Semiclassical expansion and mean-field limit

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Abstract

We consider the power series expansion in $\hbar$ of the time evolved BBGKY hierarchy of the Wigner functions for a system of N identical quantum particles under the action of a mean-field potential. The problem is to prove that each term of such an expansion converges to the corresponding term of the expansion of the solution of the infinite hierarchy associated with the Hartree equation.

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Consider a system of N identical quantum particles of unit mass interacting through a mean-field potential:

$$U(X_N) = \frac{1}{N} \sum_{i<j}^{N} \varphi\left(x_i - x_j\right), \tag{1}$$

where $X_N = \{x_1, \ldots, x_N\}$ and $x_1, \ldots, x_N$ are the positions of the N particles. We know (see [1]) that, under the assumptions that the two-body interaction φ is sufficiently smooth and the initial wave function is factorized, in the limit $N \to \infty$ each particle evolves according to the following nonlinear Schroedinger equation of the Hartree type:

$$\left(i\epsilon\partial_t + \frac{\epsilon^2}{2}\Delta - \varphi * \rho\right)\psi(t) = 0, \tag{2}$$

where

$$\rho(x,t) = |\psi(x,t)|^2. \tag{3}$$

For our convenience we denoted with ϵ the parameter $\hbar$; in fact we are going to consider a sort of semiclassical limit and so, in order to refer to $\hbar$ as a small parameter,

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we call it ϵ, but it's only matter of notation.
Further results concerning the Coulomb interaction (see [2] and [3]). have also been proved.
In all these results, however, the convergence is strongly dependent on ϵ. More precisely, the estimates which ensure the convergence fail if ϵ goes to zero. On the other hand (see [4] and [5]), one can prove that the simultaneous limit, $N \to \infty$, $\epsilon \to 0$ arbitrarily, recover the classical Vlasov equation.
Therefore one expects that the derivation of the Hartree equation (for ϵ fixed) should hold uniformly in ϵ. For an attempt in this direction see [6].
Now we approach the problem in a different way. We would like to consider the limit $N \to \infty$ by analyzing the term by term convergence of the semiclassical expansion of Wigner functions presented in [7].
The Wigner-Liouville equation for the system under consideration is

$$\left(\partial_t + V_N \cdot \nabla_{X_N}\right) W_N = T_N W_N, \tag{4}$$

where $V_N = \{v_1, \ldots, v_N\}$ and $v_1, \ldots, v_N$ are the velocities of the N particles. Clearly W_N is the Wigner function associated to the system under consideration and the operator T_N acts as follows:

$$\left(T_N W_N\right)\left(X_N, V_N\right) = i \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathrm{d}\lambda \int \mathrm{d}K_N \hat{U}\left(K_N\right) e^{iK_N \cdot X_N} K_N \cdot \nabla_{V_N} W_N\left(X_N, V_N + \epsilon\lambda K_N\right), \tag{5}$$

where $\hat{U}\left(K_N\right)$ is the Fourier transform of the potential in (1), $K_N = \{k_1, \ldots, k_N\}$ and $k_1, \ldots, k_N$ are the momenta of the N particles. We consider a factorized initial state, namely at time $t = 0$ we have:

$$W_N\left(X_N, V_N, 0\right) = W_0\left(X_N, V_N\right) = \left(f_0^{\otimes N}\right)\left(X_N, V_N\right) \tag{6}$$

where f_0 is a one particle probability density, it's sufficiently smooth and it doesn't depend on ϵ. This choice is not particularly meaningful but thanks to it computations will be simpler. Now we observe that

$$\begin{aligned} K_N \cdot \nabla_{V_N} W_N\left(X_N, V_N + \epsilon\lambda K_N\right) &= \frac{1}{\epsilon}\frac{d}{d\lambda} W_N\left(X_N, V_N + \epsilon\lambda K_N\right) = \\ &= \frac{d}{d\mu} W_N\left(X_N, V_N + \mu K_N\right), \end{aligned} \tag{7}$$

where $\mu = \epsilon\lambda$. So we can write the Taylor expansion around the point $\mu = 0$ of (7) obtaining:

