Asymptotics of Wigner functions at high frequency and near caustics

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ABSTRACT

Eigenfunction expansions of time-dependent Wigner functions are employed to motivate asymptotic expansions at high frequencies and near space-time caustics for the semiclassical Schrödinger equation with simple polynomial potentials and WKB initial data.

1 Schrödinger equation

Consider the semiclassical Schrödinger equation

$$i\varepsilon\frac{\partial\psi^\varepsilon}{\partial t}=\Big(-\frac{\varepsilon^2}{2}\Delta+V(x)\Big)\psi^\varepsilon(x,t),\quad x\in\mathbb{R},\ t>0$$

with oscillatory (WKB) initial data

$$\psi^\varepsilon(x,0)=\psi_0^\varepsilon(x)=a_0(x)e^{iS_0(x)/\varepsilon}$$

Hypotheses for the potential :

- $V(x)$ is analytic
- $\lim_{|x|\to\infty} V(x)=+\infty,\ V\in C(\mathbb{R})$
- $V\geq 0$, for some $R>0$ $\inf_{|x|>R} V(x)>0$
- $V(0)=0, V'(0)=0$ such that $V''(0)>0$
- V is polynomially bounded $|V(x)|\leq c(1+|x|^m)$(or at least grows as fast as $e^{\beta x^2}$)

2 Phase space reformulation

Wigner function in phase space [Wigner, 1932]

$$W^\varepsilon(x,k,t)=W^\varepsilon[\psi^\varepsilon](x,k,t)=\frac{1}{\pi\varepsilon}\int_{\mathbb{R}}e^{-\frac{i}{\varepsilon}2k\xi}\psi^\varepsilon(x-\xi,t)\overline{\psi}^\varepsilon(x+\xi,t)d\xi$$

Wigner equation for the evolution of W^ε

$$\begin{cases}\frac{\partial}{\partial t}W^\varepsilon(x,k,t)+\mathcal{L}^\varepsilon W^\varepsilon(x,k,t)=0,\ (x,k)\in\mathbb{R}^2, t>0\\ W^\varepsilon(x,k,t)|_{t=0}=W_0^\varepsilon(x,k)=W^\varepsilon[\psi_0^\varepsilon](x,k)\\ \mathcal{L}^\varepsilon\equiv k\frac{\partial}{\partial x}-V'(x)\frac{\partial}{\partial k}-\sum_{j=1}^\infty\varepsilon^{2j}\left(\frac{i}{2}\right)^{2j}\frac{V^{(2j+1)}(x)}{(2j+1)!}\frac{\partial^{(2j+1)}}{\partial k^{2j+1}}\end{cases}$$

Remark: For $\varepsilon\to0$ quantum Liouville operator $\mathcal{L}^\varepsilon$, formally reduces to the corresponding stationary classical Liouville operator $\mathcal{L}_c\equiv k\frac{\partial}{\partial x}-V'(x)\frac{\partial}{\partial k}$, corresponding to the Hamiltonian $H(x,k)=k^2/2+V(x)$.

3 Eigenfunction expansion of the Wigner function

W^ε admits of the eigenfunction expansion

$$W^\varepsilon(x,k,t)=\sum_{n=0}^\infty\sum_{m=0}^\infty A_{nm}^\varepsilon(t)\Phi_{nm}^\varepsilon(x,k)$$

where

$$\Phi_{nm}^\varepsilon(x,k)=W^\varepsilon[u_n^\varepsilon,u_m^\varepsilon](x,k)=\frac{1}{\pi\varepsilon}\int_{\mathbb{R}}e^{-\frac{i}{\varepsilon}2k\xi}u_n^\varepsilon(x-\xi)\overline{u}_m^\varepsilon(x+\xi)d\xi$$

are the Wigner transforms of Schrödinger eigenfunctions (Moyal eigenfunctions; [Moyal, 1949])

$$\Big(-\frac{\varepsilon^2}{2}\Delta+V(x)\Big)u_n^\varepsilon(x)=E_n^\varepsilon u_n^\varepsilon(x)$$

