UNIFORMISATION OF FOLIATIONS BY CURVES

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Summary. These lecture notes provide a full discussion of certain analytic aspects of the uniformisation theory of foliations by curves on compact Kähler manifolds, with emphasis on convexity properties and their consequences on positivity properties of the corresponding canonical bundles.

1 Foliations by curves and their uniformisation

Let X be a complex manifold. A **foliation by curves** $\mathcal{F}$ on X is defined by a holomorphic line bundle $T_{\mathcal{F}}$ on X and a holomorphic linear morphism

$$\tau_{\mathcal{F}} : T_{\mathcal{F}} \to TX$$

which is injective outside an analytic subset $Sing(\mathcal{F}) \subset X$ of codimension at least 2, called the **singular set** of the foliation. Equivalently, we have an open covering $\{U_j\}$ of X and a collection of holomorphic vector fields $v_j \in \Theta(U_j)$, with zero set of codimension at least 2, such that

$$v_j = g_{jk} v_k \qquad \text{on} \quad U_j \cap U_k,$$

where $g_{jk} \in \mathcal{O}^*(U_j \cap U_k)$ is a multiplicative cocycle defining the dual bundle $T^*_{\mathcal{F}} = K_{\mathcal{F}}$, called the **canonical bundle** of $\mathcal{F}$.

These vector fields can be locally integrated, and by the relations above these local integral curves can be glued together (without respecting the time parametrization), giving rise to the **leaves** of the foliation $\mathcal{F}$.

By the classical Uniformisation Theorem, the universal covering of each leaf is either the unit disc $\mathbb{D}$ (hyperbolic leaf) or the affine line $\mathbb{C}$ (parabolic leaf) or the projective line $\mathbb{P}$ (rational leaf).

In these notes we shall assume that the ambient manifold X is a compact connected Kähler manifold, and we will be concerned with the following problem: how the universal covering $\widetilde{L_p}$ of the leaf L_p through the point p depends on p ? For instance, we may first of all ask about the structure of the subset of X formed by those points through which the leaf is hyperbolic, resp. parabolic, resp. rational: is the set of hyperbolic leaves open in X? Is the set of parabolic leaves analytic? But, even if all the leaves are, say, hyperbolic, there are further basic questions: the uniformising map of every leaf is almost unique (unique modulo automorphisms of the disc), and after some normalization (to get uniqueness) we may ask about the way in which the uniformising map of L_p depends on the point p. Equivalently, we may put on every leaf its Poincaré metric, and we may ask about the way in which this leafwise metric varies in the directions transverse to the foliation.

Our main result will be that these universal coverings of leaves can be glued together in a vaguely "holomorphically convex" way. That is, the *leafwise* universal covering of the foliated manifold $(X, \mathcal{F})$ can be defined and it has a sort of "holomorphically convex" structure [Br2] [Br3]. This was inspired by a seminal work of Il'yashenko [Il1] [Il2], who proved a similar result when X is a Stein manifold instead of a compact Kähler one. Related ideas can also be found in Suzuki's paper [Suz], still in the Stein case. Another source of inspiration was Shafarevich conjecture on the holomorphic convexity of universal coverings of projective (or compact Kähler) manifolds [Nap].

This main result will allow us to apply results by Nishino [Nis] and Yamaguchi [Ya1] [Ya2] [Ya3] [Kiz] concerning the transverse variation of the leafwise Poincaré metric and other analytic invariants. As a consequence of this, for instance, we shall obtain that if the foliation has at least one hyperbolic leaf, then: (1) there are no rational leaves; (2) parabolic leaves fill a subset of X which is *complete pluripolar*, i.e. locally given by the poles of a plurisubharmonic function. In other words, the set of hyperbolic leaves of $\mathcal{F}$ is either empty or potential-theoretically full in X.

These results are related also to positivity properties of the canonical bundle $K_{\mathcal{F}}$, along a tradition opened by Arakelov [Ara] [BPV] in the case of algebraic fibrations by curves and developed by Miyaoka [Miy] [ShB] and then McQuillan and Bogomolov [MQ1] [MQ2] [BMQ] [Br1] in the case of foliations on projective manifolds. From this point of view, our final result is the following ruledness criterion for foliations:

Theorem 1.1. [Br3] [Br5] *Let X be a compact connected Kähler manifold and let $\mathcal{F}$ be a foliation by curves on X. Suppose that the canonical bundle $K_{\mathcal{F}}$ is* not *pseudoeffective. Then through every point $p \in X$ there exists a rational curve tangent to $\mathcal{F}$.*

Recall that a line bundle on a compact connected manifold is *pseudoeffective* if it admits a (singular) hermitian metric with positive curvature in the sense of currents [Dem]. When X is projective the above theorem follows also from results of [BMQ] and [BDP], but with a totally different proof, untranslatable in our Kähler context.

Let us now describe in more detail the content of these notes.

In Section 2 we shall recall the results by Nishino and Yamaguchi on Stein fibrations that we shall use later, and also some of Il'yashenko's results. In Section 3 and 4 we construct the leafwise universal covering of $(X, \mathcal{F})$: we give an appropriate definition of leaf L_p of $\mathcal{F}$ through a point $p \in X \setminus Sing(\mathcal{F})$ (this requires some care, because some leaves are allowed to pass through some singular points), and we show that the universal coverings $\widetilde{L_p}$ can be glued together to get a complex manifold. In Section 5 we prove that the complex manifold so constructed enjoys some "holomorphic convexity" property. This is used in Section 6 and 8, together with Nishino and Yamaguchi results, to prove (among other things) Theorem 1.1 above. The parabolic case requires also an extension theorem for certain meromorphic maps into compact Kähler manifolds, which is proved in Section 7.

All this work has been developed in our previous papers [Br2] [Br3] [Br4] and [Br5] (with few imprecisions which will be corrected here). Further results and application can be found in [Br6] and [Br7].

2 Some results on Stein fibrations

2.1 Hyperbolic fibrations

In a series of papers, Nishino [Nis] and then Yamaguchi [Ya1] [Ya2] [Ya3] studied the following situation. It is given a Stein manifold U, of dimension $n+1$, equipped with a holomorphic submersion $P : U \to \mathbb{D}^n$ with connected fibers. Each fiber $P^{-1}(z)$ is thus a smooth connected curve, and as such it has several potential theoretic invariants (Green functions, Bergman Kernels, harmonic moduli...). One is interested in knowing how these invariants vary with z, and then in using this knowledge to obtain some information on the structure of U.

For our purposes, the last step in this program has been carried out by Kizuka [Kiz], in the following form.

Theorem 2.1. [Ya1] [Ya3] [Kiz] *If U is Stein, then the fiberwise Poincaré metric on $U \xrightarrow{P} \mathbb{D}^n$ has a plurisubharmonic variation.*

This means the following. On each fiber $P^{-1}(z)$, $z \in \mathbb{D}^n$, we put its Poincaré metric, i.e. the (unique) complete hermitian metric of curvature -1 if $P^{-1}(z)$ is uniformised by $\mathbb{D}$, or the identically zero "metric" if $P^{-1}(z)$ is uniformised by $\mathbb{C}$ (U being Stein, there are no other possibilities). If v is a holomorphic nonvanishing vector field, defined in some open subset $V \subset U$ and tangent to the fibers of P, then we can take the function on V

$$F = \log \|v\|_{Poin}$$

where, for every $q \in V$, $\|v(q)\|_{Poin}$ is the norm of $v(q)$ evaluated with the Poincaré metric on $P^{-1}(P(q))$. The statement above means that, whatever v is, the function F is *plurisubharmonic*, or identically $-\infty$ if all the fibers are parabolic. Note that if we replace v by $v' = g \cdot v$, with g a holomorphic nonvanishing function on V, then F is replaced by $F' = F + G$, where $G = \log |g|$ is pluriharmonic. A more intrinsic way to state this property is: the fiberwise Poincaré metric (if not identically zero) defines on the relative canonical bundle of $U \xrightarrow{P} \mathbb{D}^n$ a hermitian metric (possibly singular) whose curvature is a *positive current* [Dem]. Note also that the plurisubharmonicity of F along the fibers is just a restatement of the negativity of the curvature of the Poincaré metric. The important fact here is the plurisubharmonicity along the directions transverse to the fibers, whence the *variation* terminology.

Remark that the poles of F correspond exactly to parabolic fibers of U. We therefore obtain the following dichotomy: either all the fibers are parabolic ($F \equiv 0$), or the parabolic fibers correspond to a complete pluripolar subset of $\mathbb{D}^n$ ($F \not\equiv 0$).

The theorem above is a generalization of, and was motivated by, a classical result of Hartogs [Ran, II.5], asserting (in modern language) that for a domain U in $\mathbb{D}^n \times \mathbb{C}$ of the form (Hartogs tube)

$$U = \{ \ (z, w) \mid \quad |w| < e^{-f(z)} \},$$

where $f : \mathbb{D}^n \to [-\infty, +\infty)$ is an upper semicontinuous function, the Steinness of U implies that f is plurisubharmonic. Indeed, in this special case the Poincaré metric is easily computed, and one checks that the plurisubharmonicity of f is equivalent to the plurisubharmonic variation of the fiberwise Poincaré metric. By Oka, also the converse holds: if f is plurisubharmonic, then U is Stein. This special case suggests that some converse statement to Theorem 2.1 could be true.

We give the proof of Theorem 2.1 only in a particular case, which is anyway the only case that we shall actually use.

We start with a fibration $P : U \to \mathbb{D}^n$ as above, but without assuming U Stein. We consider an open subset $U_0 \subset U$ such that:

(i) for every $z \in \mathbb{D}^n$, the intersection $U_0 \cap P^{-1}(z)$ is a disc, relatively compact in the fiber $P^{-1}(z)$;
(ii) the boundary ∂U_0 is real analytic and transverse to the fibers of P;
(iii) the boundary ∂U_0 is pseudoconvex in U.

Then we restrict our attention to the fibration by discs $P_0 = P|_{U_0} : U_0 \to \mathbb{D}^n$. It is not difficult to see that U_0 is Stein, but this fact will not really be used below.

Proposition 2.2. [Ya1] [Ya3] *The fiberwise Poincaré metric on* $U_0 \stackrel{P_0}{\to} \mathbb{D}^n$ *has a plurisubharmonic variation.*

Proof. It is sufficient to consider the case $n = 1$. The statement is local on the base, and for every $z_0 \in \mathbb{D}$ we can embed a neighbourhood of $\overline{P_0^{-1}(z_0)}$ in U into $\mathbb{C}^2$ in such a way that P becomes the projection to the first coordinate (see, e.g., [Suz, §3]). Thus we may assume that $U_0 \subset \mathbb{D} \times \mathbb{C}$, $P_0(z, w) = z$, and $P^{-1}(z) = D_z$ is a disc in $\{z\} \times \mathbb{C} = \mathbb{C}$, with real analytic boundary, depending on z in a real analytic and pseudoconvex way.

Take a holomorphic section $\alpha : \mathbb{D} \to U_0$ and a holomorphic vertical vector field v along α, i.e. for every $z \in \mathbb{D}$, $v(z)$ is a vector in $T_{\alpha(z)}U_0$ tangent to the fiber over z (and nonvanishing). We need to prove that $\log \|v(z)\|_{Poin(D_z)}$ is a subharmonic function on $\mathbb{D}$. By another change of coordinates, we may assume that $\alpha(z) = (z, 0)$ and $v(z) = \frac{\partial}{\partial w}|_{(z,0)}$.

For every z, let

$$g(z, \cdot) : \overline{D_z} \to [0, +\infty]$$

be the Green function of D_z with pole at 0. That is, $g(z, \cdot)$ is harmonic on $D_z \setminus \{0\}$, zero on ∂D_z, and around $w = 0$ it has the development

$$g(z, w) = \log \frac{1}{|w|} + \lambda(z) + O(|w|).$$

The constant $\lambda(z)$ (Robin constant) is related to the Poincaré metric of D_z: more precisely, we have

$$\lambda(z) = -\log \| \frac{\partial}{\partial w}|_{(z,0)} \|_{Poin(D_z)}$$

(indeed, recall that the Green function gives the radial part of a uniformisation of D_z). Hence, we are reduced to show that $z \mapsto \lambda(z)$ is *super*harmonic.

Fix $z_0 \in \mathbb{D}$. By real analyticity of ∂U_0, the function g is (outside the poles) also real analytic, and thus extensible (in a real analytic way) beyond ∂U_0. This means that if z is sufficiently close to z_0, then $g(z, \cdot)$ is actually defined on $\overline{D_{z_0}}$, and harmonic on $D_{z_0} \setminus \{0\}$. Of course, $g(z, \cdot)$ does not need to vanish on ∂D_{z_0}. The difference $g(z, \cdot) - g(z_0, \cdot)$ is harmonic on D_{z_0} (the poles annihilate), equal to $\lambda(z) - \lambda(z_0)$ at 0, and equal to $g(z, \cdot)$ on ∂D_{z_0}. By Green formula:

$$\lambda(z) - \lambda(z_0) = -\frac{1}{2\pi} \int_{\partial D_{z_0}} g(z, w) \frac{\partial g}{\partial n}(z_0, w) ds$$

and consequently:

$$\frac{\partial^2 \lambda}{\partial z \partial \bar{z}}(z_0) = -\frac{1}{2\pi} \int_{\partial D_{z_0}} \frac{\partial^2 g}{\partial z \partial \bar{z}}(z_0, w) \frac{\partial g}{\partial n}(z_0, w) ds.$$

We now compute the z-laplacian of $g(\cdot, w_0)$ when w_0 is a point of the boundary ∂D_{z_0}.

The function $-g$ is, around (z_0, w_0), a defining function for U_0. By pseudoconvexity, the Levi form of g at (z_0, w_0) is therefore nonpositive on the complex tangent space $T^{\mathbb{C}}_{(z_0,w_0)}(\partial U_0)$, i.e. on the Kernel of ∂g at (z_0, w_0) [Ran, II.2]. By developing, and using also the fact that g is w-harmonic, we obtain

$$\frac{\partial^2 g}{\partial z \partial \bar{z}}(z_0, w_0) \leq 2Re\Big\{ \frac{\frac{\partial^2 g}{\partial w \partial \bar{z}}(z_0, w_0) \cdot \frac{\partial g}{\partial z}(z_0, w_0)}{\frac{\partial g}{\partial w}(z_0, w_0)} \Big\}.$$

We put this inequality into the expression of $\frac{\partial^2 \lambda}{\partial z \partial \bar{z}}(z_0)$ derived above from Green formula, and then we apply Stokes theorem. We find

$$\frac{\partial^2 \lambda}{\partial z \partial \bar{z}}(z_0) \leq -\frac{2}{\pi} \int_{D_{z_0}} \big| \frac{\partial^2 g}{\partial w \partial \bar{z}}(z_0, w) \big|^2 i dw \wedge d\bar{w} \leq 0$$

from which we see that λ is superharmonic. □

A similar result can be proved, by the same proof, even when we drop the simply connectedness hypothesis on the fibers, for instance when the fibers of U_0 are annuli instead of discs; however, the result is that the Bergman fiberwise metric, and not the Poincaré one, has a plurisubharmonic variation. This is because on a multiply connected curve the Green function is more directly related to the Bergman metric [Ya3]. The case of the Poincaré metric is done in [Kiz], by a covering argument. The general case of Theorem 2.1 requires also to understand what happens when ∂U_0 is still pseudoconvex but no more transverse to the fibers, so that U_0 is no more a differentiably trivial family of curves. This is rather delicate, and it is done in [Ya1]. Then Theorem 2.1 is proved by an exhaustion argument.