$$\begin{aligned} \frac{d}{d\mu} W_N\left(X_N, V_N + \mu K_N\right) &= \frac{d}{d\mu} W_N\left(X_N, V_N + \mu K_N\right)|_{\mu=0} + \\ &+ \mu \frac{d^2}{d\mu^2} W_N\left(X_N, V_N + \mu K_N\right)|_{\mu=0} + \\ &+ \frac{\mu^2}{2} \frac{d^3}{d\mu^3} W_N\left(X_N, V_N + \mu K_N\right)|_{\mu=0} + o\left(\mu^3\right), \end{aligned} \tag{8}$$

that is

$$\begin{aligned} \frac{d}{d\mu} W_N\left(X_N, V_N + \mu K_N\right) &= K_N \cdot \nabla_{V_N} W_N\left(X_N, V_N\right) + \\ &+ \mu \left(K_N \cdot \nabla_{V_N}\right)^2 W_N\left(X_N, V_N\right) + \\ &+ \frac{\mu^2}{2} \left(K_N \cdot \nabla_{V_N}\right)^3 W_N\left(X_N, V_N\right) + o\left(\mu^3\right). \end{aligned} \tag{9}$$

For the sake of simplicity we analyse the expansion up to the second order in μ, or in ϵ, but we will see that in this way we are able to understand also the structure of higher order terms.
Therefore, putting the expression (9) in (5), we find the following expansion for the operator T_N:

$$T_N = T_N^{(0)} + \epsilon T_N^{(1)} + \epsilon^2 T_N^{(2)} + o\left(\epsilon^3\right), \tag{10}$$

where

$$\begin{aligned} & \left(T_N^{(0)} W_N\right)\left(X_N, V_N\right) = \\ = \quad & i \left(\int_{-\frac{1}{2}\epsilon}^{+\frac{1}{2}\epsilon} \frac{\mathrm{d}\mu}{\epsilon}\right) \int \mathrm{d}K_N \hat{U}\left(K_N\right) e^{iK_N \cdot X_N} K_N \cdot \nabla_{V_N} W_N\left(X_N, V_N\right) = \\ = \quad & i \int \mathrm{d}K_N \hat{U}\left(K_N\right) e^{iK_N \cdot X_N} K_N \cdot \nabla_{V_N} W_N\left(X_N, V_N\right) = \\ = \quad & \left(\int \mathrm{d}K_N \left(iK_N\right) \hat{U}\left(K_N\right) e^{iK_N \cdot X_N}\right) \nabla_{V_N} W_N\left(X_N, V_N\right) = \\ = \quad & \nabla_{X_N} U\left(X_N\right) \cdot \nabla_{V_N} W_N\left(X_N, V_N\right), \end{aligned} \tag{11}$$

$$
\begin{aligned}
&\left(T_N^{(1)}W_N\right)(X_N,V_N) = \\
= \quad & \frac{i}{\epsilon}\left(\int_{-\frac{1}{2}\epsilon}^{+\frac{1}{2}\epsilon}\mu\frac{\mathrm{d}\mu}{\epsilon}\right)\int \mathrm{d}K_N\hat{U}(K_N)\,e^{iK_N\cdot X_N}\left(K_N\cdot\nabla_{V_N}\right)^2 W_N(X_N,V_N) = \\
= \quad & 0,
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&\left(T_N^{(2)}W_N\right)(X_N,V_N) = \\
= \quad & \frac{i}{\epsilon^2}\left(\int_{-\frac{1}{2}\epsilon}^{+\frac{1}{2}\epsilon}\frac{\mu^2}{2}\frac{\mathrm{d}\mu}{\epsilon}\right)\int \mathrm{d}K_N\hat{U}(K_N)\,e^{iK_N\cdot X_N}\left(K_N\cdot\nabla_{V_N}\right)^3 W_N(X_N,V_N) = \\
= \quad & -\frac{1}{24}\int \mathrm{d}K_N(-i)\sum_{j_1,j_2,j_3=1}^{N}k_{j_1}k_{j_2}k_{j_3}\hat{U}(K_N)\,e^{iK_N\cdot X_N}\cdot\frac{\partial}{\partial v_{j_1}}\frac{\partial}{\partial v_{j_2}}\frac{\partial}{\partial v_{j_3}}W_N(X_N,V_N) = \\
= \quad & -\frac{1}{24}\int \mathrm{d}K_N(-i)\sum_{\alpha:|\alpha|=3}K_N^{\alpha}\hat{U}(K_N)\,e^{iK_N\cdot X_N}D_{V_N}^{\alpha}W_N(X_N,V_N) = \\
= \quad & -\frac{1}{24}\frac{1}{N}\sum_{\ell\neq m}^{N}\sum_{\alpha:|\alpha|=3}D_{x_\ell x_m}^{\alpha}\varphi(x_\ell-x_m)D_{v_\ell v_m}^{\alpha}W_N(X_N,V_N).
\end{aligned} \tag{13}
$$

