Strong Approximation of Weak Limits via Method of Averaging with Applications to Navier-Stokes Equations

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Abstract: We prove, firstly, that weak limit of some sequence from Orlicz function space can be approximated in strong sense (in norm) by the subsequence of averaged functions if the radius of averaging tends to zero slowly enough. Secondly, we consider the weakly converging sequence of approximate solutions to the Navier-Stokes equations and obtain strong convergence of some subsequence of solutions to averaged equations to the solution of the limiting equations.

Keywords: Orlicz function space, weak limit, averaging operator, strong convergence, embedding theorems.

1 Notations and basic notions from Orlicz function spaces theory.

Let $\Omega \subset \mathbb{R}^d$ be bounded domain with smooth boundary Γ, and $x = (x_1, \ldots, x_d)$ be the points of Ω. By $L^1(\Omega)$ we denote the space of absolutely integrable functions on Ω, $L^\infty(\Omega)$– the space of essentially bounded functions and $L^p(\Omega)$, $1 < p < \infty$ – the scale of Lebesgue spaces of functions which are integrable in power p.

We shall use also the Orlicz function spaces, and remind the basic notions (see [1]). Let $m(r)$ be defined on $[0, \infty)$ function, continuous from the right, non-negative, non-decreasing and such that

$$m(0) = 0, \quad m(r) \to \infty \text{ as } r \to \infty. \tag{1}$$

The convex function (Young function)

$$M(t) = \int_0^t m(r)dr \tag{2}$$

produces Orlicz class $K_M(\Omega)$ containing the functions $f(x) \in L^1(\Omega)$ such that $M(|f(x)|)$ belong to $L^1(\Omega)$,too. The linear span of $K_M(\Omega)$ endowed with the norm

$$\|f\|_{L_M(\Omega)} = \inf \left\{ \lambda > 0 \ \middle| \ \int_\Omega M\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\} \tag{3}$$

is called as Orlicz space $L_M(\Omega)$ associated with Young function $M(t)$. The closure of $L^\infty(\Omega)$ in the norm (3) yields, in general case, another Orlicz space $E_M(\Omega)$, and the inclusions are valid

$$E_M(\Omega) \subseteq K_M(\Omega) \subseteq L_M(\Omega). \tag{4}$$

One says function $M(t)$ satisfies $\Delta_2 - condition$ (cf.[1]) if there exist constants $C > 0$ and $t_0 > 0$ such that

$$M(2t) \leq C \ M(t) \quad \text{for } \forall t \geq t_0. \tag{5}$$

Three sets E_M, K_M and L_M coincide if and only if $M(t)$ satisfies Δ_2-condition.

In many cases it's possible to compare two Young functions $M_1(t)$ and $M_2(t)$ with respect to their behaviour at infinity , namely, $M_2(t)$ dominates $M_1(t)$ if there exist constants $C > 0$ and $\alpha > 0$ such that

$$M_1(\alpha t) \leq M_2(t) \quad \text{for } \forall t \geq t_0 = t_0(\alpha, C). \tag{6}$$

In this case

$$E_{M_2} \subseteq E_{M_1} \text{ and } L_{M_2} \subseteq L_{M_1}.$$

In particular, function $M(t)$ satisfying Δ_2-condition (5) is dominated by some power function $t^q, q > 1$.

If each function M_k, $k = 1, 2$ dominates the other one, then M_1 and M_2 are equivalent, $M_1 \cong M_2$, and corresponding Orlicz spaces are the same, $E_{M_1} = E_{M_2}$ and $L_{M_1} = L_{M_2}$.

Function M_2 dominates M_1 essentially if

$$\lim_{t \to \infty} \frac{M_1(\beta t)}{M_2(t)} = 0, \quad \forall \beta = \text{const} > 0. \tag{7}$$

In this case the strong embeddings

$$E_{M_2} \subset E_{M_1}, \quad L_{M_2} \subset L_{M_1} \tag{8}$$

take place.

