit further the properties of the quasi concave envelopes of solutions to (1.1) and, overacting a little bit, we could say that we push forward the above depicted method near to its natural limit.

In §3 you can find our main result, Theorem 3.8, that establishes conditions which compel \( u^* \) to be a subsolution of (1.1). Unluckily, Theorem 3.8 is not easy to manage, due to the difficulty to check the fulfilment of its main assumption, requirement (3.12). On the other hand, it is possible to give a weakened form of it, Theorem 3.1, which is more immediate and easier to apply. For the reader’s convenience, we state here the main consequence of Theorem 3.1 regarding the quasi-concavity of solutions of (1.1).

**Theorem 1.1.** Let \( \Omega = \Omega_{t_0} \setminus \Omega_{t_1} \) be a convex ring and let \( F(x,u,q,A) \) be a proper, continuous, degenerate elliptic operator in \( \Omega_{t_0} \times (t_0,t_1) \times \mathbb{R}^n \times \Gamma_F \) which satisfies a viscosity comparison principle. Assume that

\[
G_0 = \left\{ (x,p,A) \in \Omega_{t_0} \times (0,\infty) \times \Gamma_F : F\left(x,t,\theta^p,A\right) \geq 0 \right\}
\]

is convex for every choice of \((t,\theta) \in (t_0,t_1) \times S^{n-1}\).

If \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) is an admissible classical solution of (1.1) such that \(|Du| > 0 \) in \( \Omega \), then \( u \) is quasi-concave.

We refer the reader to §2.2 for the notion of admissible solution and the definition of the cone \( \Gamma_F \).
In fact, the assumption (1.2) (and also the assumption (3.10) of Theorem 3.8) may also be weakened (see Remark 3.2), but in most of the applications it is convenient to replace it with some stronger one like

\[ |G(t, x, p, A)| \]

for some \( \alpha \in \mathbb{R} \) and for every fixed \( (t, \theta) \in (t_0, t_1) \times S^{n-1} \), the function

\[ G_t, \theta, \alpha(x, p, A) = p^\alpha F(x, t, \theta, p, A) \]

is quasi-concave in \( \Omega_{t_0} \times (0, +\infty) \times \Gamma_F \)

or like the following, which is even stronger, but easier to check:

\[ |G(t, x, p, A)| \]

for some \( \alpha \in \mathbb{R} \), the function \( G(t, \theta, \alpha) \) is concave in \( \Omega_{t_0} \times (0, +\infty) \times \Gamma_F \)

for every fixed \( (t, \theta) \in (t_0, t_1) \times S^{n-1} \).

Obviously, (1.4) implies (1.3) which implies (1.2).

The main ingredients in the proof of Theorem 3.1 and Theorem 3.8 are Proposition 2.2 and Proposition 2.3, respectively. These propositions are refinements of analogous results proved in [13] and essentially describe the differential properties at relevant points of Minkowski addition of locally \( Q^2 \) functions (see §2.2 and §2.3). In fact, Proposition 2.3 implies Proposition 2.2 and hence Theorem 3.1 can be seen as a corollary of Theorem 3.8, as we will show in §3.3. However, for simplicity and clarity reasons, we chose to present them as separate results and to emphasize Theorem 3.1, which is less involved and at the same time maintains a wide range of applications.

Notice that throughout the paper we assume

\[ |Du(x)| \neq 0 \quad \text{for every } x \in \Omega. \]

This guarantees that \( t_0 < u < t_1 \) in \( \Omega \) and that for every \( t \in [t_0, t_1] \) the superlevel set \( \Omega(t) = \{ x \in \Omega : u(x) \geq t \} \cup \Omega_{t_1} \) is simply connected with

\[ \partial \Omega(t) = \{ x \in \mathbb{R}^n : u(x) = t \}. \]

This is a typical assumption for this kind of investigations. Finding geometric properties of level sets of \( u \) without (1.5) is partly an open problem; contributions to this question can be found in [7], [8] and [6].

The paper is organized as follows. In §2 we introduce notation and we recall some notions and results we will need later; moreover we prove Proposition 2.2 and Proposition 2.3 that are fundamental ingredients for the next section. In §3 we will prove our main results, that are Theorem 3.1 and Theorem 3.8, and then we will show some applications of both theorems, also discussing and proving the exact relation occurring between them. In the Appendix, we prove some technical results needed in the previous sections.

2. Preliminaries

2.1. Basic notation. In the \( n \)-dimensional Euclidean space we denote by \( \langle \cdot, \cdot \rangle \) the classical Euclidean scalar product and by \( |\cdot| \) the Euclidean norm.

\[ \langle \cdot, \cdot \rangle \]

\[ |\cdot| \]
The set of real symmetric $n \times n$ matrices is denoted by $\mathcal{S}_n$, while $\mathcal{S}_n^+$ is the subset of $\mathcal{S}_n$ of positive semidefinite matrices. For $A, B \in \mathcal{S}_n$, by $A \geq 0$ we mean $A \in \mathcal{S}_n^+$ and by $A \leq B$ we mean $B - A \geq 0$, i.e. $B - A \in \mathcal{S}_n^+$. Moreover, we denote by $\det(A)$ the determinant of $A$, by $\text{tr}(A)$ its trace and by $A^T$ its transpose matrix.

For $K \subseteq \mathbb{R}^n$, we denote by $\overline{K}$ its closure, by $\partial K$ its boundary and by $K^*$ its convex hull. Let $K_0, K_1$ be two subsets of $\mathbb{R}^n$ and let $\lambda \in [0, 1]$: we call Minkowski linear combination of $K_0, K_1$ (of ratio $\lambda$) the set

$$K_\lambda = (1 - \lambda)K_0 + \lambda K_1 = \{(1 - \lambda)x + \lambda y \mid x \in K_0, y \in K_1\}.$$ 

More generally, for $m \in \mathbb{N}$ we set

$$\Lambda_m = \left\{ (\lambda_1, \ldots, \lambda_m) : \lambda_i \in [0, 1], \sum_{i=1}^m \lambda_i = 1 \right\}$$

and, for $K_1, \ldots, K_m \subseteq \mathbb{R}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m$,

$$K_\lambda = \sum_{i=1}^m \lambda_i K_i = \left\{ \sum_{i=1}^m \lambda_i x_i : x_i \in K_i, i = 1, \ldots, m \right\}.$$

Notice that, from Carathéodory’s theorem,

$$K_* = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : \lambda \in \Lambda_{n+1}, x_i \in K \right\} = \bigcup_{\lambda \in \Lambda_{n+1}} K_\lambda. \tag{2.1}$$

Throughout the paper $\Omega, \Omega_{t_0}$ and $\Omega_{t_1}$ will be non-empty, open, convex, bounded subsets of $\mathbb{R}^n$; usually, they will be such that $\overline{\Omega_{t_1}} \subset \Omega_{t_0}$ and $\Omega = \Omega_{t_0} \setminus \overline{\Omega_{t_1}}$.

Given a function $u$ of class $C^2(\Omega)$, $Du = (D_1u, \ldots, D_nu)$ and $D^2u = (D_{ij}u)_{i,j=1}^n$ denote its gradient and its Hessian matrix respectively. Moreover, we define its hemmed Hessian matrix $\tilde{D}^2u \in \mathcal{S}_{n+1}$ as follows:

$$\tilde{D}^2u = \begin{pmatrix} D^2u & |Du|Du^T \\ |Du|Du & 0 \end{pmatrix}.$$

Finally, for $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ is the open euclidean ball of radius $r$ centered at $x$, i.e.

$$B(x, r) = \{ z \in \mathbb{R}^n : |z - x| < r \}.$$  

2.2. Viscosity solutions. In the paper we will use some notions from the theory of viscosity solutions of elliptic equations. Next we recall some basic definitions and facts and we refer the reader to [4] and [11] for more details.

An operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ is said to be proper if

$$F(x, s, q, A) \leq F(x, t, q, A) \text{ whenever } s \geq t. \tag{2.2}$$

Let $\Gamma$ be a convex cone in $\mathcal{S}_n$ with vertex at the origin and containing $\mathcal{S}_n^+$. Then $F$ is said degenerate elliptic in $\Gamma$ if

$$F(x, t, q, A) \geq F(x, t, q, B), \text{ for every } A, B \in \Gamma \text{ such that } A \geq B. \tag{2.3}$$
We put $\Gamma_F = \bigcup \Gamma$, where the union is extended to every cone $\Gamma$ such that $F$ is degenerate elliptic in $\Gamma$; when we say that $F$ is degenerate elliptic, we mean that $F$ is degenerate elliptic in $\Gamma_F \neq \emptyset$. If $F$ is a degenerate elliptic operator, we say that a function $u \in C^2(\Omega)$ is admissible for $F$ if $D^2 u(x) \in \Gamma_F$ for every $x \in \Omega$.

Two paradigmatic cases: (i) if $F$ is the Laplace operator $\Delta u$, then $\Gamma_F = S^n$ and every $C^2$ function is admissible for $F$; (ii) if $F$ is the Monge-Ampère operator $\det D^2 u$, then $\Gamma_F = S_n^+$ and convex functions only are admissible for $F$.

Let $u$ be an upper semicontinuous function and let $\phi$ be a continuous function in $\Omega$. Consider $x_0 \in \Omega$. We say that $\phi$ touches $u$ from above at $x_0$ if

$$\phi(x_0) = u(x_0) \quad \text{and} \quad \phi(x) \geq u(x),$$

in a neighbourhood of $x_0$ (i.e. $\phi(x_0) = u(x_0)$ and $\phi - u$ has a local minimum at $x_0$). Analogously, if $u$ is lower semicontinuous, we say that $\phi$ touches $u$ from below at $x_0$ if

$$\phi(x_0) = u(x_0) \quad \text{and} \quad \phi(x) \leq u(x),$$

in a neighbourhood of $x_0$ (i.e. $\phi(x_0) = u(x_0)$ and $\phi - u$ has a local maximum at $x_0$).

An upper semicontinuous function $u$ is a viscosity subsolution of the equation

$$F(x, u, Du, D^2 u) = 0,$$

if, for every $C^2$ function $\phi$ touching $u$ from above at any point $x \in \Omega$, it holds

$$F(x, u(x), D\phi(x), D^2 \phi(x)) \geq 0.$$ 

Analogously, a lower semicontinuous function $u$ is a viscosity supersolution of

$$F(x, u, Du, D^2 u) = 0$$

if, for every admissible $C^2$ function $\phi$ touching $u$ from below at any point $x \in \Omega$, it holds

$$F(x, u(x), D\phi(x), D^2 \phi(x)) \leq 0.$$ 

A viscosity solution is a continuous function which is, at the same time, subsolution and supersolution of (2.4).

In our assumptions, a classical solution is always a viscosity solution and a viscosity solution is a classical solution if it is regular enough.

An useful tool to apply our technique is the comparison principle for viscosity solution. We say that an operator $F$ satisfies the viscosity comparison principle if the following holds:

$$F(x, u(x), D\phi(x), D^2 \phi(x)) \geq 0.$$

Let $u \in C(\Omega)$ and $v \in C(\Omega)$ be, respectively, a viscosity supersolution and a viscosity subsolution of $F = 0$ such that $u \geq v$ on $\partial \Omega$; then $u \geq v$ in $\Omega$.

To find conditions which assure that $F$ satisfies the comparison principle is a difficult and current field of investigation (see, for instance, [12, 9, 10]).
2.3. Quasi-concave and $Q_2^-$ functions. A function $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is said quasi-concave if it has convex superlevel sets or, equivalently, if

$$
u \left((1 - \lambda)x_0 + \lambda x_1\right) \geq \min\{u(x_0), u(x_1)\},$$

for every $\lambda \in [0, 1]$, and every $x_0, x_1 \in \mathbb{R}^n$. If $u$ is defined only in a proper subset $\Omega$ of $\mathbb{R}^n$, we extend $u$ as $-\infty$ in $\mathbb{R}^n \setminus \Omega$ and we say that $u$ is quasi-concave in $\Omega$ if such an extension is quasi-concave in $\mathbb{R}^n$.

Moreover, $u$ is said quasi-convex if $-u$ is quasi-concave, i.e. if it has convex sublevel sets.

Obviously, if $u$ is concave (convex) then it is quasi-concave (quasi-convex). Moreover, quasi-concave (quasi-convex) functions can be easily obtained by composing a monotone increasing real function with a concave (convex) function. In particular, examples of quasi-concave functions are $p$-concave functions ($p \neq 0$) and log-concave functions (if $u$ is a non-negative function, we say that it is $p$-concave if $u^p$ is concave and $p > 0$ or if $u^p$ is convex and $p < 0$; we say instead that $u$ is log-concave if $\log u$ is concave; analogous considerations and definitions hold for quasi-convex, $p$-convex and log-convex functions).

If $u$ is an upper semicontinuous function, we denote by $u^*$ its quasi-concave envelope. Roughly speaking, $u^*$ is the upper semicontinuous function whose superlevel sets are the closed convex hulls of the corresponding superlevel sets of $u$. More precisely, let us indicate by $\Omega(t)$ the superlevel set of $u$ of value $t$, i.e.

$$\Omega(t) = \{x \in \mathbb{R}^n \mid u(x) \geq t\},$$

and by $\Omega^*(t)$ the closure of its convex hull. Then $u^*$ is defined by

$$\Omega^*(t) = \{x \in \mathbb{R}^n \mid u^*(x) \geq t\} \quad \text{for every } t \in \mathbb{R},$$

that is

$$u^*(x) = \sup\{t \in \mathbb{R} \mid x \in \Omega^*(t)\}.$$

Notice that $u^*$ is the smallest upper semicontinuous quasi-concave function larger than $u$, hence

$$(2.6) \quad u^* \geq u.$$

By the way, we recall that a function is upper semicontinuous if and only if it has closed superlevel sets.

Finally, we recall a notion, introduced in [13], which corresponds to a local strengthened version of quasi-concavity.

**Definition 2.1.** Let $u$ be a function defined in an open set $\Omega \subset \mathbb{R}^n$; we say that $u$ is a $Q_2^-$ function at a point $x \in \Omega$ if:

- $u$ is of class $C^2$ in a neighbourhood $U$ of $x$;
- its gradient does not vanish at $x$ (i.e. $|Du(x)| > 0$);
- the principal curvatures of $\{y \in \mathbb{R}^n \mid u(y) = u(x)\} \cap U$ with respect to the normal $-\frac{Du(x)}{|Du(x)|}$ are positive at $x$. 
With \( u \in Q^2(x) \) we mean that \( u \) is \( Q^2 \) at \( x \) and we write \( u \in Q^2(\Omega) \) when \( u \in Q^2(x) \) for every \( x \in \Omega \). Moreover, we write \( u \in Q^2(x) \) if \(-u \in Q^2(x)\) and so on (compare with [13] where this notion is introduced in the context of quasi-convex functions).

In other words, a \( C^2 \) function \( u \) is \( Q^2 \) at a regular point \( \bar{x} \) if its level set \( \{ x : u(x) = u(\bar{x}) \} \) is locally a regular convex surface (oriented according to \(-Du\)), whose Gauss curvature doesn’t vanish.