We can see that $T_N^{(0)}$ is nothing else than the Liouville operator, just as we expected from the classical mean-field theory, whereas $T_N^{(1)}$ is vanishing because of the integral in μ and we can easily conclude that

$$
T_N^{(2k+1)} = 0, \qquad k = 0,1,2,\dots \tag{14}
$$

Let us consider the semiclassical expansion of Wigner function, namely:

$$
W_N = W_N^{(0)} + \epsilon W_N^{(1)} + \epsilon^2 W_N^{(2)} + o\left(\epsilon^3\right). \tag{15}
$$

Therefore, inserting (15) and (10) in equation (4), we find:

$$
\begin{aligned}
& \partial_t\left(W_N^{(0)} + \epsilon W_N^{(1)} + \epsilon^2 W_N^{(2)} + o\left(\epsilon^3\right)\right) + \\
+ \quad & V_N\cdot\nabla_{X_N}\left(W_N^{(0)} + \epsilon W_N^{(1)} + \epsilon^2 W_N^{(2)} + o\left(\epsilon^3\right)\right) = \\
= \quad & \left(T_N^{(0)} + \epsilon^2 T_N^{(2)} + o\left(\epsilon^3\right)\right)\left(W_N^{(0)} + \epsilon W_N^{(1)} + \epsilon^2 W_N^{(2)} + o\left(\epsilon^3\right)\right).
\end{aligned} \tag{16}
$$

So we obtain the equation at order zero:

$$\partial_t W_N^{(0)} + V_N \cdot \nabla_{X_N} W_N^{(0)} = T_N^{(0)} W_N^{(0)}, \tag{17}$$

that is the Liouville equation with

$$W_N^{(0)}\left(X_N, V_N, 0\right) = W_0\left(X_N, V_N\right) = \left(f_0^{\otimes N}\right)\left(X_N, V_N\right). \tag{18}$$

Clearly we have that

$$W_N^{(0)}\left(X_N, V_N, t\right) = W_0\left(X_N(t), V_N(t)\right), \tag{19}$$

where $X_N(t)$ and $V_N(t)$ are the solutions of the hamiltonian system associated with the dinamic generated by the potential in (1).
Also the equation at order one is the Liouville equation because $\left(T_N^{(1)} W_N^{(0)}\right) = 0$, but now the initial datum is

$$W_N^{(1)}\left(X_N, V_N, 0\right) = 0, \tag{20}$$

so that we have

$$W_N^{(1)}\left(X_N, V_N, t\right) \equiv 0. \tag{21}$$

At second order we have again a Liouville equation with zero initial datum but solution is not trivial because there is a source term which we know from the previous step:

$$\partial_t W_N^{(2)} + V_N \cdot \nabla_{X_N} W_N^{(2)} = T_N^{(0)} W_N^{(2)} + T_N^{(2)} W_N^{(0)}. \tag{22}$$

Then we are able to compute $W_N^{(2)}$ using classical Liouville flux $S\left(t\right)$, namely:

$$W_N^{(2)}\left(X_N, V_N, t\right) = \int_0^t \mathrm{d}\tau S(t-\tau) T_N^{(2)} W_N^{(0)}(X_N, V_N, \tau). \tag{23}$$