Denote by

$$n(r) = m^{-1}(r) \tag{9}$$

the inverse function to $m(r)$, i.e. $n(m(r)) \equiv r, \forall r > 0$, and introduce Young function

$$N(t) = \int_0^t n(r)dr. \tag{10}$$

This function is called as complementary (adjoint) convex function to $M(t)$, and it's equivalent to the next one:

$$N(t) \cong \sup_{r>0} \{tr - M(r)\}. \tag{11}$$

Two Orlicz spaces L_M and L_N are supplementary, and for any $f(x) \in L_M(\Omega), g(x) \in L_N(\Omega)$ there exists the integral

$$< f, g > \equiv \int_\Omega f(x)g(x)dx \tag{12}$$

which defines the linear continuous functional on $E_N(\Omega)$ with fixed $f \in L_M(\Omega)$ and any $g \in E_M(\Omega)$. It gives the notion of weak convergence in $L_M(\Omega)$: the sequence $\{f_n(x)\}$ converges weakly to $f(x) \in L_M(\Omega)$, $f_n \rightharpoonup f$, if

$$< f_n, g > \to < f, g > \quad \text{for each } g \in E_N(\Omega) \quad as \quad n \to \infty. \tag{13}$$

At the same time it's possible to define mean convergence in $L_M(\Omega)$:

$$f_n \to f \quad \text{in mean value, if}$$

$$\int_\Omega M(|f_n - f|)dx \to 0 \quad \text{as } n \to \infty.$$

If $M(t)$ satisfies Δ_2-condition then the mean convergence is equivalent to the strong convergence, i.e. in the norm of $L_M(\Omega)$. Otherwise, mean convergence is stronger than weak, but weaker than strong one.

As an important and interesting examples we shall keep in mind three Young functions: $M_1(t) = t^p$, $1 < p < \infty$, $M_2(t) = e^t - t - 1$ and $M_3(t) = (1+t)\ln(1+t) - t$. The first function $M_1(t)$ yields the Lebesgue space $L^p(\Omega)$, the second one $M_2(t)$ produces two Orlicz spaces L_{M_2} and E_{M_2} because $M_2(t)$ doesn't satisfy Δ_2-condition, and $M_3(t)$ is slowly increasing function which is essentially dominated by $M_1(t)$. The Orlicz space $L_{M_3}(\Omega)$ is located between $L^1(\Omega)$ and any $L^p(\Omega)$, $p > 1$ while $L_{M_2}(\Omega)$ is between $L^p(\Omega)$, $\forall 1 < p < \infty$, and $L^\infty(\Omega)$.

Finally, for any $f(x) \in L^1(\Omega)$ and $h > 0$, let us denote

$$f_h(x) = \frac{1}{h^d} \int_\Omega f(y)\omega\left(\frac{x-y}{h}\right) dy \tag{15}$$

an averaging of $f(x)$ where $\omega(z)$ is the kernel of averaging:

$$\omega(z) \in C_0^\infty(\mathbb{R}^d), \quad \omega(z) \geq 0, \quad \int_{\mathbb{R}^d} \omega(z)dz = 1.$$

It's the well-known fact that

$$\|f_h - f\|_{L_M(\Omega)} \to 0 \quad \text{as } h \to 0 \tag{16}$$

if $M(t)$ satisfies Δ_2-condition, and

$$\int_\Omega M(|f_h - f|)dx \to 0 \quad \text{as } h \to 0 \tag{17}$$

for any $M(t)$, i.e.the sequence of $\{f_n\}$ approximates f in the sense of mean convergence.

2 Strong approximation of weak limits

Let us consider some sequence of functions $\{f_n(x)\}$, $n = 1, 2, \ldots$, from Orlicz space $L_M(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L_M(\Omega)$, as $n \to \infty$. For each $f_n(x)$ we construct the family of averaged functions $(f_n)_h(x)$.

Theorem 1 .*If $M(t)$ satisfies Δ_2-condition then there exists subsequence $(f_m)_{h_m}$ such that*

$$(f_m)_{h_m} \to f \quad strongly \ in \ L_M(\Omega)$$

$$as \ m \to \infty, \ h_m \to 0.$$

For any $M(t)$ there exists subsequence $(f_m)_{h_m}$ such that

$$\int_\Omega M(|f - (f_m)_{h_m}|)dx \to 0 \quad as \ m \to \infty, \ h_m \to 0,$$

i.e.mean convergence takes place.

Proof .