If \( u \in Q^2(x) \), choosing a coordinate system centered at \( x \) with the principal directions of curvature of the level surface passing through \( x \) as the first \( n - 1 \) coordinate directions and \( Du(x)/|Du(x)| \) as the last coordinate direction (that is the so called principal coordinate system), then \( \widehat{D^2}u \) takes the following simplified form (2.7)

\[
\widehat{D^2}u(x) = \begin{pmatrix}
-k_1|Du(x)| & \cdots & 0 & u_{1n} & 0 \\
0 & \ddots & 0 & \vdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & -k_{n-1}|Du(x)| & u_{n-1\,n} & 0 \\
u_{n1} & \cdots & u_{n-1\,n} & u_{nn} & |Du(x)|^2 \\
0 & \cdots & 0 & |Du(x)|^2 & 0
\end{pmatrix}
\]

where \( k_i, i = 1, \ldots, n - 1 \), denotes the \( i \)-th principal curvatures of the level surface at \( x \).

### 2.4. Minkowski addition of functions.

Let \( u_1, \ldots, u_m \) be upper semicontinuous functions defined in \( \Omega_1 \ldots \Omega_m \subset \mathbb{R}^n \), respectively, and \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m \). The **Minkowski linear combination of the functions** \( u_i \) **with ratio** \( \lambda \) is the upper semicontinuous function \( u_\lambda \) whose super-level sets \( \Omega_\lambda(t) = \{ u_\lambda \geq t \} \) are the Minkowski linear combination of the corresponding super-level sets \( \Omega_i(t) = \{ u_i \geq t \} \) of \( u_i \), i.e.

\[
\Omega_\lambda(t) = \sum_{i=1}^{m} \lambda_i \Omega_i(t), \quad \text{for every } t \in \mathbb{R},
\]

and

\[
u_\lambda(x) = \sup \{ t : x \in \Omega_\lambda(t) \}.
\]

Sometimes, we will write

\[
u_\lambda = \sum_{i=1}^{m} \lambda_i u_i,
\]

to mean that \( u_\lambda \) is the Minkowski linear combination of the functions \( u_i \)'s with coefficients \( \lambda_i \)’s. Obviously, the Minkowski addition of functions preserves the quasi-concavity property: if \( u_1, \ldots, u_m \) are quasi-concave functions, then \( u_\lambda \) is quasi-concave as well.

If one consider \( n + 1 \) copies of the same function

\[
u_1 = \nu_2 = \cdots = \nu_{n+1} = u,
\]
thanks to (2.1), \( \Omega^*(t) \) can be seen as the closed union of the convex combinations of \( \Omega(t) \),

\[
\Omega^*(t) = \bigcup_{\lambda \in \Lambda_{n+1}} \Omega_{\lambda}(t),
\]

and hence \( u^*(x) \) is the upper semicontinuous envelope of \( \sup_{\lambda \in \Lambda_{n+1}} \{u_{\lambda}(x)\} \).

In fact, we are not going to use explicitly these observations, but they are in some sense underlying to the proofs of our main theorems.

As shown in [13] the gradient and the Hessian matrix of \( u_{\lambda} \) are strictly connected with the gradient and the Hessian matrix of the functions \( u_i \).

**Proposition 2.2.** Let \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m \), \( u_1, \ldots, u_m \) be \( C^2 \) functions defined in \( \Omega_i \subseteq \mathbb{R}^n \), respectively, and \( \bar{x} = \sum_{i=1}^{m} \lambda_i x_i \) for some \( x_i \in \Omega_i \).

Moreover let \( u_{\lambda} \) be the Minkowski linear combination of \( u_i \) with ratio \( \lambda \) and let \( t = u_{\lambda}(\bar{x}) \). Assume that, for every \( i = 1, \ldots, m \):

(i) \( u_i(x_i) = t \),
(ii) \( u_i \in Q_2^2(x_i) \),
(iii) \( Du_i(x_i) \) are parallel equally oriented vectors,
(iv) the orthogonal hyperplane to \( Du_i(x_i) \) supports \( \Omega_i \) only at \( x_i \), i.e.

\[
\langle Du_i(x_i), x - x_i \rangle < 0 \quad \text{for every } x \in \Omega_i \setminus \{x_i\}.
\]

Then \( u_{\lambda} \in Q_2^2(\bar{x}) \) and it holds:

\[
(2.8) \quad \frac{1}{|Du_{\lambda}(\bar{x})|} = \sum_{i=1}^{m} \frac{\lambda_i}{|Du_i(x_i)|},
\]

\[
(2.9) \quad \frac{D^2u_{\lambda}}{|Du_{\lambda}|^3}(\bar{x}) \geq \sum_{i=1}^{m} \frac{\lambda_i}{|Du_i|^3}(x_i).
\]

**Proof.** In [13], (2.8) and (2.9) are proved for functions \( u_i \)'s that are quasi-concave (see Proposition 7.3 and Proposition 7.4 therein). To prove the same under our weaker assumptions, it is sufficient to notice that, by assumption (ii), for every \( i \in \{1, \ldots, m\} \) there exists a ball \( B_i = B(x_i, r_i) \), with \( r_i > 0 \), such that the restriction of \( u_i \) at \( B_i \) is quasi-concave, i.e. the function

\[
v_i(x) = \begin{cases} 
u_i(x) & \text{if } x \in B_i \\ -\infty & \text{otherwise}, \end{cases}
\]

is quasi-concave in \( \mathbb{R}^n \) and \( Q_2^2 \) at \( x_i \). Moreover, thanks to assumption (iv), possibly diminishing the \( r_i \)'s, we have that \( v_{\lambda} = \sum \lambda_i v_i \) coincides with \( u_{\lambda} \) in \( B_{\lambda} = \sum \lambda_i B_i \); indeed, our assumptions assure the uniqueness of the representation of \( \bar{x} \) as \( \lambda \)-convex combination of points \( x_i \) from the level sets \( \{u_i \geq t\}, i = 1, \ldots, m \), and that, thanks also to the regularity of the involved functions, this unique representation holds true in a small ball near to \( \bar{x} \).

Now we have only to apply Proposition 7.3 and Proposition 7.4 of [13] to \( v_{\lambda} \) and \( v_i \)'s. \( \square \)
Formula (2.9) is improved in the following proposition by finding an exact relation between the hemmed Hessian matrix of \( u_\lambda \) at the point \( \bar{x} \) and the hemmed Hessian matrices of the functions \( u_i \) at the points \( x_i \) (see §2.1 for the needed definitions).

**Proposition 2.3.** In the same assumptions of Proposition 2.2, we have

\[
(2.10) \quad \left( \frac{D^2 u_\lambda}{|Du_\lambda|}(\bar{x}) \right)^{-1} = \sum_{i=1}^{m} \lambda_i \left( \frac{D^2 u_i}{|Du_i|}(x_i) \right)^{-1}.
\]

**Proof.** This proposition is a refinement of [13, Theorem 7.6] and it can be proved with the same arguments of Proposition 2.2. Notice that, if \( u \) is \( Q^2 \) at \( x \), then \( \hat{D}^2 u_\lambda(x) \) is invertible, since

\[
\det \hat{D}^2 u = (-1)^n |Du(x)|^{n+3} k(x) \neq 0,
\]

where \( k(x) \) is the Gaussian curvature of \( \{ u = t \} \) at \( x \), as can be calculated by formula (2.7). \( \square \)

Notice that, (2.10) being an equality, it is sharp and then it is better than (2.9); on the other hand, the latter involves directly \( D^2 u_\lambda \) instead of \( \hat{D}^2 u_\lambda \) and then it is more handy and easier to apply. In fact, (2.9) is a straightforward consequence of (2.10), via Lemma 3.11.