2.2 Parabolic fibrations

Theorem 2.1, as stated, is rather empty when all the fibers are isomorphic to $\mathbb{C}$. However, in that case Nishino proved that if U is Stein then it is isomorphic to $\mathbb{D}^n \times \mathbb{C}$ [Nis, II]. A refinement of this was found in [Ya2].

As before, we consider a fibration $P : U \to \mathbb{D}^n$ and we do not assume that U is Stein. We suppose that there exists an embedding $j : \mathbb{D}^n \times \mathbb{D} \to U$ such that $P \circ j$ coincides with the projection from $\mathbb{D}^n \times \mathbb{D}$ to $\mathbb{D}^n$ (this can always be done, up to restricting the base). For every $\varepsilon \in [0, 1)$, we set

$$U_\varepsilon = U \setminus j(\mathbb{D}^n \times \overline{\mathbb{D}(\varepsilon)})$$

with $\overline{\mathbb{D}(\varepsilon)} = \{z \in \mathbb{C} | \quad |z| \leq \varepsilon\}$, and we denote by

$$P_\varepsilon : U_\varepsilon \to \mathbb{D}^n$$

the restriction of P. Thus, the fibers of P_ε are obtained from those of P by removing a closed disc (if $\varepsilon > 0$) or a point (if $\varepsilon = 0$).

Theorem 2.3. [Nis, II] [Ya2] *Suppose that:*

(i) for every $z \in \mathbb{D}^n$, the fiber $P^{-1}(z)$ is isomorphic to $\mathbb{C}$;
(ii) for every $\varepsilon > 0$ the fiberwise Poincaré metric on $U_\varepsilon \xrightarrow{P_\varepsilon} \mathbb{D}^n$ has a plurisubharmonic variation.

Then U is isomorphic to a product:

$$U \simeq \mathbb{D}^n \times \mathbb{C}.$$

Proof. For every $z \in \mathbb{D}^n$ we have a unique isomorphism

$$f(z,\cdot) : P^{-1}(z) \to \mathbb{C}$$

such that, using the coordinates given by j,

$$f(z,0) = 0 \qquad \text{and} \qquad f'(z,0) = 1.$$

We want to prove that f is holomorphic in z.

Set $R_\varepsilon(z) = f(z, P_\varepsilon^{-1}(z)) \subset \mathbb{C}$. By Koebe's Theorem, the distorsion of $f(z,\cdot)$ on compact subsets of $\mathbb{D}$ is uniformly bounded, and so $\mathbb{D}(\frac{1}{k}\varepsilon) \subset f(z, \mathbb{D}(\varepsilon)) \subset \mathbb{D}(k\varepsilon)$ for every $\varepsilon \in (0, \frac{1}{2})$ and for some constant k, independent on z. Therefore, for every ε and z,

$$\mathbb{C} \setminus \mathbb{D}(\frac{1}{k}\varepsilon) \subset R_\varepsilon(z) \subset \mathbb{C} \setminus \mathbb{D}(k\varepsilon).$$

In a similar way [Nis, II], Koebe's Theorem gives also the continuity of the above map f.

On the fibers of P_0, which are all isomorphic to $\mathbb{C}^*$, we put the unique complete hermitian metric of zero curvature and period (=length of closed simple geodesics) equal to $\sqrt{2}\pi$. On the fibers of P_ε, $\varepsilon > 0$, which are all hyperbolic, we put the Poincaré metric multiplied by $\log \varepsilon$, whose (constant) curvature is therefore equal to $-\frac{1}{(\log \varepsilon)^2}$. By a simple and explicit computation, the Poincaré metric on $\mathbb{C} \setminus \mathbb{D}(c\varepsilon)$ multiplied by $\log \varepsilon$ converges uniformly to the flat metric of period $\sqrt{2}\pi$ on $\mathbb{C}^*$, as $\varepsilon \to 0$. Using this and the above bounds on $R_\varepsilon(z)$, we obtain that our fiberwise metric on $U_\varepsilon \xrightarrow{P_\varepsilon} \mathbb{D}^n$ converges uniformly, as $\varepsilon \to 0$, to our fiberwise metric on $U_0 \xrightarrow{P_0} \mathbb{D}^n$ (see [Br4] for more explicit computations). Hence, from the plurisubharmonic variation of the former we deduce the plurisubharmonic variation of the latter.

Our flat metric on $P_0^{-1}(z)$ is the pull-back by $f(z,\cdot)$ of the metric $\frac{i dx \wedge d\bar{x}}{4|x|^2}$ on $R_0(z) = \mathbb{C}^*$. In the coordinates given by j, we have

$$f(z,w) = w \cdot e^{g(z,w)},$$

with g holomorphic in w and $g(z,0) = 0$ for every z, by the choice of the normalization. Hence, in these coordinates our metric takes the form

$$\left|1 + w\frac{\partial g}{\partial w}(z,w)\right|^2 \cdot \frac{idw \wedge d\bar{w}}{4|w|^2}.$$

Set $F = \log|1 + w\frac{\partial g}{\partial w}|^2$. We know, by the previous arguments, that F is plurisubharmonic. Moreover, $\frac{\partial^2 F}{\partial w \partial \bar{w}} \equiv 0$, by flatness of the metric. By semi-positivity of the Levi form we then obtain $\frac{\partial^2 F}{\partial w \partial \bar{z}_k} \equiv 0$ for every k. Hence the function $\frac{\partial F}{\partial w}$ is holomorphic, that is the function $(\frac{\partial g}{\partial w} + w\frac{\partial^2 g}{\partial w^2})(1 + w\frac{\partial g}{\partial w})^{-1}$ is holomorphic. Taking into account that $g(z,0) \equiv 0$, we obtain from this that g also is fully holomorphic. Thus f is fully holomorphic in the chart given by j, and hence everywhere. It follows that U is isomorphic to a product. □

Remark that if U is Stein then the hypothesis on the plurisubharmonic variation is automatically satisfied, by Theorem 2.1, and because if U is Stein then also U_ε are Stein, for every ε. That was the situation originally considered by Nishino and Yamaguchi.

A standard illustration of Theorem 2.3 is the following one. Take a continuous function $h : \mathbb{D} \to \mathbb{P}$, let $\Gamma \subset \mathbb{D} \times \mathbb{P}$ be its graph, and set $U = (\mathbb{D} \times \mathbb{P}) \setminus \Gamma$. Then U fibers over $\mathbb{D}$ and all the fibers are isomorphic to $\mathbb{C}$. Clearly U is isomorphic to a product $\mathbb{D} \times \mathbb{C}$ if and only if h is holomorphic, which in turn is equivalent, by a classical result (due, once a time, to Hartogs), to the Steinness of U.

2.3 Foliations on Stein manifolds

Even if we shall not need Il'yashenko's results [Il1] [Il2], let us briefly explain them, as a warm-up for some basic ideas.

Let X be a Stein manifold, of dimension n, and let $\mathcal{F}$ be a foliation by curves on X. In order to avoid some technicalities (to which we will address later), let us assume that $\mathcal{F}$ is *nonsingular*, i.e. $Sing(\mathcal{F}) = \emptyset$.

Take an embedded $(n-1)$-disc $T \subset X$ transverse to $\mathcal{F}$. For every $t \in T$, let L_t be the leaf of $\mathcal{F}$ through t, and let $\widetilde{L_t}$ be its universal covering with basepoint t. Remark that, because X is Stein, every $\widetilde{L_t}$ is isomorphic either to $\mathbb{D}$ or to $\mathbb{C}$. In [Il1] Il'yashenko proves that these universal coverings $\{\widetilde{L_t}\}_{t\in T}$ can be glued together to get a complex manifold of dimension n, a sort of "long flow box". More precisely, there exists a complex n-manifold U_T with the following properties:

(i) U_T admits a submersion $P_T : U_T \to T$ and a section $p_T : T \to U_T$ such that, for every $t \in T$, the pointed fiber $(P_T^{-1}(t), p_T(t))$ is identified (in a natural way) with $(\widetilde{L_t}, t)$;
(ii) U_T admits an immersion (i.e., local biholomorphism) $\Pi_T : U_T \to X$ which sends each fiber $(\widetilde{L_t}, t)$ to the corresponding leaf (L_t, t), as universal covering.

We shall not prove here these facts, because we shall prove later (Section 4) some closely related facts in the context of (singular) foliations on compact Kähler manifolds.

Theorem 2.4. [Il1] [Il2] *The manifold U_T is Stein.*

Proof. Following Suzuki [Suz], it is useful to factorize the immersion $U_T \to X$ through another manifold V_T, which is constructed in a similar way as U_T except that the universal coverings $\widetilde{L_t}$ are replaced by the holonomy coverings $\widehat{L_t}$.

Here is Suzuki's construction. Fix a foliated chart $\Omega \subset X$ around T, i.e. $\Omega \simeq \mathbb{D}^{n-1} \times \mathbb{D}$, $T \simeq \mathbb{D}^{n-1} \times \{0\}$, $\mathcal{F}|_\Omega =$ vertical foliation, with leaves $\{*\} \times \mathbb{D}$. Let $\mathcal{O}_\mathcal{F}(\Omega)$ be the set of holomorphic functions on Ω which are constant on the leaves of $\mathcal{F}|_\Omega$, i.e. which depend only on the first $(n-1)$ coordinates. Let $\overline{V}_T$ be the *existence domain* of $\mathcal{O}_\mathcal{F}(\Omega)$ over X: by definition, this is the maximal holomorphically separable Riemann domain

$$\overline{V}_T \to X$$

which contains Ω and such that every $f \in \mathcal{O}_\mathcal{F}(\Omega)$ extends to some $\tilde{f} \in \mathcal{O}(\overline{V}_T)$. The classical Cartan-Thullen-Oka theory [GuR] says that $\overline{V}_T$ is a Stein manifold.

The projection $\Omega \to T$ extends to a map

$$\overline{Q}_T : \overline{V}_T \to T$$

thanks to $\mathcal{O}_\mathcal{F}(\Omega) \hookrightarrow \mathcal{O}(\overline{V}_T)$. Consider a fiber $\overline{Q}_T^{-1}(t)$. It is not difficult to see that the connected component of $\overline{Q}_T^{-1}(t)$ which cuts Ω ($\subset \overline{V}_T$) is exactly the holonomy covering $\widehat{L_t}$ of L_t, with basepoint t. The reason is the following one. Firstly, if $\gamma : [0,1] \to L_t$ is a path contained in a leaf, with $\gamma(0) = t$, then any function $f \in \mathcal{O}_\mathcal{F}(\Omega)$ can be analytically prolonged along γ, by preserving the constancy on the leaves. Secondly, if γ_1 and γ_2 are two such paths with the same endpoint $s \in L_t$, then the germs at s obtained by the two continuations of f along γ_1 and γ_2 may be different. If the foliation has trivial holonomy along $\gamma_1 * \gamma_2^{-1}$, then the two germs are certainly equal; conversely, if the holonomy is not trivial, then we can find f such that the two final germs are different. This argument shows that $\widehat{L_t}$ is naturally contained into $\overline{Q}_T^{-1}(t)$. The fact that it is a connected component is just a "maximality" argument (note that $\overline{V}_T$ is foliated by the pull-back of $\mathcal{F}$, and fibers of $\overline{Q}_T$ are closed subvarieties invariant by this foliation).

We denote by $V_T \subset \overline{V}_T$ (open subset) the union of these holonomy coverings, and by Q_T the restriction of $\overline{Q}_T$ to V_T.

Let us return to U_T. We have a natural map (local biholomorphism)

$$F_T : U_T \to V_T$$

which acts as a covering between fibers (but not globally: see Examples 4.7 and 4.8 below). In particular, U_T is a Riemann domain over the Stein manifold $\overline{V}_T$.

Lemma 2.5. *U_T is holomorphically separable.*

Proof. Given $p, q \in U_T$, $p \neq q$, we want to construct $f \in \mathcal{O}(U_T)$ such that $f(p) \neq f(q)$. The only nontrivial case (V_T being holomorphically separable) is the case where $F_T(p) = F_T(q)$, in particular p and q belong to the same fiber $\widetilde{L_t}$.

We use the following procedure. Take a path γ in $\widetilde{L_t}$ from p to q. It projects by F_T to a closed path γ_0 in $\widehat{L_t}$. Suppose that $[\gamma_0] \neq 0$ in $H_1(\widehat{L_t}, \mathbb{R})$. Then we may find a holomorphic 1-form $\omega \in \Omega^1(\widehat{L_t})$ such that $\int_{\gamma_0} \omega = 1$. This 1-form can be holomorphically extended from $\widehat{L_t}$ to $V_T \subset \overline{V}_T$, because $\overline{V}_T$ is Stein and $\widehat{L_t}$ is a closed submanifold of it. Call $\widehat{\omega}$ such an extension, and $\widetilde{\omega} = F_T^*(\widehat{\omega})$ its lift to U_T. On every (simply connected!) fiber $\widetilde{L_t}$ of U_T the 1-form $\widetilde{\omega}$ is exact, and can be integrated giving a holomorphic function $f_t(z) = \int_t^z \widetilde{\omega}|_{\widetilde{L_t}}$. We thus obtain a holomorphic function f on U_T, which separates p and q: $f(p) - f(q) = \int_\gamma \widetilde{\omega} = \int_{\gamma_0} \widehat{\omega} = 1$.