Step 1. Simple example. In order to understand the problem we consider, firstly, one very simple example of the sequence $f_n(x) = \sin(nx)$, $n = 1, 2, \ldots$, $x \in \mathbb{R}^1$, $\Omega = (0, \pi)$, of periodic functions. In this case we have

$$f_n(x) \rightharpoonup 0 \text{ weakly in } L^2(0, \pi).$$

Let us take the Steklov averaging

$$(f_n)_h(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} f_n(\xi)d\xi. \tag{18}$$

It's easy to calculate

$$(f_n)_h(x) = \frac{\sin nh}{nh} \sin(nx). \tag{19}$$

It gives

a) If $h = h_n \to 0$ as $n \to \infty$, but $nh_n \to \infty$ (for example, $h_n = n^{-\alpha}$, $0 < \alpha < 1$) then

$$(f_n)_{h_n}(x) \Rightarrow 0 \text{ as } n \to \infty,$$

i.e. $(f_n)_{h_n}(x)$ tends to 0 strongly when $h_n \to 0$ slowly enough.

b) If $nh_n \to const < \infty$ (or are bounded), then $(f_n)_{h_n} \rightharpoonup 0$ weakly only.

Step 2. One-dimensional case, Steklov averaging. Now consider the case of $d = 1$, $x \in \mathbb{R}^1$, and $f_n(x) \rightharpoonup f(x), n \to \infty$, weakly in $L_M(\Omega)$, where $M(t)$ satisfies Δ_2-condition.

For the sake of simplicity we assume $f_n(x)$, $f(x)$ to be T-periodic, $T = const > 0$, and, moreover, without loss of generality, one can admit $f(x) \equiv 0$, i.e.

$$f_n(x) \rightharpoonup 0 \text{ weakly in } L_M(0, T). \tag{20}$$

Let us construct the family of functions

$$(f_n)_h(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} f_n(\xi)\xi \tag{21}$$

and the sequence

$$F_n(x) = \int\limits_{0}^{x} f_n(\xi)d\xi$$

which doesn't depend on h. Formula (21) can be rewritten as follows

$$(f_n)_h(x) = \frac{1}{2h}(F_n(x+h) - F_n(x-h)). \tag{23}$$

The sequence $\{F_n(x)\}$ possesses the estimates

$$\sup_x |F_n(x)| \leq C, \quad \|F_n'(x)\|_{L_M(0,T)} \leq C \tag{24}$$

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with constant C independent on n.

Compactness theorem implies the strong convergence $F_n(x) \to 0$ in $L_M(0, T)$, i.e.

$$\|F_{n(x)}\|_{L_M(0,T)} \leq C_n, \quad C_n \to 0 \text{ as } n \to \infty. \tag{25}$$

If we take $F_n(x + h)$ or $F_n(x - h)$ (displacements of $F_n(x)$) then

$$\|F_n(x + h)\|_{L_M} \leq C_n \cdot C, \quad \|F_n(x - h)\|_{L_M} \leq C_n \cdot C \tag{26}$$

with C independent on h.

It gives

$$\|(f_n)_h\|_{L_M} \leq C \cdot \frac{C_n}{h}. \tag{27}$$

So, if we take $h = h_n$ such that $C_n \cdot h_n^{-1} \to 0$ as $n \to \infty$, for example, $h_n = C_n^\beta$, $0 < \beta < \text{const} < 1$, then

$$(f_n)_{h_n}(x) \to 0 \quad \text{strongly in } L_M(0, T). \tag{28}$$

It proves the theorem 1 in 1-dimensional case.

Step 3. The general case. Let the sequence $\{f_n(x)\}$, $x \in \Omega \subset \mathbb{R}^d$, be weakly converging in $L_M(\Omega)$ to $f(x) \equiv 0$, where $M(t)$ satisfies Δ_2-condition.

We extend $f_n(x)$ by 0 outside of Ω and consider the family $(f_n)_h(x)$ given by the formula (15)

$$(f_n)_h(x) = \frac{1}{h^d} \int_{\mathbb{R}^d} f_n(y)\omega\left(\frac{x - y}{h}\right) dy \tag{29}$$

with arbitrary kernel of averaging $\omega(z)$.

At the beginning we fix some $h = h_0$, for example $h_0 = 1$, and consider the sequence

$$F_n(x) = \int_{\mathbb{R}^d} f_n(y)\omega(x - y) dy \equiv \int_{\mathbb{R}^d} f_n(x - z)\omega(z) dz$$

which doesn't depend on h.