## 3. The main theorems

Thanks to Proposition 2.2 and Proposition 2.3, it is possible to prove two results on quasi-concavity of solutions of (1.1). The one based on formula (2.10), namely Theorem 3.8, is of course sharper, but difficult to apply; Theorem 3.1, based on the inequality (2.9), is weaker, but easier to apply and, at the same time, sufficient to retrieve most of the known results and to obtain some new ones.

**Theorem 3.1.** Let \( \Omega = \Omega_{t_0} \setminus \Omega_{t_1} \) be a convex ring and let \( F(x, u, q, A) \) be a proper, continuous and degenerate elliptic operator in \( \Omega \times (t_0, t_1) \times \mathbb{R}^n \times \Gamma_F \).

Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be an admissible classical solution of \((1.1)\) such that \( |Du| > 0 \) in \( \Omega \). Assume \((1.2)\), then \( u^* \) is a viscosity subsolution of \((1.1)\).

**Proof.** Let \( \bar{x} \in \Omega \) and let \( \varphi \) be a \( C^2 \) function that touches \( u^* \) from above; we want to show that \( F(\bar{x}, u^*(\bar{x}), D\varphi(\bar{x}), D^2\varphi(\bar{x})) \geq 0 \). If \( u(\bar{x}) = u^*(\bar{x}) \) then \( \varphi \) touches \( u \) from above at \( \bar{x} \) too and the following hold:

\[
\varphi(\bar{x}) = u(\bar{x}),
D\varphi(\bar{x}) = Du(\bar{x}),
D^2\varphi(\bar{x}) \geq D^2 u(\bar{x}).
\]

Hence by the ellipticity property of \( F \) we have:

\[
F(\bar{x}, u^*(\bar{x}), D\varphi(\bar{x}), D^2\varphi(\bar{x})) \geq F(\bar{x}, u(\bar{x}), Du(\bar{x}), D^2 u(\bar{x})) = 0,
\]

where the last equality holds for \( u \) is a classical solution of \((1.1)\).
Assume now $u^*(\bar{x}) > u(\bar{x})$ and let $t = u^*(\bar{x})$. By Carathéodory’s theorem (see also [3]) there exist $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $0 \leq \lambda_i \leq 1$, $i = 1, \ldots, n$, $\sum_{i=1}^n \lambda_i = 1$ and $x_1, \ldots, x_n \in \partial \Omega(t)$ (not necessarily distinct) such that

$$\bar{x} = \sum_{i=1}^n \lambda_i x_i \quad \text{and} \quad u^*(\bar{x}) = u(x_1) = \ldots = u(x_n) = t.$$ 

Notice that, since the gradient of $u$ never vanishes in $\Omega$, $\bar{x}$ belongs to the boundary of $\Omega^*(t)$.

Let us consider the Minkowski linear combination of $n$ copies of the functions $u$ with coefficients $\lambda_1, \ldots, \lambda_n$; i.e. we set $u_i = u$ for $i = 1, \ldots, n$ and $u_\lambda = \sum_{i=1}^n \lambda_i u_i$. This means that each superlevel set of $u_\lambda$ is the Minkowski combination of $n$ copies of the corresponding superlevel set of $u$ with ratio $\lambda_i$. Then it holds:

$$u^*(\bar{x}) = u_\lambda(\bar{x}) = u_i(x_i) = t, \quad i = 1, \ldots, n, \quad \text{and} \quad u^*(x) \geq u_\lambda(x) \geq u_i(x) \quad \text{for every} \quad x \in \Omega.$$

This implies that the support hyperplane to $\Omega^*(t)$ at $\bar{x}$ supports $\Omega_\lambda(t)$ at $\bar{x}$ and $\Omega(t)$ at $x_i$ too and hence $D u_\lambda(\bar{x})$, $D u_i(x_i)$ are parallel equally oriented vectors for $i = 1, \ldots, n$.

Now, we would like to use Proposition 2.2. Notice that assumption (ii) may be easily not satisfied and moreover (iv) doesn’t hold true for sure since the functions $u_i$ are all coincident with $u$ and every $x_i$ belongs to the same support hyperplane $\Pi_i$. We will show later how to get rid of this problem; for the moment, in order to show clearly the crucial ideas of the proof, let us work as if we could use formulae (2.8), (2.9) without problems.

Since $\varphi$ touches $u^*$ from above at $\bar{x}$, then $\varphi$ touches $u_\lambda$ from above at $\bar{x}$, and this implies that

$$\varphi(\bar{x}) = u^*(\bar{x}) = u_\lambda(\bar{x}) = t,$$

$$D \varphi(\bar{x}) = D u_\lambda(\bar{x}),$$

$$D^2 \varphi(\bar{x}) \geq D^2 u_\lambda(\bar{x}).$$

Using the ellipticity of $F$ and formulae (2.8), (2.9) and indicating by $\theta$ the direction of $D u_\lambda(\bar{x})$, we obtain

$$F(\bar{x}, t, D \varphi(\bar{x}), D^2 \varphi(\bar{x})) \geq F(\bar{x}, t, D u_\lambda(\bar{x}), D^2 u_\lambda(\bar{x})) = F(\bar{x}, t, \theta |D u_\lambda(\bar{x})|, D^2 u_\lambda(\bar{x})) \geq$$

$$\geq F\left(\bar{x}, t, \sum_{i=1}^n \frac{\lambda_i}{|D u_i(x_i)|}, \left( \sum_{i=1}^n \lambda_i \frac{|D u_i(x_i)|}{|D u(x_i)|} \right)^{-3} \left( \sum_{i=1}^n \lambda_i \frac{|D^2 u(x_i)|}{|D u(x_i)|^3} \right) \right).$$

(3.1)
By the definition of the function $G_{t,\theta,\alpha}$ in (3.1), we have

$$F\left(\bar{x}, t, \frac{\theta}{\sum_{i=1}^{n} \lambda_i \rho_{x_i}}, \left(\sum_{i=1}^{n} \lambda_i \rho_{x_i}\right)^{-3} \left(\sum_{i=1}^{n} \lambda_i \frac{D^2 u(x_i)}{|D u(x_i)|^3}\right)\right) =$$

$$= G_{t,\theta,0} \left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i p_i, \sum_{i=1}^{n} \lambda_i A_i\right) = G_{t,\theta,0} \left(\bar{x}, \bar{\rho}, \bar{A}\right),$$

where

$$p_i = \frac{1}{|D u(x_i)|}, \quad A_i = \frac{D^2 u(x_i)}{|D u(x_i)|^3}, \quad i = 1, \ldots, n,$$

and we set $\bar{\rho} = \sum_{i=1}^{n} \lambda_i p_i$ and $\bar{A} = \sum_{i=1}^{n} \lambda_i A_i$.

On the other hand, $G_0$ coincides with the superlevel set 0 of $G_{t,\theta,0}$, i.e.

$$G_0 = \left\{(x, p, A) \in \Omega_0 \times (0, +\infty) \times \Gamma_F : G_{t,\theta,0}\left(x, p, A\right) \geq 0\right\}$$

and then

$$(x_i, p_i, A_i) \in G_0, \quad i = 1, \ldots, n,$$

since

$$G_{t,\theta,0}(x_i, p_i, A_i) = F\left(x_i, u(x_i), D u(x_i), D^2 u(x_i)\right) = 0,$$

for $u$ is a classical solution of problem (1.1).

Hence, assumption (1.2) finally yields

$$(\bar{x}, \bar{\rho}, \bar{A}) = \sum_{i=1}^{n} \lambda_i (x_i, p_i, A_i) \in G_0$$

and then, by (3.1), $F(\bar{x}, t, D\varphi(\bar{x}), D^2 \varphi(\bar{x})) \geq 0$, which shows that $u^*$ is a viscosity subsolution of (1.1).