This procedure does not work if $[\gamma_0] = 0$: in that case, every $\omega \in \Omega^1(\widehat{L_t})$ has period equal to zero on γ_0. But, in that case, we may find two 1-forms $\omega_1, \omega_2 \in \Omega^1(\widehat{L_t})$ such that the iterated integral of (ω_1, ω_2) along γ_0 is not zero (this iterated integral [Che] is just the integral along γ of $\phi_1 d\phi_2$, where ϕ_j is a primitive of ω_j lifted to $\widetilde{L_t}$). Then we can repeat the argument above: the fiberwise iterated integral of $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ is a holomorphic function on U_T which separates p and q. □

Having established that U_T is a holomorphically separable Riemann domain over $\overline{V}_T$, it is again a fundamental result of Cartan-Thullen-Oka theory [GuR] that there exists a Stein Riemann domain

$$\overline{F}_T : \overline{U}_T \to \overline{V}_T$$

which contains U_T and such that $\mathcal{O}(\overline{U}_T) = \mathcal{O}(U_T)$. The map $P_T : U_T \to T$ extends to

$$\overline{P}_T : \overline{U}_T \to T,$$

and U_T can be identified with the open subset of $\overline{U}_T$ composed by the connected components of fibers of $\overline{P}_T$ which cut $\Omega \subset \overline{U}_T$. But, in fact, much better is true:

Lemma 2.6. *Every fiber of $\overline{P}_T$ is connected, that is:*

$$\overline{U}_T = U_T.$$

Proof. If not, then, by a connectivity argument, we may find $a_0, b_0 \in \overline{P}_T^{-1}(t_0)$, $a_k, b_k \in \overline{P}_T^{-1}(t_k)$, with $a_k \to a_0$ and $b_k \to b_0$, such that:

(i) $a_0 \in \widetilde{L_{t_0}}$, $b_0 \in \overline{P}_T^{-1}(t_0) \setminus \widetilde{L_{t_0}}$;
(ii) $a_k, b_k \in \widetilde{L_{t_k}}$.

Denote by $\mathcal{M}_{t_0}$ the maximal ideal of $\mathcal{O}_{t_0}$ (on T), and for every $p \in \overline{P}_T^{-1}(t_0)$ denote by $\mathcal{I}_p \subset \mathcal{O}_p$ the ideal generated by $(\overline{P}_T)^*(\mathcal{M}_{t_0})$. At points of $\widetilde{L_{t_0}}$, this is just the ideal of functions vanishing along $\widetilde{L_{t_0}}$; whereas at points of $\overline{P}_T^{-1}(t_0) \setminus \widetilde{L_{t_0}}$, at which $\overline{P}_T$ may fail to be a submersion, this ideal may correspond to a "higher order" vanishing. Because $\overline{U}_T$ is Stein and $\overline{P}_T^{-1}(t_0)$ is a closed subvariety, we may find a holomorphic function $f \in \mathcal{O}(\overline{U}_T)$ such that:

(iii) $f \equiv 0$ on $\widetilde{L_{t_0}}$, $f \equiv 1$ on $\overline{P}_T^{-1}(t_0) \setminus \widetilde{L_{t_0}}$;
(iv) for every $p \in \overline{P}_T^{-1}(t_0)$, the differential df_p of f at p belongs to the ideal $\mathcal{I}_p \Omega_p^1$.

Let $\{z_1, \dots, z_{n-1}\}$ denote the coordinates on T lifted to $\overline{U}_T$. Then, by property (iv), we can factorize

$$df = \sum_{j=1}^{n-1} (z_j - z_j(t_0)) \cdot \beta_j$$

where β_j are holomorphic 1-forms on $\overline{U}_T$.

As in Lemma 2.5, each β_j can be integrated along the simply connected fibers of U_T (with starting point on T), giving a function $g_j \in \mathcal{O}(U_T)$. This function can be holomorphically extended to the envelope $\overline{U}_T$. By the factorization above, and (ii), we have

$$f(b_k) - f(a_k) = \sum_{j=1}^{n-1} (z_j(t_k) - z_j(t_0)) \cdot (g_j(b_k) - g_j(a_k))$$

and this expression tends to 0 as $k \to +\infty$. Therefore $f(b_0) = f(a_0)$, in contradiction with (i) and (iii). $\square$

It follows from this Lemma that $U_T = \overline{U}_T$ is Stein. $\square$

Remark 2.7. . We do not know if V_T also is Stein, i.e. if $V_T = \overline{V}_T$.

This Theorem allows to apply the results of Nishino and Yamaguchi discussed above to holomorphic foliations on Stein manifolds. For instance: the set of parabolic leaves of such a foliation is either full or complete pluripolar. A similar point of view is pursued in [Suz].

3 The unparametrized Hartogs extension lemma

In order to construct the leafwise universal covering of a foliation, we shall need an extension lemma of Hartogs type. This is done in this Section.

Let X be a compact Kähler manifold. Denote by A_r, $r \in (0,1)$, the semi-closed annulus $\{r < |w| \leq 1\}$, with boundary $\partial A_r = \{|w| = 1\}$. Given a holomorphic immersion

$$f : A_r \to X$$

we shall say that $f(A_r)$ **extends to a disc** if there exists a holomorphic map

$$g : \overline{\mathbb{D}} \to X,$$

not necessarily immersive, such that f factorizes as $g \circ j$ for some embedding $j : A_r \to \overline{\mathbb{D}}$, sending ∂A_r to $\partial \mathbb{D}$. That is, f itself does not need to extend to the full disc $\{|w| \leq 1\}$, but it extends "after a reparametrization", given by j.

Remark that if f is an embedding, and $f(A_r)$ extends to a disc, then we can find g as above which is moreover injective outside a finite subset. The image $g(\overline{\mathbb{D}})$ is a (possibly singular) disc in X with boundary $f(\partial A_r)$. Such an extension g or $g(\overline{\mathbb{D}})$ will be called **simple** extension of f or $f(A_r)$. Note that such a g is uniquely defined up to a Moëbius reparametrization of $\overline{\mathbb{D}}$.

Given a holomorphic immersion

$$f : \mathbb{D}^k \times A_r \to X$$

we shall say that $f(\mathbb{D}^k \times A_r)$ **extends to a meromorphic family of discs** if there exists a meromorphic map

$$g : W \dashrightarrow X$$

such that:

(i) W is a complex manifold of dimension $k+1$ with boundary, equipped with a holomorphic submersion $W \to \mathbb{D}^k$ all of whose fibers W_z, $z \in \mathbb{D}^k$, are isomorphic to $\overline{\mathbb{D}}$;

(ii) f factorizes as $g \circ j$ for some embedding $j : \mathbb{D}^k \times A_r \to W$, sending $\mathbb{D}^k \times \partial A_r$ to ∂W and $\{z\} \times A_r$ into W_z, for every $z \in \mathbb{D}^k$.

In particular, the restriction of g to the fiber W_z gives, after removal of indeterminacies, a disc which extends $f(z, A_r)$, and these discs depend on z in a meromorphic way. The manifold W is differentiably a product of $\mathbb{D}^k$ with $\overline{\mathbb{D}}$, but in general this does not hold holomorphically. However, note that by definition W is around its boundary ∂W isomorphic to a product $\mathbb{D}^k \times A_r$.

We shall say that an immersion $f : \mathbb{D}^k \times A_r \to X$ is an **almost embedding** if there exists a proper analytic subset $I \subset \mathbb{D}^k$ such that the restriction of f to $(\mathbb{D}^k \setminus I) \times A_r$ is an embedding. In particular, for every $z \in \mathbb{D}^k \setminus I$, $f(z, A_r)$ is an embedded annulus in X, and $f(z, A_r)$, $f(z', A_r)$ are disjoint if $z, z' \in \mathbb{D}^k \setminus I$ are different.

The following result is a sort of "unparametrized" Hartogs extension lemma [Siu] [Iv1], in which the extension of maps is replaced by the extension of their images. Its proof is inspired by [Iv1] and [Iv2]. The new difficulty is that we need to construct not only a map but also the space where it is defined. The necessity of this unparametrized Hartogs lemma for our future constructions, instead of the usual parametrized one, has been observed in [ChI].

Theorem 3.1. *Let X be a compact Kähler manifold and let $f : \mathbb{D}^k \times A_r \to X$ be an almost embedding. Suppose that there exists an open nonempty subset $\Omega \subset \mathbb{D}^k$ such that $f(z, A_r)$ extends to a disc for every $z \in \Omega$. Then $f(\mathbb{D}^k \times A_r)$ extends to a meromorphic family of discs.*

Proof. Consider the subset

$$Z = \{ \ z \in \mathbb{D}^k \setminus I \mid f(z, A_r) \text{ extends to a disc } \}.$$

Our first aim is to give to Z a complex analytic structure with countable base. This is a rather standard fact, see [Iv2] for related ideas and [CaP] for a larger perspective.

For every $z \in Z$, fix a simple extension

$$g_z : \overline{\mathbb{D}} \to X$$

of $f(z, A_r)$. We firstly put on Z the following metrizable topology: we define the distance between $z_1, z_2 \in Z$ as the Hausdorff distance in X between the discs $g_{z_1}(\overline{\mathbb{D}})$ and $g_{z_2}(\overline{\mathbb{D}})$. Note that this topology may be *finer* than the topology induced by the inclusion $Z \subset \mathbb{D}^k$: if $z_1, z_2 \in Z$ are close each other in $\mathbb{D}^k$ then $g_{z_1}(\overline{\mathbb{D}})$, $g_{z_2}(\overline{\mathbb{D}})$ are close each other near their boundaries, but their interiors may be far each other (think to blow-up).

Take $z \in Z$ and take a Stein neighbourhood $U \subset X$ of $g_z(\overline{\mathbb{D}})$. Consider the subset $A \subset \mathbb{D}^k \setminus I$ of those points z' such that the circle $f(z', \partial A_r)$ is the boundary of a compact complex curve $C_{z'}$ contained in U. Note that, by the maximum principle, such a curve is Hausdorff-close to $g_z(\overline{\mathbb{D}})$, if z' is close to z. According to a theorem of Wermer or Harvey-Lawson [AWe, Ch.19], this condition is equivalent to say that $\int_{f(z', \partial A_r)} \beta = 0$ for every holomorphic 1-form β on U (moment condition). These integrals depend holomorphically on z', for every β. We deduce (by noetherianity) that A is an analytic subset of $\mathbb{D}^k \setminus I$, on a neighbourhood of z. For every $z' \in A$, however, the curve $C_{z'}$ is not necessarily the image of a disc: recall that $g_z(\overline{\mathbb{D}})$ may be singular and may have selfintersections, and so a curve close to it may have positive genus, arising from smoothing the singularities.

Set $\mathcal{A} = \{ \ (z', x) \in A \times U \mid x \in C_{z'} \ \}$. By inspection of the proof of Wermer-Harvey-Lawson theorem [AWe, Ch.19], we see that $\mathcal{A}$ is an analytic subset of $A \times U$ (just by the holomorphic dependence on parameters of the Cauchy transform used in that proof to construct $C_{z'}$). We have a tautological fibration $\pi : \mathcal{A} \to A$ and a tautological map $\tau : \mathcal{A} \to U$ defined by the two projections. Let $B \subset A$ be the subset of those points z' such that the fiber $\pi^{-1}(z') = C_{z'}$ has geometric genus zero. This is an analytic subset of A (the function $z' \mapsto \{$ geometric genus of $\pi^{-1}(z')\}$ is Zariski lower semicontinuous).

By restriction, we have a tautological fibration $\pi : \mathcal{B} \to B$ and a tautological map $\tau : \mathcal{B} \to U \subset X$. Each fiber of π over B is a disc, sent by τ to a disc in U with boundary $f(z', \partial A_r)$. In particular, B is contained in Z.

Now, a neighbourhood of z in B can be identified with a neighbourhood of z in Z (in the Z-topology above): if $z' \in Z$ is Z-close to z then $g_{z'}(\overline{\mathbb{D}})$ is contained in U and then $z' \in B$. In this way, the analytic structure of B is transferred to Z. Note that, with this complex analytic structure, the inclusion $Z \hookrightarrow \mathbb{D}^k$ is holomorphic. More precisely, each irreducible component of Z is a *locally analytic subset* of $\mathbb{D}^k \setminus I$ (where, as usual, "locally analytic" means "analytic in a neighbourhood of it"; of course, a component does not need to be closed in $\mathbb{D}^k \setminus I$).

Let us now prove that the complex analytic space Z has a *countable* number of irreducible components.

To see this, we use the area function $\mathbf{a} : Z \to \mathbb{R}^+$, defined by

$$\mathbf{a}(z) = \text{ area of } g_z(\overline{\mathbb{D}}) = \int_{\overline{\mathbb{D}}} g_z^*(\omega)$$

($\omega =$ Kähler form of X). This function is continuous on Z. Let $c > 0$ be the minimal area of rational curves in X. Set, for every $m \in \mathbb{N}$,

$$Z_m = \{ \ z \in Z \mid \mathbf{a}(z) \in \big(m\frac{c}{2}, (m+2)\frac{c}{2}\big) \ \},$$

so that Z is covered by $\cup_{m=0}^{+\infty} Z_m$. Each Z_m is open in Z, and we claim that on each Z_m the Z-topology coincides with the $\mathbb{D}^k$-topology. Indeed, take a sequence $\{z_n\} \subset Z_m$ which $\mathbb{D}^k$-converges to $z_\infty \in Z_m$. We thus have, in X, a sequence of discs $g_{z_n}(\overline{\mathbb{D}})$ with boundaries $f(z_n, \partial A_r)$ and areas in the interval $(m\frac{c}{2}, (m+2)\frac{c}{2})$. By Bishop's compactness theorem [Bis] [Iv1, Prop.3.1], up to subsequencing, $g_{z_n}(\overline{\mathbb{D}})$ converges, in the Hausdorff topology, to a compact complex curve of the form $D \cup Rat$, where D is a disc with boundary $f(z_\infty, \partial A_r)$ and Rat is a finite union of rational curves (the bubbles). Necessarily, $D = g_{z_\infty}(\overline{\mathbb{D}})$. Moreover,

$$\lim_{n \to +\infty} \text{area}(g_{z_n}(\overline{\mathbb{D}})) = \text{area}(g_{z_\infty}(\overline{\mathbb{D}})) + \text{area}(Rat).$$

From $\mathbf{a}(z_\infty), \mathbf{a}(z_n) \in (m\frac{c}{2}, (m+2)\frac{c}{2})$ it follows that $\text{area}(Rat) < c$, hence, by definition of c, $Rat = \emptyset$. Hence $g_{z_n}(\overline{\mathbb{D}})$ converges, in the Hausdorff topology, to $g_{z_\infty}(\overline{\mathbb{D}})$, i.e. z_n converges to z_∞ in the Z-topology.

Therefore, if $L_m \subset Z_m$ is a countable $\mathbb{D}^k$-dense subset then L_m is also Z-dense in Z_m, and $\cup_{m=0}^{+\infty} L_m$ is countable and Z-dense in Z. It follows that Z has countably many irreducible components.

After these preliminaries, we can really start the proof of the theorem.

The hypotheses imply that the space Z has (at least) one irreducible component V which is open in $\mathbb{D}^k \setminus I$. Let us consider again the area function $\mathbf{a}$ on V. The following lemma is classical, and it is at the base of every extension theorem for maps into Kähler manifolds [Siu] [Iv1].