Now look at the equation at third order in ϵ, that is:

$$\partial_t W_N^{(3)} + V_N \cdot \nabla_{X_N} W_N^{(3)} = T_N^{(0)} W_N^{(3)} + T_N^{(1)} W_N^{(2)} + T_N^{(2)} W_N^{(1)}, \tag{24}$$

or

$$\partial_t W_N^{(3)} + V_N \cdot \nabla_{X_N} W_N^{(3)} = T_N^{(0)} W_N^{(3)}, \tag{25}$$

with the following initial condition

$$W_N^{(3)}\left(X_N, V_N, 0\right) \equiv 0. \tag{26}$$

Clearly we have:

$$W_N^{(3)}\left(X_N, V_N, t\right) \equiv 0. \tag{27}$$

So by the calculation that we have done we are able to conclude that, because of the particular choice of an initial datum which doesn't depend on ϵ, all the odd terms of the expansion of W_N are equal to zero. In fact they are solutions of Liouville equations with zero initial conditions. On the contrary the even terms of the expansion are not zero and we are able to compute them because they are solutions of the Liouville equations with source terms that we know from the previous steps. For example look at the equation for the fourth order term:

$$\partial_t W_N^{(4)} + V_N \cdot \nabla_{X_N} W_N^{(4)} = T_N^{(0)} W_N^{(4)} + T_N^{(1)} W_N^{(3)} + T_N^{(2)} W_N^{(2)} + T_N^{(3)} W_N^{(1)}, \tag{28}$$

or

$$\partial_t W_N^{(4)} + V_N \cdot \nabla_{X_N} W_N^{(4)} = T_N^{(0)} W_N^{(4)} + T_N^{(2)} W_N^{(2)}, \tag{29}$$

with zero initial condition. Therefore we have:

$$W_N^{(4)} (X_N, V_N, t) = \int_0^t \mathrm{d}\tau S(t-\tau) T_N^{(2)} W_N^{(2)} (X_N, V_N, \tau), \tag{30}$$

where $S(t)$ is Liouville flux and we know $W_N^{(2)}$ from (23).
We analyze now more carefully the equation for $W_N^{(2)}$:

$$\partial_t W_N^{(2)} + V_N \cdot \nabla_{X_N} W_N^{(2)} = T_N^{(0)} W_N^{(2)} + T_N^{(2)} W_N^{(0)}. \tag{31}$$

As usual, in order to perform the limit $N \to \infty$, we have to trace the equation with respect to the last $N-j$ variables ($j = 1, 2, \ldots, N$) and then we obtain a hierarchy of N equations for the marginals $W_{N,j}^{(2)}$. Let u_j be a smooth test function such that $u_j = u_j (X_j, V_j)$ and let make use of the standard notation:

$$\begin{aligned} < u_j, W_N > \quad &= \int \mathrm{d}X_N \mathrm{d}V_N u_j (X_j, V_j) W_N (X_N, V_N, t) = \\ &= \int \mathrm{d}X_j \mathrm{d}V_j u_j (X_j, V_j) \int \mathrm{d}X_{N-j} \mathrm{d}V_{N-j} W_N (X_N, V_N, t) = \\ &= \int \mathrm{d}X_j \mathrm{d}V_j u_j (X_j, V_j) W_{N,j} (X_j, V_j, t) \,. \end{aligned} \tag{32}$$

Then from a simple computation we obtain:

$$\partial_t < u_j, W_{N,j}^{(2)} > + < u_j, V_j \cdot \nabla_{X_j} W_{N,j}^{(2)} > = < u_j, T_N^{(0)} W_{N,j}^{(2)} > + < u_j, T_N^{(2)} W_{N,j}^{(0)} > . \tag{33}$$