In the previous argument we assumed functions $u_1, \ldots, u_n$ (which all coincide with the solution $u$) satisfy conditions (i) – (iv) of Proposition 2.2 to apply formulae (2.8), (2.9), but we already noticed that in fact we have some problems with assumptions (ii) and (iv). We show now how to fill this gap. Suppose that the function $u$ is not $Q^2$ at some $x_j$, $j = 1, \ldots, m$, $m \leq n$ (while it is $Q^2$ at $x_{m+1}, \ldots, x_n$, if $m < n$). Observe that, however, the principal curvatures $k_i$ of $\partial\Omega(t)$ at $x_j$ are all non-negative

$$(3.2) \quad k_i(x_j) \geq 0 \quad i = 1, \ldots, n-1, \quad j = 1, \ldots, m,$$

because there is a support hyperplane at $x_j$.

Define the function $u_i^{(\varepsilon)}$, $i = 1, \ldots, n$ as follows:

$$(3.3) \quad u_i^{(\varepsilon)}(x) = u(x) + \varepsilon \psi_i(x),$$

where $\psi_i \in C^2(\mathbb{R}^n)$ is such that $\psi_i \leq 0$ and

$$\psi_i(x) = \begin{cases} -\frac{1}{2}|x - x_i|^2 & \text{if } x \in B(x_i, \frac{\eta}{2}) \\ -1 & \text{if } x \in \mathbb{R}^n \setminus B(x_i, \eta); \end{cases}$$
where \( \eta \) is chosen so that \( B(x_i, \eta) \cap B(x_j, \eta) = \emptyset \) for all \( i \neq j \) and \( B(x_i, \eta) \subset \Omega \) for \( i = 1, \ldots, m \). Then, let \( \Omega_i(t) = \{ x \in \mathbb{R}^n \mid u_i(x) \geq t \} \).

Hence we have for \( i = 1, \ldots, n \)
\[
\begin{align*}
    u_i(x) < u(x) \quad & \text{for every } x \in \Omega \setminus \{x_i\} \\
    u_i(x) = u(x_i) \quad & \text{where } u_i(x) = \sum_{i=1}^{n} \lambda_i u_i(x_i), \\
\end{align*}
\]

(3.4)
\[
\begin{align*}
    D u_i(x) = D u(x_i), \\
    D^2 u_i(x) = D^2 u(x_i) - \varepsilon I,
\end{align*}
\]

where \( I \) is the \( n \times n \) identity matrix. Then \( u_i(x) \in Q^2 \) at \( x_i \) for every \( i \), since
\[
\begin{align*}
    k_i^2(x_j) = k_i(x_j) + \frac{\varepsilon}{|D u_i(x)|} > 0
\end{align*}
\]

where \( k_i(x_j) \) are the principal curvatures of \( \partial \Omega_i(t) \) at \( x_j \).

Moreover, by the first two of (3.4), the support hyperplane \( \Pi_i \) to \( \Omega(t) \) at \( x_i \), supports \( \Omega_i(t) \) at \( x_i \) only for every \( i = 1, \ldots, n \) (and for every \( \varepsilon \geq 0 \)), so that
\[
\langle Du_i(x), x - x_i \rangle < 0 \quad \text{for every } x \in \Omega_i(t) \setminus \{x_i\}.
\]

Hence, all the assumptions of Proposition 2.2 hold.

Let us indicate by \( u_\lambda \) the Minkowski linear combination of the functions \( u_1, \ldots, u_n \) with coefficients \( \lambda_1, \ldots, \lambda_n \). Let \( \varphi \) be any \( C^2 \) function touching \( u^* \) from above at \( \bar{x} \), then \( \varphi \) touches \( u_\lambda \) from above at \( \bar{x} \) too, since \( u_\lambda(\bar{x}) = u^*(\bar{x}) \) while \( u_\lambda(x) \leq u^*(x) \) for every \( x \in \Omega \). Now, we can repeat the argument seen before using formulae (2.8), (2.9) for \( u \), and we obtain:
\[
F \left( \bar{x}, \varphi(\bar{x}), D \varphi(\bar{x}), D^2 \varphi(\bar{x}) \right) \geq \]
\[
\geq F \left( \bar{x}, t, \frac{\theta}{\sum_{i=1}^{n} |Du_i(x)|}, \left( \sum_{i=1}^{n} \frac{\lambda_i}{|Du_i(x)|} \right)^{-3} \left( \sum_{i=1}^{n} \lambda_i |D^2 u_i(x)|^3 \right)^{\frac{1}{2}} \right).
\]

When \( \varepsilon \) tends to zero, since \( u_i(x) \to u \) in the \( C^2 \) norm, thanks to the continuity of \( F \), we obtain that inequality (3.1) holds and then we can proceed as before to conclude the proof.

Proof of Theorem 1.1. Since \( u^* = u \) on \( \partial \Omega \), Theorem 3.1 and the comparison principle imply \( u^* \leq u \). Then, by (2.6), we get \( u = u^* \) and the theorem is proved.

\[\square\]

Remark 3.2. The assumption (1.2), as well as (1.3) and (1.4), could be slightly weakened. Indeed, in the proof of Theorem 3.1, we only used the
following property:

$\sum_{i=1}^{n} \lambda_i x_i \in \Omega$, 

then $\sum_{i=1}^{n} \lambda_i (x_i, p_i, A_i) \in G_0$, 

i.e. we obviously don’t care of points $\bar{x}$ belonging to $\Omega_{t_1}$. In fact, the functional $F$ doesn’t need to be defined at all when $x \in \overline{\Omega}_{t_1}$.

Then, we could substitute $F$, in each one of that assumptions, with the functional $\tilde{F}$ which coincides with $F$ if $x \in \Omega$ and coincides with the concave envelope of $F$ when $x \in \overline{\Omega}_{t_1}$.

3.1. Examples of applications of Theorem 3.1. In this section we show some applications of Theorem 3.1. For simplicity, $f(x, u, Du)$ will be a smooth function. Already known examples (see [3]) are the following:

- The Laplace operator: $F(x, u, Du, D^2u) = \Delta u - f(x, u, Du)$.

  By taking $\alpha = 3$, choosing $\theta$ as the direction of $Du(x)$ and $t$ as the value of $u$ at $x$, we have

  $$G_{t, \theta}(x, p, A) = \text{tr}(A) - f \left(x, t, \frac{\theta}{p} \right) p^3,$$

  and it is concave if $f(x, t, \frac{\theta}{p}) p^3$ is convex.

- The $p$-Laplace operator: $F(x, u, Du, D^2u) = \Delta_p u - f(x, u, Du)$, where $\Delta_p u = \text{div}(|Du|^{p-2} Du)$. The convexity of the function $q^{p+1} f(x, u, \frac{\theta}{q})$ with respect to $x, q$ is a sufficient condition for the quasi-concavity of solutions.

New examples are the following.

- Equations of the form:

  $$\text{tr} \left( A \left( u, \frac{\nabla u}{|\nabla u|} \right) D^2u \right) = \sum_{i,j=1}^{n} a_{ij} \left( u, \frac{\nabla u}{|\nabla u|} \right) u_{ij} = f(x, u, Du),$$

  where $A(u, \theta) = (a_{ij}(u, \theta))$ is a smooth positive definite matrix for $u \in (t_0, t_1)$, $\theta \in S^{n-1}$. Not surprisingly, from Theorem 3.1, we obtain, as a sufficient condition for the quasi-concavity of $u$, the convexity of $p^2 f(x, u, \theta/p)$ with respect to $(x, p)$ for every $(u, \theta)$, the same as in the case of the Laplacian.