Lemma 3.2. *For every compact $K \subset \mathbb{D}^k$, the function $\mathbf{a}$ is bounded on $V \cap K$.*

Proof. If $z_0, z_1 \in V$, then we can join them by a continuous path $\{z_t\}_{t\in[0,1]} \subset V$, so that we have in X a continuous family of discs $g_{z_t}(\overline{\mathbb{D}})$, with boundaries $f(z_t, \partial A_r)$. By Stokes formula, the difference between the area of $g_{z_1}(\overline{\mathbb{D}})$ and $g_{z_0}(\overline{\mathbb{D}})$ is equal to the integral of the Kähler form ω on the "'tube" $\cup_{t\in[0,1]} g_{z_t}(\partial\mathbb{D}) = f(\cup_{t\in[0,1]}\{z_t\} \times \partial A_r)$. Now, for topological reasons, $f^*(\omega)$ admits a primitive λ on $\mathbb{D}^k \times A_r$. Therefore

$$\mathbf{a}(z_1) - \mathbf{a}(z_0) = \int_{\{z_1\}\times\partial A_r} \lambda - \int_{\{z_0\}\times\partial A_r} \lambda.$$

Remark that the function $z \mapsto \int_{\{z\}\times\partial A_r} \lambda$ is defined (and smooth) on the full $\mathbb{D}^k$, not only on V, and so it is bounded on every compact $K \subset \mathbb{D}^k$. The conclusion follows immediately. $\square$

We use this lemma to study the boundary of V, and to show that the complement of V is small.

Take a point $z_\infty \in (\mathbb{D}^k \setminus I) \cap \partial V$ and a sequence $z_n \in V$ converging to z_∞. By the boundedness of $\mathbf{a}(z_n)$ and Bishop compactness theorem, we obtain a disc in X with boundary $f(z_\infty, \partial A_r)$ (plus, perhaps, some rational bubbles, but we may forget them). In particular, the point z_∞ belongs to Z. Obviously, the irreducible component of Z which contains z_∞ is not open in $\mathbb{D}^k \setminus I$, because $z_\infty \in \partial V$, and so that component is a locally analytic subset of $\mathbb{D}^k \setminus I$ of *positive* codimension. It follows that the boundary ∂V is a *thin subset* of $\mathbb{D}^k$, i.e. it is contained in a countable union of locally analytic subsets of positive codimension (certain components of Z, plus the analytic subset I). Disconnectedness properties of thin subsets show that also *the complement $\mathbb{D}^k \setminus V$ $(= \partial V)$ is thin in $\mathbb{D}^k$.*

Recall now that over V we have the (normalized) tautological fibration $\pi : \mathcal{V} \to V$, equipped with the tautological map $\tau : \mathcal{V} \to X$. Basically, this provides the desired extension of f over the large open subset V. As in [Iv1], we shall get the extension over the full $\mathbb{D}^k$ by reducing to the Thullen type theorem of Siu [Siu].

By construction, $\partial\mathcal{V}$ has a neigbourhood isomorphic to $V \times A_r$, the isomorphism being realized by f. Hence we can glue to $\mathcal{V}$ the space $\mathbb{D}^k \times A_r$, using the same f. We obtain a new space $\mathcal{W}$ equipped with a fibration $\pi : \mathcal{W} \to \mathbb{D}^k$ and a map $\tau : \mathcal{W} \to X$ such that:

(i) $\pi^{-1}(z) \simeq \overline{\mathbb{D}}$ for $z \in V$, $\pi^{-1}(z) \simeq A_r$ for $z \in \mathbb{D}^k \setminus V$;
(ii) f factorizes through τ.

In other words, and recalling how $\mathcal{V}$ was defined, up to normalization $\mathcal{W}$ is simply the analytic subset of $\mathbb{D}^k \times X$ given by the union of all the discs $\{z\} \times g_z(\overline{\mathbb{D}})$, $z \in V$, and all the annuli $\{z\} \times f(z, A_r)$, $z \in \mathbb{D}^k \setminus V$.

Lemma 3.3. *There exists an embedding $\mathcal{W} \to \mathbb{D}^k \times \mathbb{P}$, which respects the fibrations over $\mathbb{D}^k$.*

Proof. Set $B_r = \{ w \in \mathbb{P} \mid |w| > r \}$. By construction, $\partial\mathcal{W}$ has a neighbourhood isomorphic to $\mathbb{D}^k \times A_r$. We can glue $\mathbb{D}^k \times B_r$ to $\mathcal{W}$ by identification of that neighbourhood with $\mathbb{D}^k \times A_r \subset \mathbb{D}^k \times B_r$, i.e. by prolonging each annulus A_r to a disc B_r. The result is a new space $\widehat{\mathcal{W}}$ with a fibration $\widehat{\pi} : \widehat{\mathcal{W}} \to \mathbb{D}^k$ such that:

(i) $\widehat{\pi}^{-1}(z) \simeq \mathbb{P}$ for every $z \in V$;
(ii) $\widehat{\pi}^{-1}(z) \simeq B_r$ for every $z \in \mathbb{D}^k \setminus V$.

We shall prove that $\widehat{\mathcal{W}}$ (and hence $\mathcal{W}$) embeds into $\mathbb{D}^k \times \mathbb{P}$ (incidentally, note the common features with the proof of Theorem 2.3).

For every $z \in V$, there exists a unique isomorphism

$$\varphi_z : \widehat{\pi}^{-1}(z) \to \mathbb{P}$$

such that

$$\varphi_z(\infty) = 0 \ , \quad \varphi_z'(\infty) = 1 \ , \quad \varphi_z(r) = \infty$$

where $\infty, r \in \overline{B_r} \subset \widehat{\pi}^{-1}(z)$ and the derivative at ∞ is computed using the coordinate $\frac{1}{w}$. Every $\mathbb{P}$-fibration is locally trivial, and so this isomorphism φ_z depends holomorphically on z. Thus we obtain a biholomorphism

$$\Phi : \widehat{\pi}^{-1}(V) \to V \times \mathbb{P}$$

and we want to prove that Φ extends to the full $\widehat{\mathcal{W}}$.

By Koebe's Theorem, the distorsion of φ_z on any compact $K \subset B_r$ is uniformly bounded (note that $\varphi_z(B_r) \subset \mathbb{C}$). Hence, for every $w_0 \in B_r$ the holomorphic function $z \mapsto \varphi_z(w_0)$ is bounded on V. Because the complement of V in $\mathbb{D}^k$ is thin, by Riemann's extension theorem this function extends holomorphically to $\mathbb{D}^k$. This permits to extends the above Φ also to fibers over $\mathbb{D}^k \setminus V$. Still by bounded distorsion, this extension is an embedding of $\widehat{\mathcal{W}}$ into $\mathbb{D}^k \times \mathbb{P}$. □

Now we can finish the proof of the theorem. Thanks to the previous embedding, we may "fill in" the holes of $\mathcal{W}$ and obtain a $\overline{\mathbb{D}}$-fibration W over $\mathbb{D}^k$. Then, by the Thullen type theorem of Siu [Siu] (and transfinite induction) the map $\tau : \mathcal{W} \to X$ can be meromorphically extended to W. This is the meromorphic family of discs which extends $f(\mathbb{D}^k \times A_r)$. □

By comparison with the usual "parametrized" Hartogs extension lemma [Iv1], one could ask if the almost embedding hypothesis in Theorem 3.1 is really indispensable. In some sense, the answer is yes. Indeed, we may easily construct a fibered immersion $f : \mathbb{D} \times A_r \to \mathbb{D} \times \mathbb{P} \subset \mathbb{P} \times \mathbb{P}$, $f(z,w) = (z, f_0(z,w))$, such that: (i) for some $z_0 \in \mathbb{D}$, $f_0(z_0, \partial A_r)$ is an embedded circle in $\mathbb{P}$; (ii) for some other $z_1 \in \mathbb{D}$, $f_0(z_1, \partial A_r)$ is an immersed but not embedded circle in $\mathbb{P}$. Then, for some neighbourhhod $U \subset \mathbb{D}$ of z_0, $f(U \times A_r)$ can be obviously extended to a meromorphic (even holomorphic) family of discs, but such a U cannot be enlarged to contain z_1, because $f_0(z_1, \partial A_r)$ bounds no disc in $\mathbb{P}$. Note, however, that $f_0(z_1, \partial A_r)$ bounds a so called holomorphic chain in $\mathbb{P}$ [AWe, Ch.19]: if $\Omega_1, ..., \Omega_m$ are the connected components of $\mathbb{P} \setminus f_0(z_1, \partial A_r)$, then $f_0(z_1, \partial A_r)$ is the "algebraic" boundary of $\sum_{j=1}^m n_j \Omega_j$, for suitable integers n_j. It is conceivable that Theorem 3.1 holds under the sole assumption that f is an immersion, provided that the manifold W is replaced by a (suitably defined) "meromorphic family of 1-dimensional chains".

4 Holonomy tubes and covering tubes

Here we shall define leaves, holonomy tubes and covering tubes, following [Br3].

Let X be a compact Kähler manifold, of dimension n, and let $\mathcal{F}$ be a foliation by curves on X. Set $X^0 = X \setminus Sing(\mathcal{F})$ and $\mathcal{F}^0 = \mathcal{F}|_{X^0}$. We could define the "leaves" of the singular foliation $\mathcal{F}$ simply as the usual leaves of the nonsingular foliation $\mathcal{F}^0$. However, for our purposes we shall need that the universal coverings of the leaves glue together in a nice way, producing what we shall call covering tubes. This shall require a sort of semicontinuity of the fundamental groups of the leaves. With the naïve definition "leaves of $\mathcal{F}$ = leaves of $\mathcal{F}^0$", such a semicontinuity can fail, in the sense that a leaf (of $\mathcal{F}^0$) can have a larger fundamental group than nearby leaves (of $\mathcal{F}^0$). To remedy to this, we give now a less naïve definition of leaf of $\mathcal{F}$, which has the effect of killing certain homotopy classes of cycles, and the problem will be settled almost by definition (but we will require also the unparametrized Hartogs extension lemma of the previous Section).

4.1 Vanishing ends

Take a point $p \in X^0$, and let L_p^0 be the leaf of $\mathcal{F}^0$ through p. It is a smooth complex connected curve, equipped with an injective immersion

$$i_p^0 : L_p^0 \to X^0,$$

and sometimes we will tacitly identify L_p^0 with its image in X^0 or X. Recall that, given a local transversal $\mathbb{D}^{n-1} \hookrightarrow X^0$ to $\mathcal{F}^0$ at p, we have a holonomy representation [CLN]

$$hol_p : \pi_1(L_p^0, p) \to Diff(\mathbb{D}^{n-1}, 0)$$

of the fundamental group of L_p^0 with basepoint p into the group of germs of holomorphic diffeomorphisms of $(\mathbb{D}^{n-1}, 0)$.

Let $E \subset L_p^0$ be a *parabolic end* of L_p^0, that is a closed subset isomorphic to the punctured closed disc $\overline{\mathbb{D}}^* = \{ 0 < |w| \leq 1 \}$, and suppose that the holonomy of $\mathcal{F}^0$ along the cycle ∂E is trivial. Then, for some $r \in (0,1)$, the inclusion $A_r \subset \overline{\mathbb{D}}^* = E$ can be extended to an embedding $\mathbb{D}^{n-1} \times A_r \to$

X^0 which sends each $\{z\} \times A_r$ into a leaf of $\mathcal{F}^0$, and $\{0\} \times A_r$ to $A_r \subset E$ (this is because A_r is Stein, see for instance [Suz, §3]). More generally, if the holonomy of $\mathcal{F}^0$ along ∂E is finite, of order k, then we can find an immersion $\mathbb{D}^{n-1} \times A_r \to X^0$ which sends each $\{z\} \times A_r$ into a leaf of $\mathcal{F}^0$ and $\{0\} \times A_r$ to $A_{r'} \subset E$, in such a way that $\{0\} \times A_r \to A_{r'}$ is a regular covering of order k. Such an immersion is (or can be chosen as) an almost embedding: the exceptional subset $I \subset \mathbb{D}^{n-1}$, outside of which the map is an embedding, corresponds to leaves which intersect the transversal, over which the holonomy is computed, at points whose holonomy orbit has cardinality strictly less than k. This is an analytic subset of the transversal. Such an almost embedding will be called *adapted to E*.

We shall use the following terminology: a meromorphic map is a *meromorphic immersion* if it is an immersion outside its indeterminacy set.

Definition 4.1. *Let $E \subset L_p^0$ be a parabolic end with finite holonomy, of order $k \geq 1$. Then E is a* **vanishing end***, of order k, if there exists an almost embedding $f : \mathbb{D}^{n-1} \times A_r \to X^0 \subset X$ adapted to E such that:*

(i) $f(\mathbb{D}^{n-1} \times A_r)$ extends to a meromorphic family of discs $g : W \dashrightarrow X$;
(ii) g is a meromorphic immersion.

In other words, E is a vanishing end if, firstly, it can be compactified in X to a disc, by adding a singular point of $\mathcal{F}$, and, secondly, this disc-compactification can be meromorphically and immersively deformed to nearby leaves, up to a ramification given by the holonomy. This definition mimics, in our context, the classical definition of vanishing cycle for real codimension one foliations [CLN].

Remark 4.2. If $g : W \dashrightarrow X$ is as in Definition 4.1, then the indeterminacy set $F = Indet(g)$ cuts each fiber W_z, $z \in \mathbb{D}^{n-1}$, along a finite subset $F_z \subset W_z$. The restricted map $g_z : W_z \to X$ sends $W_z \setminus F_z$ into a leaf of $\mathcal{F}^0$, in an immersive way, and F_z into $Sing(\mathcal{F})$. Each point of F_z corresponds to a parabolic end of $W_z \setminus F_z$, which is sent by g_z to a parabolic end of a leaf; clearly, this parabolic end is a vanishing one (whose order, however, may be smaller than k), and the corresponding meromorphic family of discs is obtained by restricting g. Remark also that, F being of codimension at least 2, we have $F_z = \emptyset$ for every z outside an analytic subset of $\mathbb{D}^{n-1}$ of positive codimension. This means (as we shall see better below) that "most" leaves have no vanishing end.

If $E \subset L_p^0$ is a vanishing end of order k, then we compactify it by adding one point, i.e. by prolonging $\overline{\mathbb{D}}^*$ to $\overline{\mathbb{D}}$. But we do such a compactification in an *orbifold sense*: the added point has, by definition, a *multiplicity* equal to k. By doing such a end-compactification for every vanishing end of L_p^0, we finally obtain a connected curve (with orbifold structure) L_p, which is by definition the **leaf** of $\mathcal{F}$ through p. The initial inclusion $i_p^0 : L_p^0 \to X^0$ can be extended to a holomorphic map

$$i_p : L_p \to X$$

which sends the discrete subset $L_p \setminus L_p^0$ into $Sing(\mathcal{F})$. Note that i_p may fail to be immersive at those points. Moreover, it may happen that two different points of $L_p \setminus L_p^0$ are sent by i_p to the same singular point of $\mathcal{F}$ (see Example 4.4 below). In spite of this, we shall sometimes identify L_p with its image in X. For instance, to say that a map $f : Z \to X$ "has values into L_p" shall mean that f factorizes through i_p.

