Geometric Analysis and PDEs

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Outline

1. Lecture I: Background from Geometry and Elliptic Theory

2. Lecture II: The Fully Nonlinear Yamabe Problem

3. Lecture III: The Functional Determinant

1. The Hessian

Let (M, g) be a Riemannian manifold, and let ∇ denote the Riemannian connection.

Definition 1. *The Hessian of $f : M \to \mathbf{R}$ is defined by*

$$\nabla^2 f(X, Y) = \nabla_X df(Y).$$

- Easy to see the Hessian is symmetric.

- In a local coordinate system $\{x^i\}$, define the *Christoffel symbols* by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

- In local coordinates,

$$\begin{aligned} (\nabla^2 f)_{ij} &= \nabla_i \nabla_j f \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k}. \end{aligned}$$

2. The Laplacian and Gradient.

Definition 2. *The* Laplacian *is the trace of the Hessian: Let* $\{e_1, \ldots, e_n\}$ *be an orthonormal basis of the tangent space at a point; then*

$$\Delta f = \sum_i \nabla^2 f(e_i, e_i).$$

- In local coordinates $\{x^i\}$,

$$\Delta f = g^{ij}\Big(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\Big),$$

where $g^{ij} = (g^{-1})_{ij}$.

- Another useful formula is

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\Big(g^{ij}\sqrt{g}\frac{\partial f}{\partial x^j}\Big),$$

where $g = det(g_{ij})$.

• The *gradient vector field* of f, denoted ∇f, is the vector field dual to the 1-form df; i.e., for each vector field X,

$$g(\nabla f, X) = df(X).$$

In local coordinates $\{x^i\}$,

$$\nabla_j f = \sum_i g^{ij} \frac{\partial f}{\partial x^i}.$$

• Green's formula:

$$\int \varphi \Delta \psi \; dV = \int -\langle \nabla \varphi, \nabla \psi \rangle \; dV,$$

where $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$ and

$$dV = \sqrt{g} \; dx^1 \wedge \dots dx^n.$$

<u>Proof</u> Just use local formulas for Δ and dV.

3. Let R denote the Riemannian curvature tensor of (M, g).

• For vector fields X, Y, Z,

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\cdot,\cdot]$ is the Lie bracket.

• In local coordinates $\{x^i\}$, the curvature tensor is given by

$$R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})\frac{\partial}{\partial x^j} = \sum_i R^i_{jkl}\frac{\partial}{\partial x^i}.$$

Lemma 1. *In local coordinates,*

$$\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = \sum_m R^m_{kij}\nabla_m f.$$

• So third derivatives do not commute unless $R = 0$, i.e., the manifold is *flat*.

6. Ricci and scalar curvatures

Definition 3. *The* Ricci curvature tensor *is the bilinear form* $Ric : T_pM \times T_pM \to \mathbf{R}$ *defined by*

$$Ric(X,Y) = \sum_i \langle R(X,e_i)Y, e_i \rangle,$$

where $\{e_1, \ldots, e_n\}$ *is an orthonormal basis of* T_pM*.*

• In local coordinates, the components of Ricci are given by

$$R_{ij} = \sum_m R^m_{ijm}.$$

• For spaces of constant curvature, the Ricci tensor is just a constant multiple of the metric:

$$\begin{aligned} \mathbf{S}^n: &\quad Ric = (n-1)g, \\ \mathbf{R}^n: &\quad Ric = 0, \\ \mathbf{H}^n: &\quad Ric = -(n-1)g. \end{aligned}$$

• The Ricci tensor is symmetric: $Ric(X,Y) = Ric(Y,X)$. Therefore, at each point $p \in M$ we can diagonalize Ric with respect to an orthonormal basis of T_pM:

$$Ric = \begin{pmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_n \end{pmatrix}$$

where $(\rho_1, \ldots, \rho_n)$ are the eigenvalues of Ric.

• To say that (M,g) has positive (negative) Ricci curvature means that all the eigenvalues of Ric are positive (negative).