- Pucci’s extremal operators: $F(x, u, Du, D^2u) = \mathcal{P}_{\lambda, \Lambda}^- (D^2u) - f(x, u, Du)$, 

  with $\mathcal{P}_{\lambda, \Lambda}^- (M) = \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i$ where $\mu_i$ are the eigenvalues of the matrix $M$ and $0 < \lambda \leq \Lambda$ are given.

  Notice that Pucci’s extremal operators are 1-homogeneous. By taking $\alpha = 3$ and since $\mathcal{P}_{\lambda, \Lambda}^-$ is a concave operator, the function $G_{t, \theta}(x, p, A) = \mathcal{P}_{\lambda, \Lambda}^- (A) - f \left(x, t, \frac{\theta}{p} \right) p^3$ is concave if $f(x, t, \frac{\theta}{p}) p^3$ is convex with respect to $x, p$. 

The previous argument can be applied to any $\alpha$-homogeneous and concave operator $F(D^2u)$. Then the equation
\[ F(D^2u) - f(x, u, Du) = 0, \]
is equivalent to
\[ F\left( \frac{D^2u}{|Du|^3} \right) - f(x, u, Du)|Du|^{-3\alpha} = 0, \]
and, in an analogous way as before,
\[ G_{t, \theta}(x, p, A) = F(A) - f\left( x, u, \frac{\theta}{p} \right) p^{3\alpha}, \]
is concave if $f(x, u, \frac{\theta}{p})p^{3\alpha}$ is convex.

3.2. A sharper theorem.

**Definition 3.3.** Let $\theta \in S^{n-1}$ be a given direction in $\mathbb{R}^n$; $\mathcal{S}^{-}_n(\theta)$ is the class of $n \times n$ symmetric real matrices that are negative definite on $\theta^\perp$, the linear space orthogonal to $\theta$. Moreover, we denote by $\mathcal{S}^{0-}_n(\theta)$ the subclass of $\mathcal{S}^{-}_n(\theta)$ of matrices that have $\theta$ as eigenvector, with corresponding null eigenvalue.

**Definition 3.4.** Let $\theta \in S^{n-1}$ be a fixed direction in $\mathbb{R}^n$ and $\Gamma$ a convex cone in $\mathcal{S}_n$. We denote by $\mathcal{A}^{-}_\theta(\Gamma)$ the following convex cone in $\mathcal{S}_{n+1}$:
\[ \mathcal{A}^{-}_\theta(\Gamma) = \{ A \in \mathcal{S}_{n+1} : A = \begin{pmatrix} A & \mu \theta^T \\ \mu \theta & 0 \end{pmatrix} \text{ with } A \in \mathcal{S}^{-}_n(\theta) \cap \Gamma, \mu > 0 \} \]
We write $\mathcal{A}^{-}_\theta$ for $\mathcal{A}^{-}_\theta(\mathcal{S}_n)$.

A straightforward calculation, see formula (4.2), shows that $\det A \neq 0$ if $A \in \mathcal{A}^{-}_\theta$, then we can give the following definition.

**Definition 3.5.** Let $\theta \in S^{n-1}$ be a fixed direction in $\mathbb{R}^n$ and $\Gamma$ a convex cone in $\mathcal{S}_n$. We say that an $(n+1) \times (n+1)$ real symmetric matrix $B$ is in $\mathcal{B}^{-}_\theta(\Gamma)$ if $B = A^{-1}$ for some $A \in \mathcal{A}^{-}_\theta(\Gamma)$, i.e. we set
\[ \mathcal{B}^{-}_\theta(\Gamma) = \{ A^{-1} : A \in \mathcal{A}^{-}_\theta(\Gamma) \}. \]
We write $\mathcal{B}^{-}_\theta$ for $\mathcal{B}^{-}_\theta(\mathcal{S}_n)$.

**Lemma 3.6.** A $(n+1) \times (n+1)$ matrix $B$ belongs to the convex cone $\mathcal{B}^{-}_\theta(\Gamma)$, if and only if there exist $\beta \in \mathbb{R}$, $B \in \mathcal{S}^{0-}_n(\theta)$ and $b \in \mathbb{R}^n$ such that
\[ B = \begin{pmatrix} B & b^T \\ b & \beta \end{pmatrix} \quad \text{and} \quad p = \langle b, \theta \rangle > 0. \]

**Proof.** See Appendix. \qed
Finally, we denote by $J$ the $n \times (n + 1)$ matrix

$$J = \begin{pmatrix} I_n & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

where $I_n$ is the $n \times n$ identity matrix.

If $B = A^{-1} \in \mathcal{B}_\theta^-(\Gamma)$, then it easily seen (see Appendix) that

$$A = J B^{-1} J^T \quad \text{and} \quad \mu = \frac{1}{p},$$

with the notation of Definition 3.5 and Lemma 3.6.

**Remark 3.7.** Notice that, if $v \in Q^2(x)$, then $D^2 v(x) \in \mathcal{A}_\theta^-$ and the hemmed Hessian matrix $\hat{D}^2 v(x)$ belongs to $\mathcal{A}_\theta^-$ with $\theta = Dv(x)/|Dv(x)|$.

Moreover

$$D^2 v(x) = |Dv(x)| J \frac{\hat{D}^2 v(x)}{|Dv(x)|} J^T.$$

Now we are finally in position to state and prove our main result.

**Theorem 3.8.** Let $\Omega = \Omega_{t_0} \setminus \Omega_{t_1}$ be a convex ring and let $F(x,u,q,A)$ be a proper, continuous, degenerate elliptic operator in $\Omega_{t_0} \times (t_0,t_1) \times \mathbb{R}^n \times \Gamma_F$ and assume that

$$H_0 = \left\{ (x,B) \in \Omega_{t_0} \times \mathcal{B}_\theta^-(\Gamma_F) : F\left(x,t,\frac{\theta}{p},\frac{1}{p}JB^{-1}J^T\right) \geq 0 \right\}$$

is convex for every fixed $(t,\theta) \in (t_0,t_1) \times S^{n-1}$,

where $p = (b,\theta)$ as in Lemma 3.6.

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is an admissible classical solution of (1.1) such that $|Du| > 0$ in $\Omega$, then $u^*$ is a viscosity subsolution of (1.1).

**Remark 3.9.** Before proving Theorem 3.8, we notice that, as in the case of Theorem 3.1, the assumption (3.10) could be substituted by some stronger assumption, easier to check, like the following ones:

$$H_{t,\theta,\alpha}(x,B) = p^\alpha F\left(x,t,\frac{\theta}{p},\frac{1}{p}JB^{-1}J^T\right)$$

is quasi-concave for every fixed $t \in (t_0,t_1)$, $\theta \in S^{n-1}$,

or

$$H_{t,\theta,\alpha}(x,B)$$

is concave in $\Omega_{t_0} \times \mathcal{B}_\theta^-(\Gamma_F)$ for every fixed $t \in (t_0,t_1)$, $\theta \in S^{n-1}$.

Obviously (3.11) implies (3.10) that implies (3.12).
Proof. The proof follows the same steps of the proof of Theorem 3.1. Let us consider \( \Omega^*(t) \) and a point \( \bar{x} \) on the boundary of \( \Omega^*(t) \):

\[
\bar{x} = \sum_{i=1}^{n} \lambda_i x_i \quad \text{and} \quad u^*(\bar{x}) = u(x_i) = t \quad i = 1, \ldots, n,
\]

where \( x_i \in \partial \Omega(t) \) and \( 0 \leq \lambda_i \leq 1, \sum_{i=1}^{n} \lambda_i = 1 \).

As before, let \( u_i = u \) for \( i = 1, \ldots, n \) and \( u_{\lambda} = \sum_{i=1}^{n} \lambda_i u \). As we showed in the proof of Theorem 3.1 we can assume that formulae (2.8), (2.10) hold, without loss of generality.

