Wave energy transport via semiclassical asymptotics: A sketch and two examples of the research activity in Pavia

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Summary of the research activity

The Plasma Physics research group at the University of Pavia has acquired an expertise in asymptotic analysis of wave equations in the high-frequency (semiclassical) limit, having in mind applications to magnetically confined plasmas for thermonuclear fusion development. Here are some issues under consideration.

- Derivation of the wave kinetic equation for the Wigner distribution [1] and its connection to the theory of Fourier integral distributions [2, 3] (see below) as well as to other type of phase space (microlocal) analysis [4, 5] such as the FBI and Bargmann transforms [1, 5, 6]. Derivation of transport equations for the wave energy density from the kinetic equation [7-9].
- High-frequency asymptotics of hyperbolic equations, both linear [9-12] and nonlinear [14, 15], with highly oscillating and localized Cauchy data modelling collimated wave beams (see below).
- Evolution of the phase fronts in terms of a curvature flow (in preparation, with Grigory V. Pereverzev, Max-Planck-Institut für Plasma Physics.)
- Application of pseudodifferential operators, their symbols and calculus, [16].
- Some work in classical electrodynamics [17].

References

[1] S. W. McDonald, Phys. Rep. **158**, 337 (1998).
[2] J. J. Duistermaat, *Fourier Integral Operators*, Birkhäuser, Boston (1996).
[3] A. Grigis and J. Sjöstrand *Microlocal Analysis for Partial Differential Operators* (Cambridge Un. Press, Cambridge 1994).
[4] P. Gérard, Comm. Partial Differential Equations **16**, 1761 (1991).
[5] A. Martinez, *An Introduction to Semiclassical and Microlocal Analysis* (Springer-Verlag, New York 2002).
[6] D. Tataru, Phase space analysis of partial differential equations, Vol.II, Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa 2004.
[7] C. Sparber, P. A. Markowich and N. J. Mauser, Asymptotic Analysis **33**, 153 (2000).
[8] M. Bornatici and Yu. A. Kravtsov, Plasma Phys. Controll. Fusion **42**, 255 (2000).
[9] O. Maj, J. Math. Phys. **46**, 083510 (2005).
[10] Yu. A. Kravtsov, *Geometrical Optics in Engineering Physics* (Alpha Science 2005).
[11] M. Bornatici and O. Maj, Plasma Phys. Controll. Fusion **45**, 707 (2003).
[12] G. V. Pereverzev, Reviews of Plasma Physics **19**, 1 (1996).
[13] V. P. Maslov, *The Complex-WKB Method for Nonlinear Equations I: Linear Theory* (Birkhäuser, Boston 1996).
[14] J.-L. Joly, G. Metivier and J. Rauch, Duke Math. J. **70**, 373 (1993).
[15] J. Rauch, *Lectures on geometric optics*, in IAS Park City Summer Math Institute (1995).
[16] M. Bornatici and O. Maj, Plasma Phys. Controll. Fusion **45**, 1511 (2003).
[17] M. Bornatici and O. Maj, Physica Scripta **73**, 160 (2006).
[18] A. Melin and J. Sjöstrand, Comm. Partial Differential Equations **1**, 313 (1976).

First-order quasi-linear hyperbolic systems with highly oscillating/localized data

We have considered systems of quasi-linear hyperbolic equations for the wave field $u(t,x) \in \mathbb{C}^N$,

$$L(u, \partial u) = \partial u/\partial t + \sum_1^d a_i(t,x,u)\cdot \partial u/\partial x_i + b(t,x,u) = 0, \qquad (t,x) \in \Omega = (0,T)\times X, \quad X \subseteq \mathbb{R}^d.$$

We search for approximate solutions in the form $\quad u_\varepsilon(t,x) = u_0(t,x) + \varepsilon\mathcal{U}(t,x,\phi(t,x)/\varepsilon), \qquad \varepsilon \to 0.$