As in one-dimensional case we conclude by the compactness theorem

$$F_m(x) \to 0 \text{ strongly in } L_M(\Omega),$$

i.e. $\|F_m\|_{L_M(\Omega)} \leq C_m$, $C_m \to 0$ as $m \to \infty$. If we take the family

$$F_{mh}(x) \equiv \int_{\mathbb{R}^d} f_m(y)\omega(\frac{x - y}{h}) dy$$

then for $h \leq h_0 = 1$

$$\|F_{mh}\|_{L_M(\Omega)} \leq C\|F_m\|_{L_M(\Omega)}$$

with constant C independent on h, i.e.

$$\|F_{mh}\|_{L_M(\Omega)} \leq C \cdot C_m.$$

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It means that if we choose $h = h_m$ such that

$$C_m h_m^{-d} \to 0 \text{ as } m \to \infty, \ h_m \to 0, \tag{30}$$

for instance, $h_m^d = C_m^\beta$, $0 < \beta < 1$, then $(f_m)_{h_m} \to 0$ strongly in $L_M(\Omega)$.

The theorem 1 is proved for the case of Young function $M(t)$ satisfying Δ_2-condition. For any $M(t)$ the same proof can be used if the mean convergence is considered instead of strong one or the strong convergence in Orlicz space $E_M(\Omega)$ where the set of smooth functions is dense (in particular, the set of averaged functions).

Remark. The main significance of theorem 1 seems to be useful for justification of the smoothing approach in computation of non-smooth solutions to partial differential equations. Big oscillations occur near singularities, and the appearance of oscillations can be connected with the weak convergence of approximate solutions to the exact one. The procedure of "smoothing of solution" means, in fact, an averaging, so, the theorem 1 indicates that the radius of averaging must be big enough according to condition (30).

3 Applications to Navier-Stokes equations

It this section we illustrate the theorem 1 by one example from the theory of Navier-Stokes equations for viscous incompressible fluid. We consider the sequence $\{\mathbf{u}_n\}$ of solutions to Navier-Stokes equations [2]:

$$\frac{\partial \mathbf{u}_n}{\partial t} + (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n + \nabla p_n = \nu \Delta \mathbf{u}_n + \mathbf{f}_n,$$
$$\text{div}\mathbf{u}_n = 0, \quad (x,t) \in Q = \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^3, \quad \Gamma = \partial\Omega, \tag{31}$$

complemented with the initial and boundary data

$$\mathbf{u_n}\Big|_{t=0} = \mathbf{u_n^0}(x), \quad x \in \Omega, \quad \mathbf{u_n}\Big|_\Gamma = 0. \tag{32}$$

Let us suppose

$$\mathbf{u_n^0}(x) \rightharpoonup \mathbf{u^0}(x) \text{ weakly in } L^2(\Omega),$$

$$\mathbf{f_n} \rightharpoonup \mathbf{f} \text{ weakly in} L^1(0,T; L^2(\Omega))$$

as $n \to \infty$. In view of well-known energy a priori estimate

$$\sup_{0<t<T} \|\mathbf{u}_n(t)\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla \mathbf{u}_n(t)\|_{L^2(\Omega)}^2 dt \leq C \tag{33}$$

with constant C independent on n, we may admit

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)) \cap L^p(0,T; L^2(\Omega)) \tag{34}$$

with any p, $1 \leq p < \infty$.

According to the theorem 1 it's possible to extract subsequence $\{(\mathbf{u}_m)_{h_m}\}$ of averaged functions (with respect to all independent variables or to spatial variables only) such that

$$(\mathbf{u}_m)_{h_m} \to \mathbf{u} \text{ strongly in } L^2(0,T;W^{1,2}(\Omega)) \cap L^p(0,T;L^2(\Omega)),$$

$$(\mathbf{f}_m)_{h_m} \to \mathbf{f} \text{ strongly in} L^1(0,T;L^2(\Omega)).$$

And the question arises here: are $\{(\mathbf{u}_m)_{h_m}\}$ the approximate solutions to the limiting equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = 0 \qquad (35)$$

in some strong sense?

To give the answer to this question we apply the operator of averaging to equations (31) and obtain the system (35) for $(\mathbf{u}_m)_{h_m}$ with new right part

$$\mathbf{F}_m = (\mathbf{f}_m)_{h_m} + [((\mathbf{u}_m)_{h_m} \cdot \nabla)(\mathbf{u_m})_{h_m} - ((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m)_{h_m}] \equiv$$

$$(\mathbf{f}_m)_{h_m} + \varphi_{\mathbf{m}}.$$

Theorem 2 . *There exists subsequence $(\mathbf{u}_m)_{h_m}$ such that the difference φ_m tends to zero in the norm of space $L^p(0,T;L^q(\Omega))$ with exponents (p,q), $p \in [1,2]$, $q \in [1,3/2]$, $1/p + 3/2q \geq 2$.*

Proof   is the simple corollary of the theorem 1 and embedding theorems.

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