• In two dimensions, the Ricci curvature is determined by the Gauss curvature K:

$$Ric = Kg.$$

Definition 4. *The* scalar curvature *is the trace of the Ricci curvature:*

$$R = \sum_i Ric(e_i, e_i),$$

where $\{e_1, \ldots, e_n\}$ *is an orthonormal basis.*

• If $\{\rho_1, \ldots, \rho_n\}$ are the eigenvalues of the Ricci curvature at a point $p \in M$, then the scalar curvature is given by

$$R = \rho_1 + \cdots + \rho_n.$$

• For the spaces of constant curvature, the scalar curvature is a constant function:

$$\begin{aligned} \mathbf{S}^n: &\quad R = n(n-1), \\ \mathbf{R}^n: &\quad R = 0, \\ \mathbf{H}^n: &\quad R = -n(n-1). \end{aligned}$$

• In two dimensions the scalar curvature is twice the Gauss curvature:

$$R = 2K.$$

Some background from Elliptic Theory

- Sobolev Spaces

These are important for discussing the PDE topics covered today.

Let (M, g) be a compact Riemannian manifold. For $1 \leq k < \infty$ and $1 \leq p \leq \infty$, introduce the norms

$$\|u\|_{k,p}^p = \sum_{0 \leq j \leq k} \int |\nabla^j u|^p \, dV,$$

where $\nabla^j u$ denotes the iterated j^{th}-covariant derivative.

Example For $k = 1$, $p = 2$,

$$\|u\|_{1,2}^2 = \int u^2 \, dV + \int |du|^2 \, dV.$$

The Sobolev space $W^{k,p}(M)$ is the completion of $C^\infty(M)$ in the norm $\| \cdot \|_{k,p}$.

Theorem 1. *(Sobolev Embedding Theorems)*

(i) *If*

$$\frac{1}{r} = \frac{1}{m} - \frac{k}{n},$$

then $W^{k,m}(M)$ *is continuously embedded in* $L^r(M)$*:*

$$\|u\|_r \leq C\|u\|_{k,m}.$$

(ii) *Suppose* $0 < \alpha < 1$ *and*

$$\frac{1}{m} \leq \frac{k - \alpha}{n}.$$

Then $W^{k,m}$ *is continuously embedded in* C^α*.*

(iii) *(Rellich-Kondrakov) If*

$$\frac{1}{r} > \frac{1}{m} - \frac{k}{n},$$

then the embedding $W^{k,m} \hookrightarrow L^r$ *is compact: i.e., a sequence which is bounded in* $W^{k,m}$ *has a subsequence which converges in* L^r*.*

Elliptic Theory

1. Linear Operators

Consider the linear differential operator L:

$$Lu = a^{ij}(x)\partial_i\partial_j u + b^k(x)\partial_k u + c(x)u,$$

where the coefficients a^{ij}, b^k, c are defined in a domain $\Omega \subset \mathbf{R}^n$.

Definition 5. *The operator L is* elliptic *in Ω if $\{a^{ij}(x)\}$ is positive definite at each point $x \in \Omega$. If there is a constant $\lambda > 0$ such that*

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $\xi \in \mathbf{R}^n$ and $x \in \Omega$, then L is strictly elliptic *in Ω. If, in addition, there is another constant $\Lambda > 0$ such that*

$$\Lambda|\xi|^2 \geq a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

then we say that L is uniformly elliptic *in Ω.*

• We can formulate a similar definition for operators defined on a Riemannian manifold; e.g., by introducing local coordinates.

• An operator is in *divergence form* if it can be written

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + b^k(x)\partial_k u + c(x)u.$$

The definition of ellipticity (strict ellipticity, etc.) is the same.

• Of course, the laplacian $L = \Delta$ is an example of a linear, uniformly elliptic operator in divergence form.

2. Weak Solutions

We say that $u \in W^{1,2}(M)$ is a *weak* solution of the equation

$$\Delta u = f(x) \tag{1}$$

if for each $\varphi \in W^{1,2}$,

$$\int -\langle \nabla u, \nabla \varphi \rangle \; dV = \int f\varphi \; dV. \tag{2}$$

• Of course, a smooth solution of (1) satisfies (2) by virtue of Green's Theorem.

• Weak solutions of elliptic equations like (1) in fact satisfy much better estimates, as we shall see.

<u>Elliptic Regularity</u>

Theorem 2. *Suppose $u \in W^{1,2}$ is a weak solution of*

$$\Delta u = f$$

on M.

(i) *If $f \in L^m$, then*

$$\|u\|_{2,m} \leq C\Big(\|f\|_m + \|u\|_m\Big). \tag{3}$$

(ii) *(Schauder estimates) If $f \in C^{\ell,\alpha}$ then*

$$\|u\|_{C^{\ell+2,\alpha}} \leq C\Big(\|f\|_{C^{\ell,\alpha}} + \|u\|_{C^{\ell,\alpha}}\Big). \tag{4}$$

Question: How are such estimates used?