- The profile $\mathcal{U}(t,x,\zeta)$ is defined for $\zeta \in \mathbb{C}^m_+$, $\mathbb{C}_+ = \mathbb{R} + i\overline{\mathbb{R}}_+$, by extending a 2π-periodic function $U(t,x,\vartheta)$ according to (series are absolutely convergent)

$$\mathcal{U}(t,x,\zeta) = \sum_{g\in\mathbb{Z}^m} \hat{U}(t,x,g)e^{i\langle g,\zeta\rangle}, \qquad \langle g,\zeta\rangle = \sum_{\mu=1}^m \big(g_\mu \mathrm{Re}(\zeta_\mu) + i|g_\mu|\mathrm{Im}(\zeta_\mu)\big),$$

  with $\hat{U}(t,x,g)$ being the Fourier coefficients of $U(t,x,\vartheta)$ and we have set $\hat{U}(t,x,0) = 0$, *strictly oscillating profiles.*
- The function $\phi \in C^\infty(\Omega, \mathbb{C}^m_+)$ is the multi-valued complex phase with components $\phi_\mu \in C^\infty(\Omega, \mathbb{C}_+)$, $\mu = 1, \ldots, m$.

Proposition 1 *Let $\mathcal{U}(t,x,\zeta)$ be any strictly oscillating profile and ϕ a multi-valued complex phase. Then, in any compact set $K \subset \Omega$ where $\mathrm{Im}(\phi_\mu) = \chi_\mu > 0$ for all μ, we have*

$$|\mathcal{U}(t,x,\phi/\varepsilon)| \leq C_n\varepsilon^n, \qquad \textit{for every } n \in \mathbb{N},$$

hence the oscillating wavefield is localized around $R = \bigcup_\mu R_\mu$, with $R_\mu = \chi_\mu^{-1}(0)$.

It is assumed that R amounts to a closed submanifold of $(0,T)\times X$ and it is called reference manifold.

Cauchy data at $t = 0$ are $h_\varepsilon(x) = h_0(x) + \varepsilon\sum_\mu H_\mu(x)e^{i\phi^0_\mu(x)} + O(\varepsilon^{+\infty})$, where each $H_\mu(x)$ is in an eigenspace of $A(t,x,\tau,\xi) = \tau + \sum a_i(t,x)\xi_i$ corresponding to the eigenvalue $f_l(t,x,\tau,\xi)$: for each μ there is a corresponding l. We require that the eigenspaces of A have constant multiplicity.

One finds that R_μ is the flow out of the initial set $(\mathrm{Im}\phi^0_\mu)^{-1}(0)$ along the characteristics of the Hamilton-Jacobi equation $f_l(t,x,d\varphi) = 0$, $\varphi(t,x) \in \mathbb{R}$. The complex phase ϕ_μ can be written as, **"paraxial expansion"**,

$$\phi_\mu(t,x) = \phi_{0,\mu}(r) + \sum_i s^i\phi_{i,\mu}(r) + \frac{1}{2}\sum_{ij} s^is^j\phi_{ij,\mu}(r)$$

where (r,s) are the submanifold coordinates near a point on R_μ, with r coordinates on R_μ. For the profiles we gets

$$\Big[\partial_t + \sum_1^d a_i(t,x,u_0(t,x))\partial_{x_i}\Big]\cdot U + \sum_{i=1}^d\sum_{\mu=1}^m \Big[\big(\partial_u a_i(t,x,u_0)(U)\partial_{x_i}\varphi_\mu(t,x)\big) + \big(\partial_{\bar{u}} a_i(t,x,u_0)(\bar{U})\partial_{x_i}\varphi_\mu\big)\Big]\cdot\partial_{\vartheta_\mu}U$$
$$+ \sum_1^d \Big[\partial_u a_i(t,x,u_0)(U) + \partial_{\bar{u}}a_i(t,x,u_0)(\bar{U})\Big]\cdot\partial_{x_i}u_0 + \Big(\partial_u b_i(t,x,u_0)(U) + \partial_{\bar{u}}b(t,x,u_0)(\bar{U})\Big) = 0, \quad \varphi_\mu = \mathrm{Re}(\phi_\mu).$$

along with the two constraints $\mathbb{E}_2U(t,x,\vartheta) = U(t,x,\vartheta)$, and $\hat{U}(t,x,0) = 0$ (where $\mathbb{E}_2$ is defined in proposition 2).