Remark that we have not defined, and shall not define, leaves L_p through $p \in Sing(\mathcal{F})$: a leaf may pass through $Sing(\mathcal{F})$, but its basepoint must be chosen outside $Sing(\mathcal{F})$.

Let us see two examples.

Example 4.3. Take a compact Kähler surface S foliated by an elliptic fibration $\pi : S \to C$, and let $c_0 \in C$ be such that the fiber $F_0 = \pi^{-1}(c_0)$ is of Kodaira type II [BPV, V.7], i.e. a rational curve with a cusp q. If $p \in F_0$, $p \neq q$, then the leaf L_p^0 is equal to $F_0 \setminus \{q\} \simeq \mathbb{C}$. This leaf has a parabolic end with trivial holonomy, which is *not* a vanishing end. Indeed, this end can be compactified to a cuspidal disc, which however cannot be meromorphically deformed *as a disc* to nearby leaves, because nearby leaves have positive genus close to q. Hence $L_p = L_p^0$.

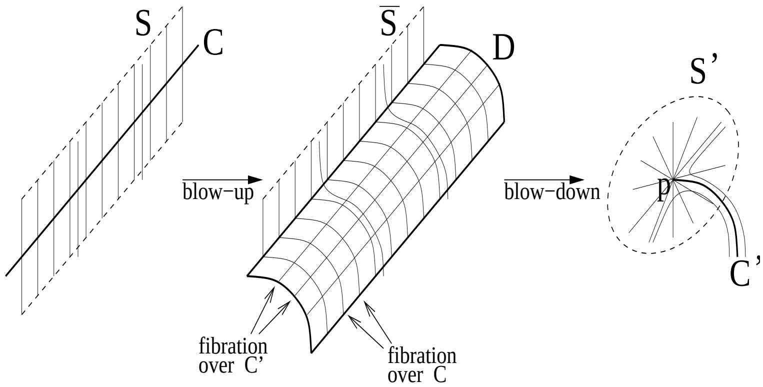
Let now $\widetilde{S} \to S$ be the composition of three blow-ups which transforms F_0 into a tree of four smooth rational curves $\widetilde{F}_0 = G_1 + G_2 + G_3 + G_6$ of respective selfintersections $-1, -2, -3, -6$ [BPV, V.10]. Let $\widetilde{\pi} : \widetilde{S} \to C$ be the new elliptic fibration/foliation. Set $p_j = G_1 \cap G_j$, $j = 2, 3, 6$. If $p \in G_1$ is different from those three points, then $L_p^0 = G_1 \setminus \{p_2, p_3, p_6\}$. The parabolic end of L_p^0 corresponding to p_2 (resp. p_3, p_6) has holonomy of order 2 (resp. 3, 6). This time, this is a vanishing end: a disc D in G_1 through p_2 (resp. p_3, p_6) ramified at order 2 (resp. 3, 6) can be deformed to nearby leaves as discs close to $2D + G_2$ (resp. $3D + G_3$, $6D + G_6$), and also the "meromorphic immersion" condition can be easily respected. Thus L_p is isomorphic to the orbifold "$\mathbb{P}$ with three points of multiplicity 2, 3, 6". Note that the universal covering (in orbifold sense) of L_p is isomorphic to $\mathbb{C}$, and the holonomy covering (defined below) is a smooth elliptic curve.

Finally, if $p \in G_j$, $p \neq p_j$, $j = 2, 3, 6$, then L_p^0 has a parabolic end with trivial holonomy, which is not a vanishing end, and so $L_p = L_p^0 \simeq \mathbb{C}$.

A more systematic analysis of the surface case, from a slightly different point of view, can be found in [Br1].

Example 4.4. Take a projective threefold M containing a smooth rational curve C with normal bundle $N_C = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Take a foliation $\mathcal{F}$ on M, nonsingular around C, such that: (i) for every $p \in C$, $T_p\mathcal{F}$ is different from T_pC; (ii) $T_{\mathcal{F}}$ has degree -1 on C. It is easy to see that there are a lot of foliations on M satisfying these two requirements. Note that, on a neighbourhood of C, we can glue together the local leaves (discs) of $\mathcal{F}$ through C, and obtain a smooth surface S containing C; condition (ii) means that the selfintersection of C in S is equal to -1.

We now perform a *flop* of M along C. That is, we firstly blow-up M along C, obtaining a threefold $\widetilde{M}$ containing an exceptional divisor D naturally $\mathbb{P}$-fibered over C. Because $N_C = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, this divisor D is in fact isomorphic to $\mathbb{P} \times \mathbb{P}$, hence it admits a second $\mathbb{P}$-fibration, transverse to the first one. Each fibre of this second fibration can be blow-down to a point (Moishezon's criterion [Moi]), and the result is a smooth threefold M', containing a smooth rational curve C' with normal bundle $N_{C'} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, over which D fibers. (At this point, M' could be no more projective, nor Kähler, but this is not an important fact in this example). The strict transform $\overline{S}$ of S in $\widetilde{M}$ cuts the divisor D along one of the fibers of the second fibration $D \to C'$, by condition (ii) above, therefore its image S' in M' is a bidimensional disc which cuts C' transversely at some point p.



Let us look at the transformed foliation $\mathcal{F}'$ on M'. The point p is a singular point of $\mathcal{F}'$, the only one on a neighbourhood of C'. The curve C' is invariant by $\mathcal{F}'$. The surface S' is tangent to $\mathcal{F}'$, and over it the foliation has a radial type singularity. In fact, in appropriate coordinates around p the foliation is generated by the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$, with $S' = \{z = 0\}$ and $C' = \{x = y = 0\}$.

If L^0 is a leaf of $(\mathcal{F}')^0$, then each component D^0 of $L^0 \cap S'$ is a parabolic end converging to p. It is a vanishing end, of order 1: the meromorphic family of discs of Definition 4.1 is obviously constructed from a flow box of $\mathcal{F}$, around a suitably chosen point of C. Generic fibers of this family are sent to discs in M' close to $D^0 \cup C'$; other fibers are sent to discs in S' passing through p, and close to $D^0 \cup \{p\}$. Remark that it can happen that $L^0 \cap S'$ has several, or even infinitely many, components; in that case the map $i : L \to M'$ sends several, or even infinitely many, points to the same $p \in M'$.

Having defined the leaf L_p through $p \in X^0$, we can now define its **holonomy covering** $\widehat{L_p}$ and its **universal covering** $\widetilde{L_p}$. The first one is the covering defined by the Kernel of the holonomy. More precisely, we start with the usual holonomy covering $(\widehat{L_p^0}, p) \to (L_p^0, p)$ with basepoint p (it is useful to think to $\widehat{L_p^0}$ as equivalence classes of paths in L_p^0 starting at p, so that the basepoint $p \in \widehat{L_p^0}$ is the class of the constant path). If $E \subset L_p^0$ is a vanishing end of order k, then its preimage in $\widehat{L_p^0}$ is a (finite or infinite) collection of parabolic ends $\{\widehat{E_j}\}$, each one regularly covering E with degree k. Each such map $\overline{\mathbb{D}}^* \simeq \widehat{E_j} \to \overline{\mathbb{D}}^* \simeq E$ can be extended to a map $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$, with a ramification at 0 of order k. By definition, $\widehat{L_p}$ is obtained by compactifying all these parabolic ends of $\widehat{L_p^0}$, over all the vanishing ends of L_p^0. Therefore, we have a covering map

$$(\widehat{L_p}, p) \to (L_p, p)$$

which ramifies over $L_p \setminus L_p^0$. However, from the orbifold point of view such a map is a *regular* covering: $w = z^k$ is a regular covering if $z = 0$ has multiplicity 1 and $w = 0$ has multiplicity k. Note that we do not need anymore a orbifold structure on $\widehat{L_p}$, in the sense that all its points have multiplicity 1.

In a more algebraic way, the orbifold fundamental group $\pi_1(L_p, p)$ is a quotient of $\pi_1(L_p^0, p)$, through which the holonomy representation hol_p factorizes. Then $\widehat{L_p}$ is the covering defined by the Kernel of this representation of $\pi_1(L_p, p)$ into $Diff(\mathbb{D}^{n-1}, 0)$.

The universal covering $\widetilde{L_p}$ can be now defined as the universal covering of $\widehat{L_p}$, or equivalently as the universal covering, in orbifold sense, of L_p. We then have natural covering maps

$$(\widetilde{L_p}, p) \to (\widehat{L_p}, p) \to (L_p, p).$$

Recall that there are few exceptional orbifolds (teardrops) which do not admit a universal covering. It is a pleasant fact that in our context such orbifolds do not appear.

4.2 Holonomy tubes

We now analyze how the maps $p \mapsto \widehat{L_p}$ and then $p \mapsto \widetilde{L_p}$ depend on p. Propositions 4.5 and 4.6 below say that, in some sense, the dependence on p is holomorphic: holonomy coverings and universal coverings can be holomorphically glued together, producing fibered complex manifolds.

Let $T \subset X^0$ be a (local) transversal to $\mathcal{F}^0$.

Proposition 4.5. *There exists a complex manifold V_T of dimension n, a holomorphic submersion*

$$Q_T : V_T \to T,$$

a holomorphic section

$$q_T : T \to V_T,$$

and a meromorphic immersion

$$\pi_T : V_T \dashrightarrow X$$

such that:

(i) for every $t \in T$, the pointed fiber $(Q_T^{-1}(t), q_T(t))$ is isomorphic to $(\widehat{L_t}, t)$;
(ii) the indeterminacy set $Indet(\pi_T)$ of π_T cuts each fiber $Q_T^{-1}(t) = \widehat{L_t}$ along the discrete subset $\widehat{L_t} \setminus \widehat{L_t^0}$;
(iii) for every $t \in T$, the restriction of π_T to $Q_T^{-1}(t) = \widehat{L_t}$ coincides, after removal of indeterminacies, with the holonomy covering $\widehat{L_t} \to L_t \xrightarrow{i_t} i_t(L_t) \subset X$.

Proof. We firstly prove a similar statement for the regular foliation $\mathcal{F}^0$ on X^0. We use Il'yashenko's methodology [Il1]; an alternative but equivalent one can be found in [Suz], we have already seen it at the beginning of the proof of Theorem 2.4. In fact, in the case of a regular foliation the construction of V_T^0 below is a rather classical fact in foliation theory, which holds in the much more general context of smooth foliations with real analytic holonomy.

Consider the space $\Omega_T^{\mathcal{F}^0}$ composed by continuous paths $\gamma : [0,1] \to X^0$ tangent to $\mathcal{F}^0$ and such that $\gamma(0) \in T$, equipped with the uniform topology. On $\Omega_T^{\mathcal{F}^0}$ we put the following equivalence relation: $\gamma_1 \sim \gamma_2$ if $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and the loop $\gamma_1 * \gamma_2^{-1}$, obtained by juxtaposing γ_1 and γ_2^{-1}, has trivial holonomy.

Set

$$V_T^0 = \Omega_T^{\mathcal{F}^0} / \sim$$

with the quotient topology. Note that we have natural continuous maps

$$Q_T^0 : V_T^0 \to T$$

and

$$\pi_T^0 : V_T^0 \to X^0$$

defined respectively by $[\gamma] \mapsto \gamma(0) \in T$ and $[\gamma] \mapsto \gamma(1) \in X^0$. We also have a natural section

$$q_T^0 : T \to V_T^0$$

which associates to $t \in T$ the equivalence class of the constant path at t. Clearly, for every $t \in T$ the pointed fiber $((Q_T^0)^{-1}(t), q_T^0(t))$ is the same as $(\widehat{L_t^0}, t)$, by the very definition of holonomy covering, and π_T^0 restricted to that fiber is the holonomy covering map. Therefore, we just have to find a complex structure on V_T^0 such that all these maps become holomorphic.

We claim that V_T^0 is a Hausdorff space. Indeed, if $[\gamma_1], [\gamma_2] \in V_T^0$ are two nonseparated points, then $\gamma_1(0) = \gamma_2(0) = t$, $\gamma_1(1) = \gamma_2(1)$, and the loop

$\gamma_1 * \gamma_2^{-1}$ in the leaf L_t^0 can be uniformly approximated by loops $\gamma_{1,n} * \gamma_{2,n}^{-1}$ in the leaves $L_{t_n}^0$ $(t_n \to t)$ with trivial holonomy (so that $[\gamma_{1,n}] = [\gamma_{2,n}]$ is a sequence of points of V_T^0 converging to both $[\gamma_1]$ and $[\gamma_2]$). But this implies that also the loop $\gamma_1 * \gamma_2^{-1}$ has trivial holonomy, by the identity principle: if $h \in Diff(\mathbb{D}^{n-1}, 0)$ is the identity on a sequence of open sets accumulating to 0, then h is the identity everywhere. Thus $[\gamma_1] = [\gamma_2]$, and V_T^0 is Hausdorff.

Now, note that $\pi_T^0 : V_T \to X^0$ is a local homeomorphism. Hence we can pull back to V_T^0 the complex structure of X^0, and in this way V_T^0 becomes a complex manifold of dimension n with all the desired properties. Remark that, at this point, π_T^0 has not yet indeterminacy points, and so V_T^0 is a so-called Riemann Domain over X^0.

In order to pass from V_T^0 to V_T, we need to add to each fiber $\widehat{L_t^0}$ of V_T^0 the discrete set $\widehat{L_t} \setminus \widehat{L_t^0}$.

Take a vanishing end $E \subset L_t^0$, of order k, let $f : \mathbb{D}^{n-1} \times A_r \to X^0$ be an almost embedding adapted to E, and let $g : W \dashrightarrow X$ be a meromorphic family of discs extending f, immersive outside $F = Indet(g)$. Take also a parabolic end $\widehat{E} \subset \widehat{L_t^0}$ projecting to E, with degree k. By an easy holonomic argument, the immersion $g|_{W \setminus F} : W \setminus F \to X^0$ can be lifted to V_T^0, as a proper embedding

$$\widetilde{g} : W \setminus F \to V_T^0$$

which sends the central fiber $W_0 \setminus F_0$ to $\widehat{E}$. Each fiber $W_z \setminus F_z$ is sent by $\widetilde{g}$ to a closed subset of a fiber $\widehat{L_{t(z)}^0}$, and each point of F_z corresponds to a parabolic end of $\widehat{L_{t(z)}^0}$ projecting to a vanishing end of $L_{t(z)}^0$.