1. To prove the regularity of weak solutions.

   Weak solutions are often easier to find, for example, by variational methods.

2. To prove *a priori* estimates of solutions, that is, estimates which are necessarily satisfied by any solution of a given equation.

   Often *a priori* estimates can be combined with a topological argument to establish existence.

An Example

To illustrate some of these results we consider an equation that will be an important model for much of the subsequent material.

Theorem 3. *Suppose $u \geq 0$ is a (weak) solution of*

$$\Delta u + c(x)u = K(x)u^p, \qquad (5)$$

where c, K are smooth functions, and

$$1 \leq p < \frac{n+2}{n-2}.$$

If

$$\int u^{\frac{2n}{(n-2)}} \, dV \leq B, \qquad (6)$$

then u satisfies

$$\sup_M u \leq C(p, B).$$

- In fact, we can estimate u with respect to any Hölder norm, all in terms of p and B.

- The assumption $p < \frac{n+2}{n-2}$ is <u>crucial</u>.

Proof. Using the preceding elliptic regularity theorem, we know that u satisfies

$$\begin{aligned} \|u\|_{2,m} &\le C\Big(\|\Delta u\|_m + \|u\|_m\Big) \\ &\le C\Big(\|u^p\|_m + \|u\|_m\Big) \\ &\le C\Big(\|u\|_{mp}^p + \|u\|_m\Big). \end{aligned} \tag{7}$$

Denote

$$m_0 = \frac{2n}{n-2},$$

and choose m so that

$$mp = m_0.$$

It follows from (7) that

$$\|u\|_{2,m} \le C(p, B).$$

We now use the Sobolev embedding theorem, which says

$$\|u\|_r \le C\|u\|_{2,m}$$

where

$$\frac{1}{r} = \frac{1}{m} - \frac{2}{n} = \frac{n-2m}{mn},$$

or,

$$r = \frac{mn}{n-2m} = \frac{(\frac{m_0}{p})n}{n-2(\frac{m_0}{p})}.$$

So, we've passed from one Lebesgue-space estimate to another. Have things improved? The answer is yes, as long as

$$\frac{(\frac{m_0}{p})n}{n-2(\frac{m_0}{p})} > m_0.$$

Solving this inequality, we see that it will hold provided p satisfies

$$p < \frac{n+2}{n-2}.$$

In this case, we iterate this process an arbitrary number of times, and conclude that

$$\|u\|_{2,m} \leq C(m,p,B) \quad \forall m >> 1.$$

Once m is large enough, though, we can once more appeal to the Sobolev embedding theorem, part (ii), and conclude that u is Hölder continuous–in particular, u is bounded as claimed.

$\square$

• For higher order regularity we turn to the Schauder estimates, since we actually proved that u is Hölder continuous. Iterating the Schauder estimates, we can prove the Hölder continuity of derivatives of all orders.

• As we mentioned above, and will soon see by explicit example, the preceding result is false if $p = (n+2)/(n-2)$. However, it can be "localized": that is, if

$$\int_{B(x_0,r)} u^{\frac{2n}{(n-2)}}\, dV \leq \epsilon_0$$

for some $\epsilon_0 > 0$ small enough, then

$$\sup_{B(x_0,r/2)} u \leq C(r).$$

A Corollary of this result is that weak solutions of (5) are regular, for all $1 \leq p \leq (n+2)/(n-2)$.

Background from Conformal Geometry

Definition 6. *Let* (M, g) *be a Riemannian manifold. A metric* h *is* pointwise conformal *to* g *(or just* conformal*) if there is a function* f *such that*

$$h = e^f g.$$

• The function e^f is referred to as the *conformal factor*. We used the exponential function to emphasize the fact that we need to multiply by a positive function (since h must be positive definite). However, in some cases it will be more convenient to write the conformal factor differently.

• We can introduce an equivalence relation on the set of metrics: $h \sim g$ iff h is pointwise conformal to g.

The equivalence class of a metric g is called its *conformal class*, and will be denoted by $[g]$. Note that

$$[g] = \{e^f g \mid f \in C^\infty(M)\}.$$

Definition 7. *Let (M, g) and (N, h) be two Riemannian manifolds. A diffeomorphism $\varphi : M \to N$ is called* conformal *if*

$$\varphi^* h = e^f g.$$

We say that (M, g) and (N, h) are *conformally equivalent*.