Now look at the term $< u_j, T_N^{(2)} W_{N,j}^{(0)} >$:

$$\begin{aligned}
< u_j, T_N^{(2)} W_{N,j}^{(0)} > \quad &= \frac{1}{24}\frac{1}{N}\int \mathrm{d}X_N \int \mathrm{d}V_N \sum_{\ell \neq m}^{N} \sum_{\alpha:|\alpha|=3} D^{\alpha}_{v_\ell v_m} u_j\left(X_j, V_j\right) \cdot \\
&\quad \cdot D^{\alpha}_{x_\ell x_m} \varphi(x_\ell - x_m) W_N^{(0)}\left(X_N, V_N, t\right) = \\
&= \frac{1}{24}\frac{1}{N}\int \mathrm{d}X_j \int \mathrm{d}V_j \sum_{\ell \neq m}^{j} \sum_{\alpha:|\alpha|=3} D^{\alpha}_{v_\ell v_m} u_j\left(X_j, V_j\right) \cdot \\
&\quad \cdot D^{\alpha}_{x_\ell x_m} \varphi(x_\ell - x_m) W_{N,j}^{(0)}\left(X_j, V_j, t\right) + \\
&+ \frac{1}{24}\frac{N-j}{N}\int \mathrm{d}X_{j+1} \int \mathrm{d}V_{j+1} \sum_{\ell=1}^{j} \sum_{\alpha:|\alpha|=3} D^{\alpha}_{v_\ell} u_j\left(X_j, V_j\right) \cdot \\
&\quad \cdot D^{\alpha}_{x_\ell} \varphi(x_\ell - x_{j+1}) W_{N,j+1}^{(0)}\left(X_{j+1}, V_{j+1}, t\right).
\end{aligned} \tag{34}$$

The first term in the r.h.s of equation (34) is expected to be $O\left(\frac{j^2}{N}\right)$ hence vanishing in the limit. In regard to the second term, we expect that

$$\begin{aligned}
< u_j, T_N^{(2)} W_{N,j}^{(0)} > \overset{N\to\infty}{\longrightarrow} \quad & \frac{1}{24}\int \mathrm{d}X_{j+1} \int \mathrm{d}V_{j+1} \sum_{\ell=1}^{j} \sum_{\alpha:|\alpha|=3} D^{\alpha}_{v_\ell} u_j\left(X_j, V_j\right) \cdot \\
&\cdot D^{\alpha}_{x_\ell} \varphi(x_\ell - x_{j+1}) f^{\otimes j+1}\left(X_{j+1}, V_{j+1}, t\right),
\end{aligned} \tag{35}$$

where $f(x, v, t)$ solves the Vlasov equation with initial datum f_0. Here we are using the classical mean-field theory (see [8], [9], [10], [11]).

Now we analyze the term $< u_j, T_N^{(0)} W_{N,j}^{(2)} >$:

$$\begin{aligned}
< u_j, T_N^{(0)} W_{N,j}^{(2)} > \quad &= -\frac{1}{N} \int \mathrm{d}X_N \int \mathrm{d}V_N \sum_{\ell \neq m}^{N} \nabla_{v_\ell} u_j \,(X_j, V_j) \,\boldsymbol{.} \\
&\quad \boldsymbol{.}\, \nabla_{x_\ell} \varphi(x_\ell - x_m) W_N^{(2)} \,(X_N, V_N, t) = \\
&= -\frac{1}{N} \int \mathrm{d}X_j \int \mathrm{d}V_j \sum_{\ell \neq m}^{j} \nabla_{v_\ell} u_j \,(X_j, V_j) \,\boldsymbol{.} \\
&\quad \boldsymbol{.}\, \nabla_{x_\ell} \varphi(x_\ell - x_m) W_{N,j}^{(2)} \,(X_j, V_j, t) + \\
&\quad - \frac{N-j}{N} \int \mathrm{d}X_{j+1} \int \mathrm{d}V_{j+1} \sum_{\ell = 1}^{j} \nabla_{v_\ell} u_j \,(X_j, V_j) \,\boldsymbol{.} \\
&\quad \boldsymbol{.}\, \nabla_{x_\ell} \varphi(x_\ell - x_{j+1}) W_{N,j+1}^{(2)} \,(X_{j+1}, V_{j+1}, t) \,.
\end{aligned} \tag{36}$$

