1 Introduction

The classic Rogers & Shephard inequality states that:

\[ V(K + (-K)) \leq 2^n V(K) \]

where \( K \) belongs to \( \mathbb{R}^n \), the class of convex bodies contained in the euclidean \( n \)-space \( \mathbb{R}^n \), and the set \( K + (-K) \) is the difference body of \( K \).

Notice that this inequality is sharp, indeed when \( K \) is a simplex equality holds. Another inequality due again to Rogers & Shephard is:

\[ V(\text{conv}(K \cup (-K))) \leq 2^p V(K) \]

and the constant is still optimal (see [4]). Shephard asserts that:

- The problem is invariant under nonsingular linear transformation of \( \mathbb{R}^n \), but it is not invariant under translation for \( p > 1 \).
- The \( p \)-Brunn-Minkowski inequality implies the lower bound

\[ \sqrt[p]{V(K)} \leq V(K^p \cdot (-K)) \]

According to (1), (2) for \( p = 1 \) and \( p = \infty \) the value of \( c_{np} \) is known; \( c_{1p} = \left( \frac{2}{n+1} \right)^{\frac{1}{n}} \), \( c_{np} = 2^p \). A simple but not sharp bound for \( c_{np} \) is

\[ c_{np} \leq \min \left\{ \left( \frac{2}{n+1} \right)^{\frac{1}{n}}, \left( \frac{2}{n+1} \right)^{\frac{1}{n}} \right\} \]

which follows from (1), (2) and the monotonicity of the \( p \)-sum.

4 The main result

Theorem 1. For \( n = 2 \) the maximum of \( F_2(K) \) in the class of all convex bodies containing the origin is attained if \( K \) is a triangle with one vertex in the origin.

Corollary 2.

\[ c_{2p} = 2(2p-1) \left( \frac{1}{2} \right)^{\frac{p}{p-1}} \left( \frac{1}{n+1} \right)^{\frac{1}{p-1}} \]

5 The idea of the proof

The proof of Theorem 1 is based on continuous movement of convex bodies. In particular we use the so called shadow systems introduced by Rogers & Shephard in [6].

A shadow system can be seen as the family of projections of a \((n+1)\)-dimensional convex body. More exactly a shadow system \( (K[u]) \) is the intersection between a fixed hyperplane \( \mathbb{R}^n \) and the cylinder \( Z(K; a+u) \) containing the fixed body \( K \in \mathbb{R}^{n+1} \) with generators in the direction \( a+u \), where \( a \) is a unit vector normal to \( \mathbb{R}^n \).

A linear parameter system \( (C(t)/I(t); K(t)) \) along the direction \( a \), \( t \in [0,1] \), is the convex hull of the set \( \{a+\lambda\tau:v(\lambda) \in C(t)\} \) where \( \lambda \) is an arbitrary index set, \( v(\lambda) \in B(\lambda) \) is a real function defined in \( J \). The function \( \lambda \) represents the ‘speed’ of the linear parameter system.

It can be shown that each linear parameter system is a shadow system.

A parallel chord movement is a linear parameter system with a speed function \( \lambda \) constant on each chord parallel to the direction of the movement \( v \).

Notice that the problem is invariant under nonsingular linear transformation of \( \mathbb{R}^n \), but it is not invariant under translation for \( p > 1 \).

The proof uses a theorem by Rogers & Shephard which assumes the convexity of the volume of a linear parameter system \( C(t) \) as a function of the parameter \( t \) (see [5]). Another important ingredient is the following result:

Theorem 3. Let \( K(t); L(t) \) be two linear parameter systems in \( \mathbb{R}^n \) along the direction \( v \), then their \( p \)-sum body \( K^{pl}(t) + L^{pl}(t) \) is a linear parameter system along the same direction \( v \).

Observe that the case \( p = 1 \) and \( p = \infty \) were already known (see [1] for \( p = 1 \) and [5] for \( p = \infty \)). Moreover S. Campi and P. Gronchi show an analogous result for \( p \)-zonotopes instead of the \( p \)-sum, \( p \in [1,\infty) \) (see [2]).

As a consequence of Theorem 3 and the previous considerations, if \( (K(t)) \) is a parallel chord movement, then \( F_2(K)(t) \) is a convex function of the parameter \( t \), for all \( p \geq 1 \).

From this fact we gain a clue concerning maximizers of \( F_p \) if \( K \) is an ‘inner body’ of a parallel chord movement, then it is not a good candidate as extremal body.

In the plane we can be more precise. In fact for every polygon \( P \) with \( m+1 \) vertices, \( m \geq 4 \), there exists a parallel chord movement \( \{P(t),P(t); \tau(t)\} \) so that \( P(0) = P(1) \) are polygons with \( m \) vertices, and \( P = P(t) \) for some \( t \in (0,1) \).

Then triangles are among maximizers of \( F_2 \) in the class of polygons and by a standard density argument, it follows that they maximize \( F_2 \) in \( \mathbb{R}^2 \).

Recalling that the functional we are considering is not invariant under translations, we can use parallel chord movements again to show that triangles with one vertex in the origin maximize the functional \( F_p \).

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References