Choosing \( v = u_\lambda \) in (3.9), by (2.10) and (2.8) then we have

\[
D^2 u_\lambda(\bar{x}) = \frac{1}{\sum_{i=1}^{n} \sum_{i=1}^{n} \lambda_i} \left( \sum_{i=1}^{n} \lambda_i \left( \frac{D^2 u(x_i)}{|D u(x_i)|} \right)^{-1} \right)^{-1} J^T.
\]

Notice that \( D^2 u_\lambda(\bar{x})^{-1} \in \mathcal{B}_{\Gamma F}^{-} \) and \( D^2 u_\lambda(\bar{x}) \in \mathcal{A}_{\Gamma F}^{-} \) with

\[
\theta = \frac{D u_\lambda(\bar{x})}{|D u_\lambda(\bar{x})|}.
\]

Let \( \varphi \in C^2(\Omega) \) that touches \( u^* \) from above at \( \bar{x} \), then \( \varphi \) touches \( u_\lambda \) from above at \( \bar{x} \). This implies that

\[
\varphi(\bar{x}) = u^*(\bar{x}) = u_\lambda(\bar{x}) = t,
\]

\[
D \varphi(\bar{x}) = D u_\lambda(\bar{x}),
\]

\[
D^2 \varphi(\bar{x}) \geq D^2 u_\lambda(\bar{x}).
\]

Using the previous relations, equations (2.8), (3.13) and the ellipticity of \( F \), we get

\[
F(\bar{x}, t, D \varphi(\bar{x}), D^2 \varphi(\bar{x})) \geq F(\bar{x}, t, D u_\lambda(\bar{x}), D^2 u_\lambda(\bar{x})) =
\]

\[
= F(\bar{x}, t, \frac{\theta}{\sum_{i=1}^{n} \lambda_i}, \frac{1}{\sum_{i=1}^{n} \lambda_i} J \left( \sum_{i=1}^{n} \lambda_i \left( \frac{D^2 u(x_i)}{|D u(x_i)|} \right)^{-1} \right)^{-1} J^T) =
\]

\[
= H_{t, \theta, 0} \left( \sum_{i=1}^{n} \lambda_i x_i, \lambda_i B_i \right) = H_{t, \theta, 0} \left( \bar{x}, \mathcal{B} \right),
\]

where \( B_i = \left( \frac{D^2 u(x_i)}{|D u(x_i)|} \right)^{-1} \) and \( \mathcal{B} = \sum_{i=1}^{n} \lambda_i B_i \) (hence \( p_i = |D u(x_i)|^{-1} \) and \( \mathcal{P} = \sum_{i=1}^{n} \lambda_i p_i \) and \( H_{t, \theta, 0} \) is defined as in (3.10).

Since \( H_{t, \theta, 0}(x_i, B_i) = F(x_i, t, D u(x_i), D^2 u(x_i)) = 0, i = 1, \ldots, n \), for \( u \) is a solution of (1.1), and

\[
H_0 = \left\{ (x, B) \in \Omega_0 \times \mathcal{B}_{\Gamma F}^{-} : H_{t, \theta, 0}(x, B) \geq 0 \right\},
\]

then \( (x_i, B_i) \in H_0 \) for \( i = 1, \ldots, n \) and assumption (3.10) yields \( (\bar{x}, \mathcal{B}) \in H_0 \). The proof is so concluded.

As a corollary, we have the following refinement of Theorem 3.3.
Theorem 3.10. In the same assumptions of Theorem 3.8, if $F$ satisfies a viscosity comparison principle, then $u$ is quasi-concave.

3.3. The relationship between Theorem 3.1 and Theorem 3.8. This section is devoted to analyze the relationship between Theorem 3.1 and Theorem 3.8. As already announced, in Theorem 3.12 below, we will prove that Theorem 3.8 in fact implies Theorem 3.1. From an analytic point of view, this fact relies on the differential properties of the support function $H$ of the level sets of a $Q^2_-$ function $u$; these properties are derived in [13] for the class of $Q^2_+$ quasiconvex functions, which are the opposite of $Q^2_-$ quasiconcave functions. From an algebraic point of view, what we need is just the following lemma.

Lemma 3.11. In the same assumptions and notation of Lemma 3.6, the map

$$\Phi : B \to p^2JB^{-1}J^T,$$

from $\mathcal{C}_o^-(\Gamma)$ to $\mathcal{C}_n^-(\theta) \cap \Gamma$, is concave.

Proof. See Appendix. \hfill \Box

Notice that, by (3.9), the above lemma shows that (2.10) implies (2.9).

Theorem 3.12. Let $F(x,t,q,A)$ be a proper, continuous and degenerate elliptic operator in $\Omega \times (t_0, t_1) \times \mathbb{R}^n \times \Gamma_F$. If $F$ satisfies (1.4) then $F$ satisfies (3.12).

Proof. Let $x_1, \ldots, x_m \in \Omega$, $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m$ and $B_1, \ldots, B_m \in \mathcal{C}_o^-(\Gamma_F)$. Set $x_\lambda = \sum_i \lambda_i x_i$, $B_\lambda = \sum_i \lambda_i B_i$, then $b_\lambda = \sum_i \lambda_i b_i$ and $p_\lambda = \langle b_\lambda, \theta \rangle = \sum_i \lambda_i p_i$, with the notation of Lemma 3.6. Then from the previous lemma, for fixed $\theta$ and $t$, we have

$$\frac{p_\lambda^2}{p_\lambda}Jb_\lambda^{-1}J^T \geq \sum_i \lambda_i p_i^2 Jb_i^{-1}J^T,$$

whence

$$\frac{1}{p_\lambda}Jb_\lambda^{-1}J^T \geq \frac{\sum_i \lambda_i p_i^2 Jb_i^{-1}J^T}{(\sum_i \lambda_i p_i)^3},$$

and both terms in the previous inequality belong to $\Gamma_F$. Since $F$ is proper on $\Gamma_F$, we get

$$\frac{p_\lambda^2}{p_\lambda}F\left(x_\lambda, t, \frac{\theta}{p_\lambda}, \frac{1}{p_\lambda}Jb_\lambda^{-1}J^T\right) \geq \left(\sum_i \lambda_i p_i\right)\alpha F\left(\sum_i \lambda_i x_i, t, \frac{\theta}{\sum_i \lambda_i p_i}, \frac{\sum_i \lambda_i p_i^2 Jb_i^{-1}J^T}{(\sum_i \lambda_i p_i)^3}\right).$$

By taking $A = \frac{p_\lambda^2}{p_\lambda}Jb_\lambda^{-1}J^T$ and applying (1.4), we obtain

$$\frac{p_\lambda^2}{p_\lambda}F\left(x_\lambda, t, \frac{\theta}{p_\lambda}, \frac{1}{p_\lambda}Jb_\lambda^{-1}J^T\right) \geq \min_{i=1,\ldots,n} \left\{p_i^0 F\left(x_i, t, \frac{\theta}{p_i}, \frac{1}{p_i}Jb_i^{-1}J^T\right)\right\}.$$
is quasi-concave in $\Omega \times \mathcal{B}^{-\theta}$, that is (3.12). \hfill $\square$

We conclude by showing some examples of operators $F : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_n \to \mathbb{R}$ (with associated elliptic equations in $\mathbb{R}^2$) satisfying the assumptions of Theorem 3.8 but not those of Theorem 3.1.