Supposing that the first term in the r.h.s. of (36) is $O\left(\frac{j^2}{N}\right)$ and that

$$W_{N,j+1}^{(2)} \rightharpoonup f_{j+1}^{(2)}, \qquad \text{when } N \to \infty, \tag{37}$$

for some function $f_{j+1}^{(2)}$, we would have:

$$\begin{aligned}
< u_j, T_N^{(0)} W_{N,j}^{(2)} > \quad \stackrel{N \to \infty}{\longrightarrow} \quad & - \int \mathrm{d}X_{j+1} \int \mathrm{d}V_{j+1} \sum_{\ell = 1}^{j} \nabla_{v_\ell} u_j \,(X_j, V_j) \,\boldsymbol{.} \\
& \boldsymbol{.}\, \nabla_{x_\ell} \varphi(x_\ell - x_{j+1}) f_{j+1}^{(2)} \,(X_{j+1}, V_{j+1}, t) \,.
\end{aligned} \tag{38}$$

Let us remember now that in equation (33) we have also the terms:

$$\partial_t < u_j, W_{N,j}^{(2)} > \qquad \text{and} \qquad < u_j, V_j \,\boldsymbol{.}\, \nabla_{X_j} W_{N,j}^{(2)} > . \tag{39}$$

With respect to them we are able to affirm that they converge respectively to:

$$\partial_t < u_j, f_j^{(2)} > \qquad \text{and} \qquad < u_j, V_j \,\boldsymbol{.}\, \nabla_{X_j} f_j^{(2)} >, \tag{40}$$

under the assumption (37).
In conclusion we obtained formally the following infinite hierarchy:

$$\begin{aligned}
&\partial_t < u_j, f_j^{(2)} > + < u_j, V_j \cdot \nabla_{X_j} f_j^{(2)} >= \\
= \quad & -\frac{1}{24}\int \mathrm{d}X_j \int \mathrm{d}V_j \; u_j\left(X_j, V_j\right) \cdot \\
&\cdot \int \mathrm{d}x_{j+1} \int \mathrm{d}v_{j+1} \sum_{\ell=1}^{j} \sum_{\alpha:|\alpha|=3} D^\alpha_{x_\ell} \varphi(x_\ell - x_{j+1}) D^\alpha_{v_\ell} f^{\otimes j+1}\left(X_{j+1}, V_{j+1}, t\right) + \\
&+ \int \mathrm{d}X_j \int \mathrm{d}V_j \; u_j\left(X_j, V_j\right) \cdot \\
&\cdot \int \mathrm{d}x_{j+1} \int \mathrm{d}v_{j+1} \sum_{\ell=1}^{j} \nabla_{x_\ell} \varphi(x_\ell - x_{j+1}) \nabla_{v_\ell} f^{(2)}_{j+1}\left(X_{j+1}, V_{j+1}, t\right).
\end{aligned} \tag{41}$$

Now we want to check that (41) is exactly what we expect from the well known result about quantum mean-field limit (see [1]): the hierarchy corresponding to the the second term of the expansion in powers of ϵ of the solution of the infinite hierarchy associated with the Hartree equation.
First of all we write the Wigner-Liouville equation associated with the Hartree equation (2), namely:

$$\left(\partial_t + v \cdot \nabla_x\right) f = Tf, \tag{42}$$

where $f(x, v)$ is such that:

$$|\psi(x)|^2 = \rho(x) = \int \mathrm{d}v \; f(x, v) \tag{43}$$

and the operator T acts as follows:

$$\begin{aligned}
\left(Tf\right)(x, v) \quad &= i \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathrm{d}\lambda \int \mathrm{d}k \widehat{\varphi * \rho}(k) e^{ikx} \left(k \cdot \nabla_v\right) f(x, v + \epsilon\lambda k) = \\
&= i \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathrm{d}\lambda \int \mathrm{d}k \hat{\varphi}(k) \hat{\rho}(k) e^{ikx} \left(k \cdot \nabla_v\right) f(x, v + \epsilon\lambda k),
\end{aligned} \tag{44}$$