• Note h and g are pointwise conformal if and only if the identity map is conformal.

Example 1 Let $\delta_\lambda(x) = \lambda^{-1}x$ be the dilation map on $\mathbf{R}^n$, where $\lambda > 0$. Then δ_λ is easily seen to be conformal; in fact,

$$\delta_\lambda^* ds^2 = \lambda^{-2} ds^2,$$

where ds^2 is the Euclidean metric.

Example 2 Let $P = (0, \ldots, 0, 1)$ be the north pole of $\mathbf{S}^n \subset \mathbf{R}^{n+1}$. Let $\sigma : \mathbf{S}^n \setminus \{P\} \to \mathbf{R}^{n+1}$ denote stereographic projection, defined by

$$\sigma(\zeta^1, \ldots, \zeta^n, \xi) = \Big(\frac{\zeta^1}{1-\xi}, \ldots, \frac{\zeta^n}{1-\xi}\Big).$$

Then $\sigma : (\mathbf{S}^n \setminus \{P\}, g_0) \to (\mathbf{R}^{n+1}, ds^2)$ is conformal, where g_0 is the standard metric on $\mathbf{S}^n$.

Since the composition of conformal maps is again conformal, we can use σ to construct conformal maps of $\mathbf{S}^n$ to itself: for $\lambda > 0$, let

$$\varphi_\lambda = \sigma^{-1} \circ \delta_\lambda \circ \sigma : \mathbf{S}^n \to \mathbf{S}^n.$$

Then

$$\varphi_\lambda^* g_0 = \Psi_\lambda^2 g_0,$$

where

$$\Psi_\lambda(\zeta, \xi) = \frac{2\lambda}{(1+\xi) + \lambda^2(1-\xi)}.$$

Note

$$(\zeta,\xi) = (\mathbf{0},\mathbf{1}) \Rightarrow \mathbf{\Psi}_\lambda \to \infty \quad \text{as } \lambda \to \infty,$$
$$(\zeta,\xi) \neq (\mathbf{0},\mathbf{1}) \Rightarrow \mathbf{\Psi}_\lambda \to \mathbf{0} \quad \text{as } \lambda \to \infty.$$

• The set of conformal maps of a given Riemannian manifold is a Lie group; the construction above shows that the conformal group of the sphere is *non-compact*. This fact distinguishes the sphere:

Theorem 4. *(Lelong-Ferrand) A compact Riemmanian manifold with non-compact conformal group is conformally equivalent to the sphere with its standard metric.*

This fact is the source of many of the analytic difficulties we will encounter in the PDEs we are about to describe.

Curvature and conformal changes of metric

Let $h = e^{-2u}g$ be conformal metrics, and let $Ric(h), R(h)$ denote the Ricci and scalar curvatures of h, and $Ric(g), R(g)$ denote the Ricci and scalar curvatures of g. Then

$$\begin{aligned} Ric(h) =& Ric(g) + (n-2)\nabla^2 u + \Delta u g \\ & + (n-2) du \otimes du - (n-2)|\nabla u|^2 g, \end{aligned}$$

$$\begin{aligned} R(h) =& e^{2u}\Big\{ R(g) + 2(n-1)\Delta u \\ & - (n-1)(n-2)|\nabla u|^2 \Big\}, \end{aligned}$$

where $\nabla^2 u$ and Δu denote the Hessian and laplacian of u with respect to g.

The Uniformization Theorem

Let (M, g) be a closed (no boundary), compact, two-dimensional Riemannian manifold. Let K denotes its Gauss curvature.

Theorem 5. *(The Uniformization Theorem) There is a conformal metric* $h = e^{-2u}g$ *with constant Gauss curvature.*

• Let $K_h = const.$ denote the Gauss curvature of the metric h. The sign of K_h is determined by the Gauss-Bonnet formula:

$$\begin{aligned} 2\pi\chi(M) &= \int K_h \; dA_h \\ &= K_h \cdot Area(h). \end{aligned}$$

• Note the geometric/topological significance of this result: Since h has constant curvature, the universal cover $\tilde{M}$ is isometric to either $\mathbf{S}^2, \mathbf{R}^2$, or $\mathbf{H}^2$, each case being determined by the sign of the Euler characteristic.