One relevant example is the Mean Curvature operator:

$$\mathcal{M}u = (1 + |Du|^2)\Delta u - \sum_{i,j=1}^2 D_i u D_{ij} u D_j u.$$ 

This can be generalized in the following way:

$$(i) \quad F(Du, D^2 u) = \Delta u + \left( |Du|^2 \Delta u - \sum_{i,j=1}^2 D_i u D_{ij} u D_j u \right) (\alpha |Du| + \beta |Du|^\gamma),$$

where $\alpha$, $\beta$ and $\gamma$ are real parameters such that $\alpha \geq 0$, $\beta \geq 0$, $\alpha^2 + \beta^2 > 0$ and $\gamma \in (-\infty, -3) \cup [0, +\infty)$. $\mathcal{M}u$ corresponds to the choice $\alpha = 0$, $\beta = 1$, $\gamma = 0$.

Notice that for a $C^2$ function $u(x, y)$ of two variables, \( \frac{1}{|Du|^2} \sum_{i,j=1}^2 D_i u D_{ij} u D_j u \) coincides with the second normal derivative $u_{\theta\theta}$, where $\theta = Du/|Du|$. Then, if we denote by $T$ the direction $\theta \perp$ tangential to the level curve passing through $(x, y)$, we have

$$|Du|^2 \Delta u - \sum_{i,j=1}^2 D_i u D_{ij} u D_j u = u_{TT} |Du|^2.$$ 

If $u \in Q_2^2(x_0, y_0)$, the second tangential derivative of $u$ is given by

$$u_{TT} = -k |Du|$$

where $k$ is the curvature (positive) of the level line at the considered point.

Then, if we set $B = (\Delta^2 u/|Du|)^{-1}$ (hence $p = |Du|^{-1}$) and we choose the so called principal coordinate system centered at $(x_0, y_0)$ (with $T$ and $\theta$ as coordinate directions) we can write

$$u_{TT} = \frac{\text{tr}(B^{-1})}{p} = \frac{1}{pb_{11}}.$$ 

Hence in the case (i) we have

$$G_{t,\theta}(p, A) = \frac{1}{p^3} \text{tr}(A) + \frac{1}{p^3 + \gamma} a_{11} \left( \frac{\alpha}{p} + \beta \right)$$

and

$$H_{t,\theta}(B) = \frac{1}{p} \text{tr}(JB^{-1}J^T) + \frac{1}{p^3 + \gamma} b_{11} \left( \frac{\alpha}{p} + \beta \right).$$

It is now easily seen that the set

$$G_0 = \{ (p, A) \in (0, +\infty) \times \mathcal{A}_\theta : G_{t,\theta}(p, A) \geq 0 \}$$
is not convex. By setting \( \text{tr}(A) = a_{11} + a_{22} = z \) and \( a_{11} = y \), the convexity of \( G_0 \) is equivalent to the convexity of the set

\[
\left\{ (y, p, z) \in \mathbb{R}^3 : p > 0, y < 0, z \geq -\frac{\alpha y}{p^{3+\gamma}} - \frac{\beta y}{p^{2+\gamma}} \right\},
\]

which is the restriction to the quarter of space where \( p > 0 \) and \( y < 0 \) of the epigraph of the function

\[
g(y, p) = -\frac{\alpha y}{p^{3+\gamma}} - \frac{\beta y}{p^{2+\gamma}}.
\]

Then \( G_0 \) is convex if and only if the function \( g \) is convex for \( p > 0 \) and \( y < 0 \); since \( g \) is not convex for \( \alpha, \beta \) and \( \gamma \) as specified, then the assumption (1.2) is not satisfied.

On the contrary, it is not hard to see that \( H_{t, \theta} \) satisfies assumption (3.12), with \( \alpha = 3 \). Indeed \( H_{t, \theta, 3} \) is splitted in two addenda: the first one corresponds to the map \( \Phi \), which is concave by Lemma 3.11; the second one is

\[
\frac{1}{b_{11}p^\gamma} \left( \frac{\alpha}{p} + \beta \right),
\]

which is concave in \((p, b_{11}) \in (0, +\infty) \times (-\infty, 0)\), hence in the variable \( B \in \mathcal{B}_{\theta^\gamma} \), when \( \alpha, \beta \) and \( \gamma \) are as specified.

Going back to the equation of Mean Curvature

\[
\mathcal{M} u = f(x, u, Du),
\]

thanks to the previous discussion and to Theorem 3.10 we obtain that the solution of (1.1) is quasi-concave if the function \( h : (x, p) \in \Omega \times (0, +\infty) \to \mathbb{R} \) defined by

\[
h(x, p) = \frac{1}{p^3} f(x, u, \frac{\theta}{p})
\]

is convex; the same condition as in [3].

Another example satisfying the assumptions of Theorem 3.8 but not those of Theorem 3.1 is the following

\[
(ii) \quad F(Du, D^2u) = \Delta u + \varepsilon \left( \Delta u - \frac{1}{|Du|^2} \sum_{i,j=1}^2 D_i u D_{ij} D_j u \right)^3
\]

where \( \varepsilon > 0 \) is a real parameter.

The example (ii) can be treated in a similar way as (i).

4. Appendix

Proof of Lemma 3.6

Without loss of generality, we can assume \( \theta = (0, \ldots, 0, 1) \); then \( A \) takes the
following form

\[
A = \begin{pmatrix}
\times & 0 \\
\vdots & \vdots \\
\times & 0 \\
\times \cdots \times & \mu \\
0 \cdots 0 & \mu \ 0
\end{pmatrix},
\]

where the \((n-1) \times (n-1)\) matrix \((a^{ij})\) represents the restriction of \(A\) on \(\theta^\perp\) and therefore is negative definite (here and later, we do not care about the values at the positions denoted by \(\times\)). So it easily seen that

\[
\text{det} A = -\mu^2 \text{det}(a^{ij}) \neq 0.
\]

Since by definition \(B = A^{-1}\), the inversion of (4.1) give us the following representation of \(B\)

\[
B = \begin{pmatrix}
0 & \times \\
\vdots & \vdots \\
0 & \times \\
\times \cdots \times & 1/\mu & \beta
\end{pmatrix},
\]

Then

\[
B = \begin{pmatrix}
l_{ij} & 0 \\
0 & 0
\end{pmatrix},
\]

where \((l_{ij})\) is a definite negative \((n-1) \times (n-1)\) matrix with inverse matrix \((a^{ij})\), and \(p = \langle b, \theta \rangle = b_n = 1/\mu\).

\[
\boxed{\square}
\]

**Proof of Lemma 3.11**

By (3.8), for any vector \(\xi = (\xi_1, \ldots, \xi_n)\), some boring computations lead to

\[
p^2 \xi J B^{-1} J^T \xi = \sum_{i,j=1}^{n} \xi_i \left[ (b_0 \delta_i^j - b_0 \delta_n^j) a^{ij} \left( b_0 \delta_j^m - b_0 \delta_n^m \right) - \beta \delta_i^j \delta_n^m \right] \xi_m
\]

\[
= V L^{-1} V^T - \beta \xi_n^2,
\]

where \(\delta_i^j\) denotes the Kronecker symbol, \(L = (l_{ij}) \in S_{n-1}^\perp\) and \(V = (v_1, \ldots, v_{n-1})\) with

\[
v_i = \sum_{l=1}^{n} \xi_l (b_0 \delta_i^l - b_0 \delta_n^l), \quad i = 1, \ldots, n-1.
\]

Obviously \(-\beta \xi_n^2\) depends linearly on \(B\), while the term \(V^T L^{-1} V\) is concave in \((L, V) \in S_{n-1}^\perp \times \mathbb{R}^{n-1}\), see [1] Appendix. This proves that the map \(\Phi\) is concave in the \((B, b, \beta)\) variables, i.e. it is concave in \(B\).

\[
\boxed{\square}
\]
REFERENCES


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