where, as previously, we denoted with $\hat{g}$ the Fourier transform of a function g. We define:

$$f_j\left(X_j, V_j\right) = \left(f^{\otimes j}\right)\left(X_j, V_j\right), \tag{45}$$

where f is the solution of equation (42) with initial data given by f_0.
Through a standard computation we obtain the following equations (one for each value of j):

$$\left(\partial_t + V_j \cdot \nabla_{X_j}\right) f_j = \sum_{\ell=1}^{j} T_\ell f_j, \quad \text{with } j = 1, 2, \ldots \tag{46}$$

where we denoted with the symbol T_ℓ the operator T acting on ℓ-variable, namely:

$$\begin{aligned}
\left(T_\ell f_j\right)(X_j, V_j) \quad &= i \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathrm{d}\lambda \int \mathrm{d}k \widehat{\varphi * \rho}(k) e^{ikx_\ell} \left(k \cdot \nabla_{v_\ell}\right) f_j \left(X_j, V_{\ell-1}, v_\ell + \epsilon\lambda k, V_{j-\ell}\right) = \\
&= i \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathrm{d}\lambda \int \mathrm{d}k \hat{\varphi}(k) \hat{\rho}(k) e^{ikx_\ell} \left(k \cdot \nabla_{v_\ell}\right) f\left(x_\ell, v_\ell + \epsilon\lambda k\right) \prod_{r \neq \ell}^{j} f(x_r, v_r) = \\
&= i \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathrm{d}\lambda \int \mathrm{d}k \hat{\varphi}(k) \hat{\rho}(k) e^{ikx_\ell} \left(k \cdot \nabla_{v_\ell}\right) f_j \left(X_j, V_{\ell-1}, v_\ell + \epsilon\lambda k, V_{j-\ell}\right).
\end{aligned} \tag{47}$$

Clearly for each of the equations of (46) we have the following factorized initial data:

$$f_j\left(X_j, V_j, 0\right) = f^0\left(X_j, V_j\right) = f_0^{\otimes j}\left(X_j, V_j\right). \tag{48}$$

Now, using Taylor expansion as in the case of N particles interacting through a mean-field potential, we obtain the following expansion in power series of ϵ of the operator T_ℓ:

$$T_\ell = T_\ell^{(0)} + \epsilon T_\ell^{(1)} + \epsilon^2 T_\ell^{(2)} + o(\epsilon^3), \tag{49}$$

where

$$\begin{aligned}
\left(T_\ell^{(0)} f_j\right)(X_j, V_j) \quad &= i \int \mathrm{d}k \hat{\varphi}(k) \hat{\rho}(k) e^{ikx_\ell} \left(k \cdot \nabla_{v_\ell}\right) f_j\left(X_j, V_j\right) = \\
&= \int \mathrm{d}k \left(ik\hat{\varphi}(k)\right) \hat{\rho}(k) e^{ikx_\ell} \nabla_{v_\ell} f_j\left(X_j, V_j\right) = \\
&= \left(\nabla_{x_\ell}\varphi * \rho\right)(x_\ell) \nabla_{v_\ell} f_j\left(X_j, V_j\right).
\end{aligned} \tag{50}$$

Let us observe that the last line of (50) is equal to:

$$\int \mathrm{d}x_{j+1} \int \mathrm{d}v_{j+1} \nabla_{x_\ell}\varphi\left(x_\ell - x_{j+1}\right) e^{ikx_\ell - x_{j+1}} \nabla_{v_\ell} f_{j+1}\left(X_{j+1}, V_{j+1}\right). \tag{51}$$