We now modify the notion of L_0-coherent spaces [14]. Here, $L_0 = \partial_t + \sum_1^d a_i(t,x,u_0(t,x))\partial_{x_i}$.

Definition 1 *A linear subspace $\Phi \subset C^\infty(\Omega)$ over $\mathbb{C}$, is approximately L_0-coherent to order k on the submanifold R_μ if for every $\psi \in \Phi$ one of the following two statements holds. (i) For all $(t,x) \in R_\mu$, $d\psi \neq 0$ and $\partial_s^\alpha f_l(t,x,d\psi) = 0$ for all α with $|\alpha| \leq k$. (ii) For all $(t,x) \in R_\mu$, $\partial_s^\alpha f_l(t,x,d\psi) \neq 0$ at least for one α with $|\alpha| \leq k$; this in particular implies $d\psi \neq 0$.*

Condition 1 *The phases $\phi_1, \ldots, \phi_m$ span (over the field of complex numbers) a subspace $\Phi \subset C^\infty(\Omega)$ approximately L_0-coherent <u>to order 2</u> on each submanifolds R_μ. Moreover, if $\langle g,\phi\rangle = 0$ then $g = 0$, that is, the map $\mathbb{Z}^m \ni g \mapsto \langle g,\phi\rangle \in \Phi$ is injective.*

Proposition 2 *Let $\phi = (\phi_1, \ldots, \phi_m)$ satisfy condition 1, then for every μ the set*

$$\widetilde{\mathscr{C}}^{\phi,2}_\mu = \{g \in \mathbb{Z}^m\setminus\{0\} : \partial_s^\alpha f_l(t,x,\langle g, d\phi\rangle) = 0, \ \textit{in } R_\mu, \ |\alpha| \leq 2\},$$

is well-defined in $\mathbb{Z}^m$ and non empty, where l is the label of the unique eigenvalue corresponding to μ; equivalently the characteristic set

$$\mathscr{C}^{\phi,2}_\mu = \{(t,x,g) \in R_\mu \times (\mathbb{Z}^m\setminus\{0\}) : \partial_s^\alpha f_l(t,x,\langle g,d\phi\rangle) = 0, \ |\alpha| \leq 2\},$$

is constant on R_μ, i.e., $\mathscr{C}^{\phi,2}_\mu = R_\mu \times \widetilde{\mathscr{C}}^{\phi,2}_\mu$. Furthermore, let us define

$$\widetilde{\mathscr{C}}^{\phi,2}_l = \bigcup_\mu \widetilde{\mathscr{C}}^{\phi,2}_\mu,$$

with the union being over all μ corresponding to l, and, on recalling that $R = \bigcup_\mu R_\mu$, let us also define the operator

$$\mathbb{E}_2\Big(\sum_{g\in\mathbb{Z}^m}\hat{U}(t,x,g)e^{i\langle g,\zeta\rangle}\Big) = \sum_{g\in\mathbb{Z}^m}\pi_2^\phi(t,x,g)\cdot\hat{U}(t,x,g)\ e^{i\langle g,\zeta\rangle},$$

acting on formal series, where

$$\pi_2^\phi(t,x,g) = \begin{cases} \pi_l(t,x,\langle g,d\phi\rangle), & \textit{for } g \in \widetilde{\mathscr{C}}^{\phi,2}_l, \\ I, & \textit{for } g \notin \widetilde{\mathscr{C}}^{\phi,2}_l \textit{ such that } \mathrm{Im}\langle g,\phi\rangle \neq 0 \textit{ on } R, \\ 0, & \textit{otherwise,} \end{cases}$$

$\pi_l(t,x,\tau,\xi)$ being the projector on the l-th eigenspace of A and I the identity matrix. Then, if the eigenvectors $e_l(t,x,\langle g,d\phi\rangle)$ of the matrix $A(t,x,\langle g,d\phi\rangle)$ are bounded together with all their derivatives, the operator $\mathbb{E}_2$ is well defined : $C^\infty(\Omega\times\mathbb{C}^m_+, \mathbb{C}^N) \to C^\infty(\Omega\times\mathbb{C}^m_+, \mathbb{C}^N)$.