Remark 3.1: $\Phi_{nm}^\varepsilon(x,k)$ are defined by the system of both eigenvalue equations

$$\mathcal{L}^\varepsilon\Phi_{nm}^\varepsilon(x,k)=\frac{i}{\varepsilon}(E_n^\varepsilon-E_m^\varepsilon)\Phi_{nm}^\varepsilon(x,k)$$

$$\mathcal{M}^\varepsilon\Phi_{nm}^\varepsilon(x,k)=\frac{1}{2}(E_n^\varepsilon+E_m^\varepsilon)\Phi_{nm}^\varepsilon(x,k)$$

where

$$\mathcal{M}^\varepsilon=-\frac{\varepsilon^2}{8}\triangle_{xk}+H(x,k)+\sum_{j=1}^\infty\varepsilon^{2j}\left(\frac{i}{2}\right)^{2j}\frac{V^{(2j)}(x)}{(2j)!}\partial_k^{(2j)}+\frac{\varepsilon^2}{8}\partial_k^2$$

Remark 3.2: Employing asymptotics of Schrödinger eigenfunctions u_n^ε about the eigenfunctions of the corresponding harmonic oscillator (potential $V_H(x)=x^2/2$) [Simon, 1983], we derive formal asymptotic expansions of the Moyal eigenfunctions

$$\Phi_{nm}^\varepsilon(x,k)\sim\Psi_{nm}^\varepsilon(x,k)+\sum_{l=1}^\infty\varepsilon^{\frac{l}{2}}Z_{nm}^{\varepsilon,(l)}(x,k)$$

where

$$\Psi_{nm}^\varepsilon(x,k)=W^\varepsilon[\psi_n^\varepsilon,\psi_m^\varepsilon](x,k)\ ,$$

$\{\psi_n^\varepsilon(x)\}_{n=0,1,\ldots}$ being the eigenfunctions of the corresponding harmonic oscillator.

4 Asymptotics of the Wigner functions

4.1 Near Wigner functions of the harmonic oscillator

The eigenfunction expansion of $W^\varepsilon(x,k,t)$ and the asymptotic approximation of $\Phi_{nm}^\varepsilon(x,k)$, lead to the ansatz:

$$W^\varepsilon(x,k,t)\sim W_H^\varepsilon(x,k,t)+\sum_{l=1}^\infty\varepsilon^{l/2}Z^{\varepsilon,(l)}(x,k,t)$$

where $W_H^\varepsilon(\mathcal{X},\mathcal{K},t)$ and $Z^{\varepsilon,(l)}(\mathcal{X},\mathcal{K},t)$ ($\mathcal{X}=\frac{x}{\sqrt{\varepsilon}},\ \mathcal{K}=\frac{k}{\sqrt{\varepsilon}}$) are solutions of

$$\begin{cases}\frac{\partial}{\partial t}W_H^\varepsilon(\mathcal{X},\mathcal{K},t)+L_HW_H^\varepsilon(\mathcal{X},\mathcal{K},t)=0\\ W_H^\varepsilon(\mathcal{X},\mathcal{K},t)|_{t=0}=W_0^\varepsilon(\mathcal{X},\mathcal{K})\end{cases}$$

$$\begin{cases}\frac{\partial}{\partial t}Z^{\varepsilon,(l)}(\mathcal{X},\mathcal{K},t)+L_HZ^{\varepsilon,(l)}(\mathcal{X},\mathcal{K},t)=D^{(l)}(\mathcal{X},\mathcal{K},t),\ l\geq1\\ Z^{\varepsilon,(l)}(\mathcal{X},\mathcal{K},t)|_{t=0}=0\end{cases}$$

where

$$L_H=\mathcal{K}\frac{\partial}{\partial\mathcal{X}}-\mathcal{X}\frac{\partial}{\partial\mathcal{K}}$$