Now we can glue W to V_T^0 using $\widetilde{g}$: this corresponds to compactify all parabolic ends of fibers of V_T^0 which project to vanishing ends and which are close to $\widehat{E}$. By doing this operation for every E and $\widehat{E}$, we finally construct our manifold V_T, fibered over T with fibers $\widehat{L_t}$. The map π_T extending (meromorphically) π_T^0 is then deduced from the maps g above. □

The manifold V_T will be called **holonomy tube** over T. The meromorphic immersion π_T is, of course, *very complicated*: it contains all the dynamics of the foliation, so that it is, generally speaking, very far from being, say, finite-to-one. Note, however, that most fibers do not cut the indeterminacy set of π_T, so that π_T sends that fibers to leaves of $\mathcal{F}^0$; moreover, most leaves have trivial holonomy (it is a general fact [CLN] that leaves with non trivial holonomy cut any transversal along a thin subset), and so on most fibers π_T is even an isomorphism between the fiber and the corresponding leaf of $\mathcal{F}^0$. But be careful: a leaf may cut a transversal T infinitely many times, and so V_T will contain infinitely many fibers sent by π_T to the same leaf, as holonomy coverings (possibly trivial) with different basepoints.

4.3 Covering tubes

The following proposition is similar, in spirit, to Proposition 4.5, but, as we shall see, its proof is much more delicate. Here the Kähler assumption becomes really indispensable, via the unparametrized Hartogs extension lemma. Without the Kähler hypothesis it is easy to find counterexamples (say, for foliations on Hopf surfaces).

Proposition 4.6. *There exists a complex manifold U_T of dimension n, a holomorphic submersion*

$$P_T : U_T \to T,$$

a holomorphic section

$$p_T : T \to U_T,$$

and a surjective holomorphic immersion

$$F_T : U_T \to V_T$$

such that:

(i) for every $t \in T$, the pointed fiber $(P_T^{-1}(t), p_T(t))$ is isomorphic to $(\widetilde{L_t}, t)$;
(ii) for every $t \in T$, F_T sends the fiber $(\widetilde{L_t}, t)$ to the fiber $(\widehat{L_t}, t)$, as universal covering.

Proof. We use the same methodology as in the first part of the previous proof, with $\mathcal{F}^0$ replaced by the fibration V_T and $\Omega_T^{\mathcal{F}^0}$ replaced by $\Omega_T^{V_T}$ = space of continuous paths $\gamma : [0,1] \to V_T$ tangent to the fibers and starting from $q_T(T) \subset V_T$. But now the equivalence relation $\sim$ is given by homotopy, not holonomy: $\gamma_1 \sim \gamma_2$ if they have the same extremities and the loop $\gamma_1 * \gamma_2^{-1}$ is homotopic to zero in the fiber containing it. The only thing that we need to prove is that the quotient space

$$U_T = \Omega_T^{V_T} \big/ \sim$$

is Hausdorff; then everything is completed as in the previous proof, with F_T associating to a homotopy class of paths its holonomy class. The Hausdorff property can be spelled as follows ("nonexistence of vanishing cycles"):

(*) if $\gamma : [0,1] \to \widehat{L_t} \subset V_T$ is a loop (based at $q_T(t)$) uniformly approximated by loops $\gamma_n : [0,1] \to \widehat{L_{t_n}} \subset V_T$ (based at $q_T(t_n)$) homotopic to zero in $\widehat{L_{t_n}}$, then γ is homotopic to zero in $\widehat{L_t}$.

Let us firstly consider the case in which γ is a simple loop. We may assume that $\Gamma = \gamma([0,1])$ is a real analytic curve in $\widehat{L_t}$, and we may find an embedding

$$f : \mathbb{D}^{n-1} \times A_r \to V_T$$

sending fibers to fibers and such that $\Gamma = f(0, \partial A_r)$. Thus $\Gamma_n = f(z_n, \partial A_r)$ is homotopic to zero in its fiber, for some sequence $z_n \to 0$. For evident reasons, if z_n' is sufficiently close to z_n, then also $f(z_n', \partial A_r)$ is homotopic to zero in its fiber. Thus, we have an open nonempty subset $U \subset \mathbb{D}^{n-1}$ such that, for every $z \in U$, $f(z, \partial A_r)$ is homotopic to zero in its fiber. Denote by D_z the disc in the fiber bounded by such $f(z, \partial A_r)$.

We may also assume that Γ is disjoint from the discrete subset $\widehat{L_t} \setminus \widehat{L_t^0}$, so that, after perhaps restricting $\mathbb{D}^{n-1}$, the composite map

$$f' : \pi_T \circ f : \mathbb{D}^{n-1} \times A_r \to X$$

is holomorphic, and therefore it is an almost embedding. We already know that, for every $z \in U$, $f'(z, A_r)$ extends to a disc, image by π_T of D_z. Therefore, by Theorem 3.1, $f'(\mathbb{D}^{n-1} \times A_r)$ extends to a meromorphic family of discs

$$g : W \dashrightarrow X.$$

It may be useful to observe that such a g is a meromorphic immersion. Indeed, setting $F = Indet(g)$, the set of points of $W \setminus F$ where g is not an immersion is (if not empty) an hypersurface. Such a hypersurface cannot cut a neighbourhood of the boundary ∂W, where g is a reparametrization of the immersion f'. Also, such a hypersurface cannot cut the fiber W_z when $z \in U$ is generic (i.e. $W_z \cap F = \emptyset$), because on a neighbourhood of such a W_z the map g is a reparametrization of the immersion π_T on a neighbourhood of D_z. It follows that such a hypersurface is empty.

As in the proof of Proposition 4.5, $g|_{W \setminus F}$ can be lifted, holomorphically, to V_T^0, and then g can be lifted to V_T, giving an embedding $\widetilde{g} : W \to V_T$. Then $\widetilde{g}(W_0)$ is a disc in $\widehat{L_t}$ with boundary Γ, and consequently γ is homotopic to zero in the fiber $\widehat{L_t}$.

Consider now the case in which γ is possibly not simple. We may assume that γ is a smooth immersion with some points of transverse selfintersection, and idem for γ_n. We reduce to the previous simple case, by a purely topological argument.

Take the immersed circles $\Gamma = \gamma([0, 1])$ and $\Gamma_n = \gamma_n([0, 1])$. Let $R_n \subset \widehat{L_{t_n}}$ be the open bounded subset obtained as the union of a small tubular neighbourhood of Γ_n and all the bounded components of $\widehat{L_{t_n}} \setminus \Gamma_n$ isomorphic to the disc. Thus, each connected component of $R_n \setminus \Gamma_n$ is either a disc with boundary in Γ_n (union of arcs between selfintersection points), or an annulus with one boundary component in Γ_n and another one in ∂R_n; this last one is not the boundary of a disc in $\widehat{L_{t_n}} \setminus R_n$. We have the following elementary topological fact: if Γ_n is homotopic to zero in $\widehat{L_{t_n}}$, then it is homotopic to zero also in R_n.

Let $R \subset \widehat{L_t}$ be defined in a similar way, starting from Γ. By the first part of the proof, if $D_n \subset R_n \setminus \Gamma_n$ is a disc with boundary in Γ_n, then for $t_n \to t$ such a disc converges to a disc $D \subset \widehat{L_t}$ with boundary in Γ, i.e. to a disc $D \subset R \setminus \Gamma$. Conversely, but by elementary reasons, any disc $D \subset R \setminus \Gamma$ with boundary in Γ can be deformed to discs $D_n \subset R_n \setminus \Gamma_n$ with boundaries in $\mathit{\Gamma}_n$. We deduce that R is diffeomorphic to R_n, or more precisely that the pair $(R, \mathit{\Gamma})$ is diffeomorphic to the pair $(R_n, \mathit{\Gamma}_n)$, for n large. Hence from $\mathit{\Gamma}_n$ homotopic to zero in R_n we infer $\mathit{\Gamma}$ homotopic to zero in R, and a fortiori in $\widehat{L_t}$. This completes the proof of the Hausdorff property (*). □

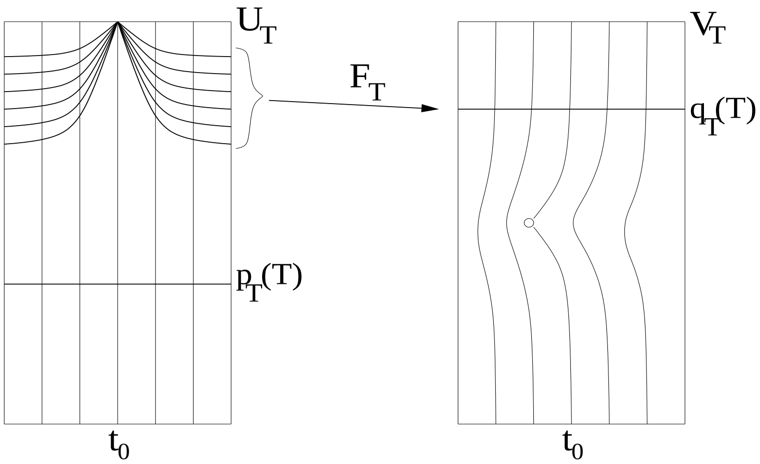
The manifold U_T will be called **covering tube** over T. We have a meromorphic immersion

$$\mathit{\Pi}_T = \pi_T \circ F_T : U_T \dashrightarrow X$$

whose indeterminacy set $Indet(\mathit{\Pi}_T)$ cuts each fiber $P_T^{-1}(t) = \widetilde{L_t}$ along the discrete subset which is the preimage of $\widehat{L_t} \setminus \widehat{L_t^0}$ under the covering map $\widetilde{L_t} \to \widehat{L_t}$. For every $t \in T$, the restriction of $\mathit{\Pi}_T$ to $P^{-1}(t)$ coincides, after removal of indeterminacies, with the universal covering $\widetilde{L_t} \to L_t \stackrel{i_t}{\to} i_t(L_t) \subset X$.

The local biholomorphism $F_T : U_T \to V_T$ is a fiberwise covering, but globally it may have a quite wild structure. Let us see two examples.

Example 4.7. We take again the elliptic fibration $S \stackrel{\pi}{\to} C$ of Example 4.3. Let $T \subset S$ be a small transverse disc centered at $t_0 \in F_0 \setminus \{q\}$. Then, because the holonomy is trivial, $\widehat{L_t} = L_t$ for every t. We have already seen that $L_{t_0} = F_0 \setminus \{q\}$, and obviously for $t \neq t_0$, L_t is the smooth elliptic curve through t. The covering tube V_T is simply $\pi^{-1}(\pi(T)) \setminus \{q\}$. Remark that its central fiber is simply connected, but the other fibers are not. All the fibers of U_T are isomorphic to $\mathbb{C}$ (in fact, one can see that $U_T \simeq T \times \mathbb{C}$). The map $F_T : U_T \to V_T$, therefore, is injective on the central fiber, but not on the other ones.



To see better what is happening, take the basepoints $q_T(T) \subset V_T$ and consider the preimage $F_T^{-1}(q_T(T)) \subset U_T$. This preimage has infinitely many components: one of them is $p_T(T) \subset U_T$, and each other one is the graph over $T \setminus \{t_0\}$ of a 6-valued section of U_T. This follows from the fact that the monodromy of the elliptic fibration around a fiber of type II has order 6 [BPV, V.10]. The map F_T sends this 6-valued graph to $q_T(T \setminus \{t_0\})$, as a regular 6-fold covering. There is a "virtual" ramification of order 6 over $q_T(t_0)$, which is however pushed-off U_T, to the point at infinity of the central fiber.

Example 4.8. We take again an elliptic fibration $S \xrightarrow{\pi} C$, but now with a fiber $\pi^{-1}(c_0) = F_0$ of Kodaira type I_1, i.e. a rational curve with a node q. As before, V_T coincides with $\pi^{-1}(\pi(T)) \setminus \{q\}$, but now the central fiber is isomorphic to $\mathbb{C}^*$. Again $U_T \simeq T \times \mathbb{C}$. The map $F_T : U_T \to V_T$ is a $\mathbb{Z}$-covering over the central fiber, a $\mathbb{Z}^2$-covering over the other fibers. The preimage of $q_T(T)$ by F_T has still infinitely many components. One of them is $p_T(T)$. Some of them are graphs of (1-valued) sections over T, passing through the (infinitely many) points of $F_T^{-1}(q_T(t_0))$. But most of them are graphs of ∞-valued sections over $T \setminus \{t_0\}$ (like the graph of the logarithm). Indeed, the monodromy of the elliptic fibration around a fiber of type I_1 has infinite order [BPV, V.10]. If $t \neq t_0$, then $F_T^{-1}(q_T(t))$ is a lattice in $\widetilde{L_t} \simeq \mathbb{C}$, with generators 1 and $\lambda(t) \in \mathbb{H}$. For $t \to t_0$, this second generator diverges to $+i\infty$, and the lattices reduces to $\mathbb{Z} = F_T^{-1}(q_T(t_0))$. The monodromy acts as $(n, m\lambda(t)) \mapsto (n+m, m\lambda(t))$. Then each connected component of $F_T^{-1}(q_T(T))$ intersects $\widetilde{L_t}$ either at a single point $(n, 0)$, fixed by the monodromy, or along an orbit $(n + m\mathbb{Z}, m\lambda(t))$, $m \neq 0$.

More examples concerning elliptic fibrations can be found in [Br4].

Remark 4.9. As we recalled in Section 2, similar constructions of U_T and V_T have been done, respectively, by Il'yashenko [Il1] and Suzuki [Suz], in the case where the ambient manifold X is a Stein manifold. However, the Stein case is much simpler than the compact Kähler one. Indeed, the meromorphic maps $g : W \dashrightarrow X$ with which we work are automatically *holomorphic* if X is Stein. Thus, in the Stein case there are no vanishing ends, i.e. $L_p = L_p^0$ for every p and leaves of $\mathcal{F}$ = leaves of $\mathcal{F}^0$. Then the maps π_T and Π_T are *holomorphic* immersions of V_T and U_T into X^0 (and so V_T and U_T are Riemann Domains over X^0). Also, our unparametrized Hartogs extension lemma still holds in the Stein case, but with a much simpler proof, because we do not need to worry about "rational bubbles" arising in Bishop's Theorem.

In fact, there is a common framework for the Stein case and the compact Kähler case: the framework of *holomorphically convex* (not necessarily compact) Kähler manifolds. Indeed, the only form of compactness that we need, in this Section and also in the next one, is the following: for every compact $K \subset X$, there exists a (larger) compact $\hat{K} \subset X$ such that every holomorphic disc in X with boundary in K is fully contained in $\hat{K}$. This property is obviously satisfied by any holomorphically convex Kähler manifold, with $\hat{K}$ equal to the usual holomorphically convex hull of K.

A more global point of view on holonomy tubes and covering tubes will be developed in the last Section, on parabolic foliations.