Theorem 3 *Let $U \subset \Omega$ be a neighbourhood of any point on R and let $u_\varepsilon(t,x)$ be the approximated solution given above with condition 1 satisfied. Then, there exists an ε-bounded function $S(\varepsilon,t,x,u_\varepsilon)$ together with an even integer $\nu \geq 2$ such that $|L(u_\varepsilon, \partial u_\varepsilon)| \leq \varepsilon^{3/\nu}S(\varepsilon,t,x,u_\varepsilon)$, uniformly in U. Here, ε-boundness means that $\sup_U \varepsilon^{|\alpha|}|\partial^\alpha S(\varepsilon,t,x,u_\varepsilon)| \leq C$ uniformly for $\varepsilon \in (0,\varepsilon_0]$ and for any multi-index α.*

An interesting outcome. The above sketched analysis has been put forward with the aim of combining widely-applied approximate solutions [9-11] with the nonlinear geometric optics [14, 15]; this, in particular, allows us to consider wave objects in a form between that of wave beams and short pulses. In addition, by working out simple analytically tractable cases we can see that the phenomenon of phase resonance [15] can be destroyed by the localization of the wavefield: the new phase generated by resonance can have a reference manifold not intersecting R and, thus, it give no contribution to significant order.

The wave kinetic equation and Fourier integrals

The aim of this work is to give an alternative derivation of the wave kinetic equation which does not make use of neither the Wigner-Weyl formalism [1] nor microlocal techniques [4], and resembles the theory of Fourier integrals [2, 3]. It is expected that, in addition to establish a connection between the kinetic theory of waves and the Maslov-Hörmander theory, such an approach can clarify some issues connected to the wave energy transport [1].

Outline. Let P be a semiclassical pseudo-differential operator in $X \subseteq \mathbb{R}^n$ of real principal type with a *completely integrable* principal symbol $p(x,\xi)$. This, in particular, means that there is a submersion

$$f : T^*X \to A \subseteq \mathbb{R}^n, \qquad f(x,\xi) = \big(f_1(x,\xi), \ldots, f_n(x,\xi)\big), \qquad f_1(x,\xi) = p(x,\xi),$$

with $f_1, \ldots, f_n$ in involution. Then, each "fiber" $f^{-1}(a)$, $a \in A$, is a Lagrangian submanifold of T^*X; more specifically,

$$\mathrm{graph}(f) \subset T^*X \times A,$$

defines a family of Lagrangian submanifolds in the sense of Melin and Sjöstrand [18].

A family of Lagrangian manifolds can be locally parametrized by means of a phase function $\varphi(x,\vartheta,a)$ defined (locally) on $X \times \mathbb{R}^k \times A$, with $k \leq n$ auxiliary variables (the equivalent of a Morse family). If, the family is non-degenerate it defines a class of Fourier integral distributions [18].

Idea: there exists a transform $u \mapsto T_f u = u_f$ such that

$$u(x) = \int e^{i\varphi(x,\vartheta,a)}u_f(x,\vartheta,a)d\vartheta da \in \mathcal{D}'(X),$$

where $u_f(x,\vartheta,a)$ is a distribution that can be approximated by a sequence of symbols with fiber-variables (ϑ,a). This reduces to the Fourier transform for $X = A = \mathbb{R}^n$ and $f(x,\xi) = \xi = a$, i.e., free-particle momentum. Then, a suitable average of $u_f(x,\vartheta,a)$ should be related to the wave energy density on T^*X.