$$D^{(l)}(\mathcal{X},\mathcal{K},t)=-\mathcal{B}_l(\mathcal{X},\tfrac{\partial}{\partial\mathcal{K}})W_H^\varepsilon(\mathcal{X},\mathcal{K},t)-\sum_{\nu=1}^{l-1}\mathcal{B}_\nu(\mathcal{X},\tfrac{\partial}{\partial\mathcal{K}})Z^{\varepsilon,(l-\nu)}(\mathcal{X},\mathcal{K},t)$$

$$\mathcal{B}_\nu(\mathcal{X},\tfrac{\partial}{\partial\mathcal{K}})=-V^{(\nu+2)}(0)\sum_{j=0}^{[(\nu-1)/2]+1}\left(\tfrac{i}{2}\right)^{2j}\frac{1}{(2j+1)!}\frac{\mathcal{X}^{\nu+1-2j}}{(\nu+1-2j)!}\frac{\partial^{(2j+1)}}{\partial\mathcal{K}^{2j+1}},\quad\nu\geq1$$

Remark 4.1: Then initial value problem for W_H^ε, involving classical Liouville operator $\partial/\partial t+L_H$ can be integrated applying the method of characteristics.

Important observation 4.2: Such an expansion, valid for small times, is not appropriate near caustics, since L_H fails, in general, to produce the correct Lagrangian manifold due to linearization of Hamiltonian flow.

4.2 Near solutions of classical Liouville equation

Observe that

$$\mathcal{L}^\varepsilon=\mathcal{L}_c-\sum_{j=1}^\infty\varepsilon^{2j}\Theta_j(x,\frac{\partial}{\partial k})\longrightarrow\mathcal{L}_c\quad,\text{as }\varepsilon\to0\,,$$

where

$$\mathcal{L}_c\equiv k\frac{\partial}{\partial x}-V'(x)\frac{\partial}{\partial k}\quad\text{(classical Liouville operator)}$$

$$\Theta_j(x,\frac{\partial}{\partial k})\equiv\left(\frac{i}{2}\right)^{2j}\frac{V^{(2j+1)}(x)}{(2j+1)!}\frac{\partial^{(2j+1)}}{\partial k^{2j+1}}$$

Then, for small ε, a natural expansion which respects the evolution of the Lagrangian manifold has the form

$$W^\varepsilon(x,k,t)\sim W_c^\varepsilon(x,k,t)+\sum_{l=1}^\infty\varepsilon^{2l}Z_c^{\varepsilon,(l)}(x,k,t)$$

where

$$\begin{cases}\frac{\partial}{\partial t}W_c^\varepsilon(x,k,t)+\mathcal{L}_cW_c^\varepsilon(x,k,t)=0\\ W_c^\varepsilon(x,k,t)|_{t=0}=W_0^\varepsilon(x,k)\end{cases}$$

and

$$\begin{cases}\frac{\partial}{\partial t}Z_c^{\varepsilon,(l)}(x,k,t)+\mathcal{L}_cZ_c^{\varepsilon,(l)}(x,k,t)=\Theta^{(l)}(x,k,t)=\sum_{j=1}^l\Theta_j(x,\frac{\partial}{\partial k})Z_c^{\varepsilon,(l-j)}(x,k,t),\ l\geq1\\ Z_c^{\varepsilon,(l)}(x,k,t)|_{t=0}=0\end{cases}$$

Remark 4.3: See [Steinruck, 1990] for non-oscillatory initial data $\psi_0^\varepsilon(x)$, and [Pulvirenti, 2006] for oscillatory $\psi_0^\varepsilon(x)$ using asymptotics of $W_0^\varepsilon(x,k)$ in terms of Dirac functions.