4.4 Rational quasi-fibrations

We conclude this Section with a result which can be considered as an analog, in our context, of the classical Reeb Stability Theorem for real codimension one foliations [CLN].

Proposition 4.10. *Let X be a compact connected Kähler manifold and let $\mathcal{F}$ be a foliation by curves on X. Suppose that there exists a rational leaf L_p (i.e., $\widetilde{L_p} = \mathbb{P}$). Then all the leaves are rational. Moreover, there exists a compact connected Kähler manifold Y, $\dim Y = \dim X - 1$, a meromorphic map $B : X \dashrightarrow Y$, and Zariski open and dense subsets $X_0 \subset X$, $Y_0 \subset Y$, such that:*

(i) B is holomorphic on X_0 and $B(X_0) = Y_0$;
(ii) $B : X_0 \to Y_0$ is a proper submersive map, all of whose fibers are smooth rational curves, leaves of $\mathcal{F}$.

Proof. It is sufficient to verify that all the leaves are rational; then the second part follows by standard arguments of complex analytic geometry, see e.g. [CaP].

By connectivity, it is sufficient to prove that, given a covering tube U_T, if some fiber is rational then all the fibers are rational. We can work, equivalently, with the holonomy tube V_T. Now, such a property was actually already verified in the proof of Proposition 4.6, in the form of "nonexistence of vanishing cycles". Indeed, the set of rational fibers of V_T is obviously open. To see that it is also closed, take a fiber $\widehat{L_t}$ approximated by fibers $\widehat{L_{t_n}} \simeq \mathbb{P}$. Take an embedded cycle $\Gamma \subset \widehat{L_t}$, approximated by cycles $\Gamma_n \subset \widehat{L_{t_n}}$. Each Γ_n bounds in $\widehat{L_{t_n}}$ *two* discs, one on each side. As in the proof of Proposition 4.6, we obtain that Γ also bounds in $\widehat{L_t}$ *two* discs, one on each side. Hence $\widehat{L_t}$ is rational. □

Such a foliation will be called **rational quasi-fibration**. A meromorphic map B as in Proposition 4.10 is sometimes called *almost holomorphic*, because the image of its indeterminacy set is a proper subset of Y, of positive codimension, contained in $Y \setminus Y_0$. If $\dim X = 2$ then B is necessarily holomorphic, and the foliation is a rational fibration (with possibly some singular fibers). In higher dimensions one may think that the foliation is obtained from a rational fibration by a meromorphic transformation which does not touch generic fibers (like flipping along a codimension two subset).

Note that, as the proof shows, for a rational quasi-fibration every holonomy tube and every covering tube is isomorphic to $T \times \mathbb{P}$, provided that the transversal T is sufficiently small (every $\mathbb{P}$-fibration is locally trivial).

There are certainly many interesting issues concerning rational quasi-fibrations, but basically this is a chapter of Algebraic Geometry. In the following, we shall forget about them, and we will concentrate on foliations with parabolic and hyperbolic leaves.

5 A convexity property of covering tubes

Let X be a compact Kähler manifold, of dimension n, and let $\mathcal{F}$ be a foliation by curves on X, different from a rational quasi-fibration. Fix a transversal $T \subset X^0$ to $\mathcal{F}^0$, and consider the covering tube U_T over T, with projection $P_T : U_T \to T$, section $p_T : T \to U_T$, and meromorphic immersion $\Pi_T : U_T \dashrightarrow X$. Each fiber of U_T is either $\mathbb{D}$ or $\mathbb{C}$.

We shall establish in this Section, following [Br2] and [Br3], a certain convexity property of U_T, which later will allow us to apply to U_T the results of Section 2 of Nishino and Yamaguchi.

We fix also an embedded closed disc $S \subset T$ ($S \simeq \overline{\mathbb{D}}$, and the embedding in T is holomorphic up to the boundary), and we denote by U_S, P_S, p_S, Π_S the corresponding restrictions. Set $\partial U_S = P_S^{-1}(\partial S)$. We shall assume that S satisfies the following properties:

(a) U_S, as a subset of U_T, intersects $Indet(\Pi_T)$ along a discrete subset, necessarily equal to $Indet(\Pi_S)$, and ∂U_S does not intersect $Indet(\Pi_T)$;
(b) for every $z \in \partial S$, the area of the fiber $P_S^{-1}(z)$ is infinite.

In (b), the area is computed with respect to the pull-back by Π_S of the Kähler form ω of X. Without loss of generality, we take ω real analytic. We will see later that these assumptions (a) and (b) are "generic", in a suitable sense.

Theorem 5.1. *For every compact subset $K \subset \partial U_S$ there exists a real analytic bidimensional torus $\Gamma \subset \partial U_S$ such that:*

(i) Γ is transverse to the fibers of $\partial U_S \xrightarrow{P_S} \partial S$, and cuts each fiber $P_S^{-1}(z)$, $z \in \partial S$, along a circle $\Gamma(z)$ which bounds a disc $D(z)$ which contains $K \cap P_S^{-1}(z)$ and $p_S(z)$;
(ii) Γ is the boundary of a real analytic Levi-flat hypersurface $M \subset U_S$, filled by a real analytic family of holomorphic discs D^θ, $\theta \in \mathbb{S}^1$; each D^θ is the image of a section $s^\theta : S \to U_S$, holomorphic up to the boundary, with $s^\theta(\partial S) \subset \Gamma$;
(iii) M bounds in U_S a domain Ω, which cuts each fiber $P_S^{-1}(z)$, $z \in S$, along a disc $\Omega(z)$ which contains $p_S(z)$ ($\Omega(z) = D(z)$ when $z \in \partial S$).

This statement should be understood as expressing a variant of Hartogs-convexity [Ran, II.2], in which the standard Hartogs figure is replaced by $p_S(S) \cup (\cup_{z \in \partial S} D(z))$, and its envelope is replaced by Ω. By choosing a large compact K, condition (i) says that $\cup_{z \in \partial S} D(z)$ almost fills the lateral boundary ∂U_S; conditions (ii) and (iii) say that the family of discs $D(z)$, $z \in \partial S$, can be pushed inside S, getting a family of discs $\Omega(z)$, $z \in S$, in such a way that the boundaries $\partial\Omega(z)$, $z \in S$, vary with z in a "holomorphic" manner ("variation analytique" in the terminology of [Ya3]). It is a sort of "geodesic" convexity of U_S, in which the extremal points of the geodesic are replaced by Γ and the geodesic is replaced by M.

Theorem 5.1 will be proved by solving a nonlinear Riemann-Hilbert problem, see [For] and [AWe, Ch. 20] and reference therein for some literature on this subject. An important difference with this classical literature, however, is that the torus Γ is not fixed a priori: we want just to prove that *some* torus Γ, enclosing the compact K as in (i), is the boundary of a Levi-flat hypersurface M as in (ii); we do not pretend that *every* torus Γ has such a property. Even if, as we shall see below, we have a great freedom in the choice of Γ.

We shall use the continuity method. The starting point is the following special (but not so much) family of tori.

Lemma 5.2. *Given $K \subset \partial U_S$ compact, there exists a real analytic embedding*

$$F : \partial S \times \overline{\mathbb{D}} \to \partial U_S,$$

sending fibers to fibers, such that:

(i) $\partial S \times \{0\}$ is sent to $p_S(\partial S) \subset \partial U_S$;
(ii) $\partial S \times \{|w| = t\}$, $t \in (0,1]$, is sent to a real analytic torus $\Gamma_t \subset \partial U_S$ transverse to the fibers of P_S, so that for every $z \in \partial S$, $\Gamma_t(z) = \Gamma_t \cap P_S^{-1}(z)$ is a circle bounding a disc $D_t(z)$ containing $p_S(z)$;
(iii) $D_1(z)$ contains $K \cap P_S^{-1}(z)$, for every $z \in \partial S$;
(iv) for every $t \in (0,1]$ the function

$$\mathbf{a}_t : \partial S \to \mathbb{R}^+ \quad , \qquad \mathbf{a}_t(z) = \mathrm{area}(D_t(z))$$

is constant (the constant depending on t, of course).

Proof. Recall that the area in the fibers is computed with respect to the pull-back of the Kähler form ω. Because the fibers over ∂S have infinite area, we can certainly find a smooth torus $\Gamma' \subset \partial U_S$ which encloses K and $p_S(\partial S)$, and such that all the discs $D'(z)$, bounded by $\Gamma'(z)$, have the same area, say equal to k. We may approximate Γ' with a real analytic torus Γ''; the corresponding discs $D''(z)$ have now variable area, but close to k, say between k and $k + \varepsilon$.

For every $z \in \partial S$ we have in $\overline{D''(z)} \setminus p_S(z)$ a canonical foliation by circles, the standard circles under the uniformisation $(\overline{D''(z)}, p_S(z)) \simeq (\overline{\mathbb{D}}, 0)$. For every $t \in (0,1]$, let $\Gamma_t(z)$ be the circle of that foliation which bounds a disc of area equal to kt. Then, because all the data (Γ'', ω,...) are real analytic, the union $\Gamma_t = \cup_{z \in \partial S} \Gamma_t(z)$ is a real analytic torus, and these tori glue together in a real analytic way, producing the map F. If the initial perturbation is sufficiently small, Γ_1 encloses K. And the function $\mathbf{a}_t$ is constantly equal to kt. □

Given F as in Lemma 5.2, we shall say that a real analytic embedding

$$G : S \times \overline{\mathbb{D}} \to U_S$$

is a **Levi-flat extension** of F if G sends fibers to fibers and:

(i) $G(S \times \{0\}) = p_S(S)$;
(ii) $G(S \times \{|w| = t\})$, $t \in (0,1]$, is a real analytic Levi-flat hypersurface $M_t \subset U_S$ with boundary Γ_t, filled by images of holomorphic sections over S with boundary values in Γ_t.

Our aim is to construct such a G. Then $\Gamma = \Gamma_1$ and $M = M_1$ gives Theorem 5.1.

The continuity method consists in analyzing the set of those $t_0 \in (0,1]$ such that a similar G can be constructed over $S \times \overline{\mathbb{D}(t_0)}$. We need to show that this set is nonempty, open and closed.

Nonemptyness is a consequence of classical results [For]. Just note that a neighbourhood of $p_S(S)$ can be embedded in $\mathbb{C}^2$, in such a way that P_S becomes the projection to the first coordinate, and $p_S(S)$ becomes the closed unit disc in the first axis. Hence Γ_t, t small, becomes a torus in $\partial\mathbb{D} \times \mathbb{C}$ enclosing $\partial\mathbb{D} \times \{0\}$. Classical results on the Riemann-Hilbert problem in $\mathbb{C}^2$ imply that, for $t_0 > 0$ sufficiently small, there exists a Levi-flat extension on $S \times \overline{\mathbb{D}(t_0)}$.

Openness is a tautology. By definition, a real analytic embedding defined on $S \times \overline{\mathbb{D}(t_0)}$ is in fact defined on $S \times \mathbb{D}(t_0 + \varepsilon)$, for some $\varepsilon > 0$, and obviously if G is a Levi-flat extension on $S \times \overline{\mathbb{D}(t_0)}$, then it is a Levi-flat extension also on $S \times \overline{\mathbb{D}(t_0 + \varepsilon')}$, for every $\varepsilon' < \varepsilon$.

The heart of the matter is closedness. In other words, we need to prove that:

if a Levi-flat extension exists on $S \times \mathbb{D}(t_0)$, then it exists also on $S \times \overline{\mathbb{D}(t_0)}$.

The rest of this Section is devoted to the proof of this statement.

5.1 Boundedness of areas

We shall denote by D_t^θ, $\theta \in \mathbb{S}^1$, the closed holomorphic discs filling M_t, $0 < t < t_0$. Each D_t^θ is the image of a section $s_t^\theta : S \to U_S$, holomorphic up to the boundary, with boundary values in Γ_t.

Consider the areas of these discs. These areas are computed with respect to $\Pi_S^*(\omega) = \omega_0$, which is a real analytic Kähler form on $U_S \setminus Indet(\Pi_S)$. Because $H^2(U_S \setminus Indet(\Pi_S), \mathbb{R}) = 0$ (for U_S is a contractible complex surface and $Indet(\Pi_S)$ is a discrete subset), this Kähler form is exact:

$$\omega_0 = d\lambda$$

for some real analytic 1-form λ on $U_S \setminus Indet(\Pi_S)$. If D_t^θ is disjoint from $Indet(\Pi_S)$, then its area $\int_{D_t^\theta} \omega_0$ is simply equal, by Stokes formula, to $\int_{\partial D_t^\theta} \lambda$. If D_t^θ intersects $Indet(\Pi_S)$, this is no more true, but still we have the inequality

$$\text{area}(D_t^\theta) = \int_{D_t^\theta} \omega_0 \leq \int_{\partial D_t^\theta} \lambda.$$

The reason is the following: by the meromorphic map Π_S the disc D_t^θ is mapped not really to a disc in X, but rather to a disc *plus* some rational bubbles coming from indeterminacy points of Π_S; then $\int_{\partial D_t^\theta} \lambda$ is equal to the area (in X) of the disc *plus* the areas of these rational bubbles, whence the inequality above. Remark that, by our standing assumptions, the boundary of D_t^θ is contained in ∂U_S and hence it is disjoint from $Indet(\Pi_S)$.

Now, the important fact is that, thanks to the crucial condition (iv) of Lemma 5.2, we may get a *uniform* bound of these areas.

Lemma 5.3. *There exists a constant $C > 0$ such that for every $t \in (0, t_0)$ and every $\theta \in \mathbb{S}^1$:*

$$\mathrm{area}(D_t^\theta) \leq C.$$

Proof. By the previous remarks, we just have to bound the integrals $\int_{\partial D_t^\theta} \lambda$. The idea is the following one. For t fixed the statement is trivial, and we need just to understand what happens for $t \to t_0$. Look at the curves $\partial D_t^\theta \subset \Gamma_t$. They are graphs of sections over ∂S. For $t \to t_0$ these graphs could oscillate more and more. But, using condition (iv) of Lemma 5.2, we will see that these oscillations do not affect the integral of λ. This would be evident if the tori Γ_t were lagrangian (i.e. $\omega_0|_{\Gamma_t} \equiv 0$, i.e. $\lambda|_{\Gamma_t}$ closed), so that the integrals of λ would have a cohomological meaning, not affected by the oscillations. Our condition (iv) of Lemma 5.2 expresses a sort of half-lagrangianity in the direction along which oscillations take place, and this is sufficient to bound the integrals.

Fix real analytic coordinates $(\varphi, \psi, r) \in \mathbb{S}^1 \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$ around Γ_{t_0} in ∂U_S such that:

(i) $P_S : \partial U_S \to \partial S$ is given by $(\varphi, \psi, r) \mapsto \varphi$;
(ii) $\Gamma_t = \{r = t - t_0\}$ for every t close to t_0.