The Yamabe Problem

Let (M, g) be a closed, compact, Riemannian manifold of dimension $n \geq 3$.

In higher dimensions there are obstructions to being even *locally* conformal to a constant curvature metric. This leads to

Question: How do we generalize the Uniformization Theorem to higher dimensions?

A major theme of these lectures is the various ways one might answer this question (there are yet others).

The Yamabe Problem Find a conformal metric $h = e^{-2u}g$ whose *scalar curvature* is constant.

• By the formulas above, solving the Yamabe problem is equivalent to solving the semilinear PDE

$$2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R(g) = \mu e^{-2u}$$

for some constant μ.

• This formula can be simplified if we write $h = v^{4/(n-2)}g$, where $v > 0$. Then v should satisfy

$$-\frac{4(n-1)}{(n-2)}\Delta v + R(g)v = \lambda v^{\frac{n+2}{n-2}}.$$

• Notice the exponent! This equation is of the form

$$\Delta v + c(x)v = K(x)v^p,$$

where $p = (n+2)/(n-2)$. This is the delicate case we discussed above.

The case of the sphere

Recall the conformal maps of the sphere described above, $\varphi_\lambda : \mathbf{S}^n \to \mathbf{S}^n$. Then $h = \varphi_\lambda^* g_0 = \Psi_\lambda^2 g_0$ has the same scalar curvature as the standard metric. Therefore, writing

$$h = v_\lambda^{4/(n-2)} g_0,$$

where

$$v_\lambda = \Psi_\lambda^{\frac{(n-2)}{2}},$$

we have a family $\{v_\lambda\}$ of solutions to

$$-\frac{4(n-1)}{(n-2)} \Delta v_\lambda + n(n-1) v_\lambda = n(n-1) v_\lambda^{\frac{n+2}{n-2}}.$$

As we observed above, if P is the North pole, then

$$v_\lambda(P) \to \infty \quad \text{as } \lambda \to \infty,$$

whereas if $x \neq P$, then

$$v_\lambda(x) \to 0 \quad \text{as } \lambda \to \infty.$$

• Good news: There are many solutions of the Yamabe equation.

• Bad news: It will be impossible to prove *a priori* estimates.

• Of course, the non-compactness of the set of solutions arises precisely because of the influence of the conformal group. Thus, on manifolds other than the sphere, one would expect that the set of solutions is compact.

Put another way, ideally we would like to show that non-compactness implies the underlying manifold is $(\mathbf{S}^n, g_0)$.

A variational approach

There is an approach to solving the Yamabe problem by the methods of the calculus of variations.

Define the functional $\mathcal{Y} : W^{1,2} \to \mathbf{R}$ by

$$\mathcal{Y}(v) = \frac{\int \left(\frac{4(n-1)}{(n-2)} |\nabla v|^2 + R(g) v^2 \right) \, dV}{\left(\int v^{\frac{2n}{(n-2)}} \, dV \right)^{(n-2)/n}}.$$

• Using the formulas above, one can check that

$$\mathcal{Y}(v) = Vol(h)^{-(n-2)/n} \int R(h) \; dV(h),$$

where $h = v^{4/(n-2)} g$.

• The quantity on the right-hand side is called the *total scalar curvature* of h.

Lemma 2. *A function $v \in W^{1,2}$ is a critical point of $\mathcal{Y}$ iff v is a weak solution of the Yamabe equation.*

• By critical point, we mean that

$$\frac{d}{dt}\mathcal{Y}(v + t\varphi)\Big|_{t=0} = 0$$

for all $\varphi \in W^{1,2}$.

• Recall that weak solutions are regular.

• By the Sobolev embedding theorem, the number

$$Y(M, [g]) = \inf_{v \in W^{1,2}} \mathcal{Y}(v) > -\infty.$$

This number is called the *Yamabe invariant* of the conformal class of g.

Overview of the Lectures

1. A fully nonlinear Yamabe problem.

   The Yamabe problem is a natural extension of the uniformization theorem to higher dimensions. A generalization of the Yamabe problem leads to a "fully nonlinear uniformization problem." This will be the subject of Lecture 2.

2. The functional determinant.

   One approach to the Uniformization Theorem for surfaces is to consider a variational problem from spectral theory. A generalization to four dimensions leads to a "higher order uniformization problem." This will be described in the final lecture.