5 Caustics

For simple Gaussian-Fresnel initial wave function

$$\psi_0^\varepsilon(x)=e^{-\frac{x^2}{2}}e^{i\frac{x^2}{2\varepsilon}}\ ,$$

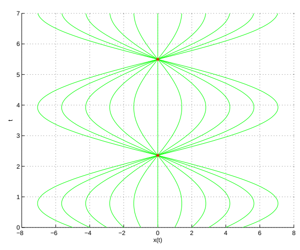
the initial Wigner function is Gaussian in phase space

$$W_0^\varepsilon(x,k)=\frac{1}{\sqrt{\pi}\varepsilon}e^{-x^2}e^{-\frac{(k-x)^2}{\varepsilon^2}}$$

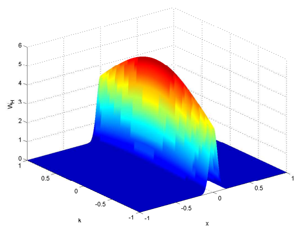
5.1 Harmonic Oscillator ($V_H(x)=\frac{x^2}{2}$). Focal points

Bicharacteristics $x(q,p,t)=q\cos(t)+p\sin(t),\ k(q,p,t)=p\cos(t)-q\sin(t)$

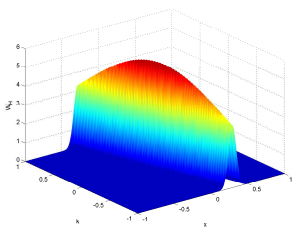
Focal points: $(x_\nu,t_\nu)=(0,\nu\pi-\frac{\pi}{4}),\ \nu=1,2,\ldots$



Caustic & Rays



Harmonic Oscillator Wigner function (t=3π/4)

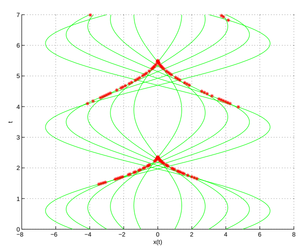


Harmonic Oscillator Wigner function (t=200)

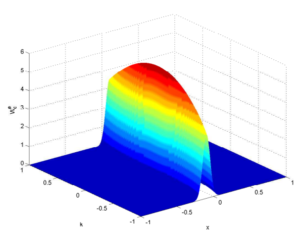
Amplitude computed approximately: $|\psi^\varepsilon(x,t)|^2=\int_{\mathbb{R}}W^\varepsilon(x,k,t)dk\sim\int_{\mathbb{R}}W_H^\varepsilon(x,k,t)dk$

At the focal points: $|\psi^\varepsilon(x=0,t=t_\nu)|\sim0(\varepsilon^{-1/2})$

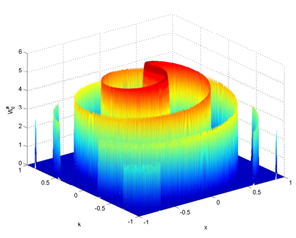
5.2 Quartic Oscillator ($V(x)=\frac{x^2}{2}+\mu\frac{x^4}{4},\quad\mu>0$). Cusps



Caustic & Rays



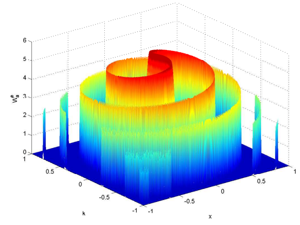
Wigner function (t=3π/4, μ = 0.1)



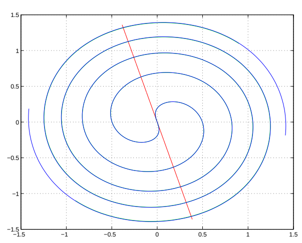
Wigner function (t=200, μ = 0.1)

Approximate bicharacteristics via multiple scale asymptotics of the Hamiltonian system, for small μ.

Cusp points: $(x_\nu,t_\nu)=(0,\nu\pi-\frac{\pi}{4}),\ \nu=1,2,\ldots$



Wigner function-Approximate bicharacteristics (t=200, μ = 0.1)



Lagrangian manifolds (t=200, μ = 0.1)

$|\psi^\varepsilon(x,t)|^2\sim\int_{\mathbb{R}}W_a^\varepsilon(x,k,t)dk$

At the cusp points: $|\psi^\varepsilon(x=0,t=t_\nu)|\sim0(\varepsilon^{-1/3})$

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