Therefore $T_\ell^{(0)}$ acts on function of $j+1$ particles. So that from now on we will denote the zero order term of the expansion of the operator T_ℓ with $C^{(0)}_{\ell,j+1}$.
For the first order term of the expansion we have that

$$\left(T_\ell^{(1)} f_j\right)(X_j, V_j) \equiv 0, \tag{52}$$

and it's easy to conclude that we have also:

$$T_\ell^{(2k+1)} \equiv 0 \qquad \text{with} \qquad k = 1, 2, \dots \tag{53}$$

On the contrary the second term of the expansion of T_ℓ in power series of ϵ is not zero and we have:

$$\begin{aligned} \left(T_\ell^{(2)} f_j\right)(X_j, V_j) \quad &= \frac{i}{24}\int \mathrm{d}k \hat{\varphi}(k)\hat{\rho}(k) e^{ikx_\ell} \left(k \cdot \nabla_{v_\ell}\right)^3 f_j\left(X_j, V_j\right) = \\ &= -\frac{1}{24}\sum_{\ell=1}^{j}\sum_{\alpha:|\alpha|=3}\int \mathrm{d}x_{j+1}\int \mathrm{d}v_{j+1} D^\alpha_{x_\ell}\varphi\left(x_\ell - x_{j+1}\right) D^\alpha_{v_\ell} f_{j+1}\left(X_{j+1}.V_{j+1}\right). \end{aligned} \tag{54}$$

We can see that also $T_\ell^{(2)}$, as $T_\ell^{(0)}$, acts on function of $j+1$ particles and then from now on we will denote the second order term of the expansion of T_ℓ with $C^{(2)}_{\ell,j+1}$.
Therefore we have that the equations in (46) form a hierarchy of equations which we will call Hartree hierarchy.
Supposing that we have the following expansion for the solution of the Hartree hierarchy:

$$f_j = f_j^{(0)} + \epsilon f_j^{(1)} + \epsilon^2 f_j^{(2)} + o(\epsilon^3), \tag{55}$$

we can write the equations at each order in ϵ. At order zero we obtain:

$$\left(\partial_t + V_j \cdot \nabla_{X_j}\right) f_j^{(0)} = \sum_{\ell=1}^{j} C^{(0)}_{\ell,j+1} f^{(0)}_{j+1}, \tag{56}$$

with

$$f_j^{(0)}\left(X_j, V_j, 0\right) = f_0^{\otimes j}. \tag{57}$$

This is nothing else than the hierarchy associated to the Vlasov equation as we expected from classical mean-field theory. At second order we obtain:

$$\left(\partial_t + V_j \cdot \nabla_{X_j}\right) f_j^{(2)}\left(X_j, V_j\right) = \sum_{\ell=1}^{j}\left(C^{(0)}_{\ell,j+1} f^{(2)}_{j+1} + C^{(2)}_{\ell,j+1} f^{(0)}_{j+1}\right), \tag{58}$$

with zero initial condition. Let us remember how the operators $C^{(0)}_{\ell,j+1}$ and $C^{(2)}_{\ell,j+1}$ act respectively on $f^{(2)}_{j+1}$ and $f^{(0)}_{j+1}$:

$$\left(C^{(0)}_{\ell,j+1} f^{(2)}_{j+1}\right)(X_j, V_j) = \int \mathrm{d}x_{j+1} \int \mathrm{d}v_{j+1} \nabla_{x_\ell} \varphi\left(x_\ell - x_{j+1}\right) e^{ikx_\ell - x_{j+1}} \nabla_{v_\ell} f^{(2)}_{j+1}\left(X_{j+1}, V_{j+1}\right), \tag{59}$$

and

$$\left(C^{(0)}_{\ell,j+1} f^{(2)}_{j+1}\right)(X_j, V_j) == -\frac{1}{24} \sum_{\ell=1}^{j} \sum_{\alpha:|\alpha|=3} \int \mathrm{d}x_{j+1} \int \mathrm{d}v_{j+1} D^{\alpha}_{x_\ell} \varphi\left(x_\ell - x_{j+1}\right) D^{\alpha}_{v_\ell} f^{(0)}_{j+1}\left(X_{j+1}.V_{j+1}\right). \tag{60}$$

Observing (59) and (60), we can conclude that we obtained at second order in ϵ the equations we expected from the formal limit that we discussed previously (cfr. (41)). The hope is to prove rigorously the previous convergence we established at a formal level only.

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