Each curve ∂D_t^θ, $t < t_0$ close to t_0, is therefore expressed by

$$\partial D_t^\theta = \{\psi = h_t^\theta(\varphi),\ r = t - t_0\}$$

for some real analytic function $h_t^\theta : \mathbb{S}^1 \to \mathbb{S}^1$. Because the discs D_t^θ form a continuous family, all these functions h_t^θ have the same degree, and we may suppose that it is zero up to changing ψ to $\psi + \ell\varphi$.

The 1-form λ, restricted to ∂U_S, in these coordinates is expressed by

$$\lambda = a(\varphi, \psi, r)d\varphi + b(\varphi, \psi, r)d\psi + c(\varphi, \psi, r)dr$$

for suitable real analytic functions a, b, c on $\mathbb{S}^1 \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$. Setting $b_0(\varphi, r) = \int_{\mathbb{S}^1} b(\varphi, \psi, r)d\psi$, we can write $b(\varphi, \psi, r) = b_0(\varphi, r) + \frac{\partial b_1}{\partial \psi}(\varphi, \psi, r)$, for some real analytic function b_1 (the indefinite integral of $b - b_0$ along ψ), and therefore

$$\lambda = a_0(\varphi, \psi, r)d\varphi + b_0(\varphi, r)d\psi + c_0(\varphi, \psi, r)dr + db_1$$

with $a_0 = a - \frac{\partial b_1}{\partial \varphi}$ and $c_0 = c - \frac{\partial b_1}{\partial r}$.

Remark now that $b_0(\varphi, r)$ is just equal to $\int_{\partial D_t(z)} \lambda$, for $r = t - t_0$ and $\varphi =$ the coordinate of $z \in \partial S$. By Stokes formula, this is equal to the area of the disc $D_t(z)$, and by condition (iv) of Lemma 5.2 this does not depend on φ. That is, the function b_0 depends only on r, and not on φ:

$$b_0(\varphi, r) = b_0(r).$$

In particular, if we restrict λ to a torus Γ_t we obtain, up to an exact term, a 1-form

$$a_0(\varphi, \psi, t - t_0)d\varphi + b_0(t - t_0)d\psi$$

which is perhaps not closed (this would be the lagrangianity of Γ_t), but its component along ψ is closed. And note that the oscillations of the curves ∂D_t^θ are directed along ψ.

If we now integrate λ along ∂D_t^θ we obtain

$$\int_{\partial D_t^\theta} \lambda = \int_{\mathbb{S}^1} a_0(\varphi, h_t^\theta(\varphi), t - t_0)d\varphi + b_0(t - t_0) \cdot \int_{\mathbb{S}^1} \frac{\partial h_t^\theta}{\partial \varphi}(\varphi)d\varphi.$$

The first integral is bounded by $C = \sup |a_0|$, and the second integral is equal to zero because the degree of h_t^θ is zero. □

Take now any sequence of discs

$$D_n = D_{t_n}^{\theta_n}, \ n \in \mathbb{N},$$

with $t_n \to t_0$. Our next aim is to prove that $\{D_n\}$ converges (up to subsequencing) to some disc $D_\infty \subset U_S$, with boundary in Γ_{t_0}. The limit discs so obtained will be then glued together to produce the Levi-flat hypersurface M_{t_0}.

5.2 Convergence around the boundary

We firstly prove that everything is good around the boundary. Recall that every disc D_n is the image of a section $s_n = s_{t_n}^{\theta_n} : S \to U_S$ with boundary values in $\Gamma_n = \Gamma_{t_n}$.

Lemma 5.4. *There exists a neighbourhood $V \subset S$ of ∂S and a section*

$$s_\infty : V \to U_S$$

such that $s_n|_V$ converges uniformly to s_∞ (up to subsequencing).

Proof. We want to apply Bishop compactness theorem [Bis] [Chi] to the sequence of analytic subsets of bounded area $D_n \subset U_S$. This requires some care due to the presence of the boundary.

Let us work on some slightly larger open disc $S' \subset T$ containing the closed disc S. Every torus $\Gamma_t \subset U_{S'}$ has a neighbourhood $W_t \subset U_{S'}$ over which we have a well defined Schwarz reflection with respect to Γ_t (which is totally real and of half dimension in $U_{S'}$). Thus, the complex curve $D_t^\theta \cap W_t$ with boundary in Γ_t can be doubled to a complex curve without boundary A_t^θ, properly embedded in W_t. Moreover, using the fact that the tori Γ_t form a real analytic family up to t_0, we see that the size of the neighbourhoods W_t is uniformly bounded from below. That is, there exists a neighbourhood W of Γ_{t_0} in $U_{S'}$ which is contained in every W_t, for t sufficiently close to t_0, and therefore every A_t^θ restricts to a properly embedded complex curve in W, still denoted by A_t^θ. Set

$$\widehat{D}_t^\theta = D_t^\theta \cup A_t^\theta.$$

Because the Schwarz reflection respects the fibration of $U_{S'}$, it is clear that $\widehat{D}_t^\theta$ is still the image of a section $\widehat{s}_t^\theta$, defined over some open subset $R_t^\theta \subset S'$ which contains S. The area of A_t^θ is roughly the double of the area of $D_t^\theta \cap W$, and therefore the properly embedded analytic subsets $\widehat{D}_t^\theta \subset U_S \cup W$ also have uniformly bounded areas.

Having in mind this uniform extension of the discs D_t^θ into the neighbourhood W of Γ_{t_0}, we now apply Bishop Theorem to the sequence $\{D_n\}$. Remark that $\partial D_n \subset \Gamma_n$ cannot exit from W, as $n \to +\infty$, because Γ_n converges to Γ_{t_0}. Up to subsequencing, we obtain that $\{D_n\}$ Hausdorff-converges to a complex curve $D_\infty \subset U_S$ with boundary in Γ_{t_0}. Moreover, and taking into account that D_n are graphs over S, we see that D_∞ has a graph-type irreducible component plus, possibly, some vertical components. More precisely (compare with [Iv1, Prop. 3.1]):

(i) $D_\infty = D_\infty^0 \cup E_1 \cup \ldots \cup E_m \cup F_1 \cup \ldots \cup F_\ell$;
(ii) D_∞^0 is the image of a section $s_\infty : V \to U_S$, over some open subset $V \subset S$ which contains ∂S;
(iii) each E_j is equal to $P_S^{-1}(p_j)$, for some $p_j \in S \setminus \partial S$ (interior bubble);
(iv) each F_j is equal to the closure of a connected component of $P_S^{-1}(q_j) \setminus \Gamma_{t_0}(q_j)$, for some $q_j \in \partial S$ (boundary bubble);
(v) for every compact $K \subset V \setminus \{p_1, \ldots, p_m, q_1, \ldots, q_\ell\}$, $s_n|_K$ converges uniformly to $s_\infty|_K$, as $n \to +\infty$.

We have just to prove that there are no boundary bubbles, i.e. that the set $\{q_1, \ldots, q_\ell\}$ is in fact empty. Then the conclusion follows by taking a smaller V, which avoids interior bubbles.

Consider the family of Levi-flat hypersurfaces $M_t \subset U_S$ with boundary Γ_t, for $t < t_0$. Each M_t is a "lower barrier", which prevents the approaching of D_n to the bounded component of $P_S^{-1}(q) \setminus \Gamma_{t_0}(q)$, for every $q \in \partial S$. More precisely, for any compact R in that bounded component we may select $t_1 < t_0$ such that $\cup_{0 \le t < t_1} \Gamma_t$ contains R, and so $\cup_{0 \le t < t_1} M_t$ is a neighbourhood of R in U_S. For n sufficiently large (so that $t_n > t_1$), $D_n \subset M_{t_n}$ is disjoint from that neighbourhood of R. Hence the sequence D_n cannot accumulate to the bounded component of $P_S^{-1}(q) \setminus \Gamma_{t_0}(q)$.

But neither D_n can accumulate to the unbounded component of $P_S^{-1}(q) \setminus \Gamma_{t_0}(q)$, because that component has infinite area, by our standing assumptions. Therefore, as desired,

$$\{q_1, \dots, q_\ell\} = \emptyset.$$

$\square$

Remark that by the same barrier argument we have also

$$\{p_1, \dots, p_m\} = \emptyset$$

but this fact will not be used below. The proof above shows in fact the following: there is a maximal V over which s_∞ is defined, and the image $s_\infty(V)$ in U_S is a properly embedded complex curve. For every $z \in V$ the sequence $s_n(z)$ is convergent to $s_\infty(z)$, whereas for every $z \notin V$ the sequence $s_n(z)$ is divergent in the fiber $P_S^{-1}(z)$.

5.3 Convergence on the interior

In order to extend the convergence above from V to the full S, we need to use the map Π_S into X. Consider the discs

$$f_n = \Pi_S \circ s_n : S \to X$$

in the compact Kähler manifold X. They have bounded area, and, once a time, we apply to them Bishop compactness theorem [Bis] [Iv1, Prop.3.1]. We obtain a holomorphic map

$$f_\infty : S \cup B \to X$$

which obviously coincides with $\Pi_S \circ s_\infty$ on the neighbourhood V of ∂S of Lemma 5.4. The set B is a union of trees of rational curves, each one attached to some point of S outside V. We will prove that $f_\infty|_S$ can be lifted to U_S, providing the extension of the section s_∞ to the full S.

The map f_∞ is an immersion around ∂S. Let us even suppose that it is an embedding (anyway, this is true up to moving a little ∂S inside S, and this does not affect the following reasoning). In some sufficiently smooth tubular neighbourhood $X_0 \subset X$ of $f_\infty(\partial S)$, we have a properly embedded complex surface with boundary Y, given by the image by Π_S of a neighbourhood of $s_\infty(\partial S)$ in U_S. The boundary ∂Y of Y in X_0 is filled by the images by Π_S of part of the tori Γ_t, t close to t_0; denote them by Γ_t' (with a good choice of X_0, each Γ_t' is a real annulus). Thus f_n, n large, sends S to a disc in X whose (embedded) boundary is contained in $\Gamma_n' = \Gamma_{t_n}'$, and f_∞ sends $S \cup B$ to a disc with rational bubbles in X whose (embedded) boundary is contained in $\Gamma_\infty' = \Gamma_{t_0}'$. Inspired by [IvS], but avoiding any infinite dimensional tool due to our special context, we now prove that f_∞ and f_n, for some large n, can be holomorphically interpolated by discs with boundaries in ∂Y.

Lemma 5.5. *There exists a complex surface with boundary W, a proper map $\pi : W \to \mathbb{D}$, a holomorphic map $g : W \to X$, such that:*

(i) for every $w \neq 0$, the fiber $W_w = \pi^{-1}(w)$ is isomorphic to S, and g sends that fiber to a disc in X with boundary in ∂Y;
(ii) for some $e \neq 0$, g coincides on $W_e = \pi^{-1}(e)$ with f_n, for some n (large);
(iii) $W_0 = \pi^{-1}(0)$ is isomorphic to $S \cup B$, and g on that fiber coincides with f_∞.

Proof. Let us work on the complex manifold $\widehat{X} = X \times \mathbb{D}(t_0, \varepsilon)$, where the second factor is a small disc in $\mathbb{C}$ centered at t_0. The real surfaces Γ'_t in X_0 can be seen as a single real analytic submanifold of dimension three Γ' in $\widehat{X}_0 = X_0 \times \mathbb{D}(t_0, \varepsilon)$, by considering Γ'_t as a subset of $X_0 \times \{t\}$. Remark that Γ' is totally real. Similarly, the discs $f_n(S)$ can be seen as discs in $X \times \{t_n\} \subset \widehat{X}$, and the disc with bubbles $f_\infty(S \cup B)$ can be seen as a disc with bubbles in $X \times \{t_0\} \subset \widehat{X}$; all these discs have boundaries in Γ'. In $\widehat{X}_0$ we also have a complex submanifold of dimension three with boundary $\widehat{Y} = Y \times \mathbb{D}(t_0, \varepsilon)$, which is "half" of the complexification of Γ'.

Around the circle $f_\infty(\partial S) \subset \widehat{X}$, we may find holomorphic coordinates $z_1, \ldots, z_{n+1}$, with $|z_j| < \delta$ for $j \leq n$, $1 - \delta < |z_{n+1}| < 1 + \delta$, such that:

(i) $f_\infty(\partial S) = \{z_1 = \ldots = z_n = 0,\ |z_{n+1}| = 1\}$;
(ii) $\Gamma' = \{z_1 = \ldots = z_{n-2} = 0,\ Im\ z_{n-1} = Im\ z_n = 0,\ |z_{n+1}| = 1\}$;
(iii) $\widehat{Y} = \{z_1 = \ldots = z_{n-2} = 0,\ |z_{n+1}| \leq 1\}$.

We consider, in these cordinates, the Schwarz reflection $(z_1, \ldots, z_n, z_{n+1}) \mapsto (\bar{z}_1, \ldots \bar{z}_n, \frac{1}{\bar{z}_{n+1}})$. It is a antiholomorphic involution, which fixes in particular every point of Γ'. Using it, we may double a neighbourhood Z_0 of $f_\infty(S \cup B)$ in $\widehat{X}$: we take Z_0 and $\overline{Z}_0$ (i.e., Z_0 with the opposite complex structure), and we glue them together using the Schwarz reflection. Call Z this double of Z_0. Then Z naturally contains a tree of rational curves R_∞ which comes from doubling $f_\infty(S \cup B)$, because this last has boundary in the fixed point set of the Schwarz reflection. Similarly, each $f_n(S)$ doubles to a rational curve $R_n \subset Z$, close to R_∞ for n large. Moreover, in some neighbourhood $N \subset Z$ of the median circle of R_∞ (arising from $f_\infty(\partial S)$), we have a complex threefold $\widetilde{Y} \subset N$, arising by doubling $\widehat{Y}$ or by complexifying Γ', which contains $R_\infty \cap N$ and every $R_n \cap N$.

Now, the space of trees of rational curves in Z close to R_∞ has a natural structure of complex analytic space $\mathcal{R}$, see e.g. [CaP] or [IvS]. Those trees which, in N, are contained in $\widetilde{Y}$ form a complex analytic subspace $\mathcal{R}_0 \subset \mathcal{R}$. The curve R_n above correspond to points of $\mathcal{R}_0$ converging to a point corresponding to R_∞. Therefore we can find a disc in $\mathcal{R}_0$ centered at R_∞ and passing through some R_n. This gives a holomorphic family of trees of rational curves in Z interpolating R_∞ and R_n. Restricting to $Z_0 \subset Z$ and projecting to X, we obtain the desired family of discs g. □

Note the the doubling trick used in the previous lemma is not so far from the similar trick used in the proof of Lemma 3.3. In both cases, a problem concerning discs is reduced to a more tractable problem concerning rational curves.

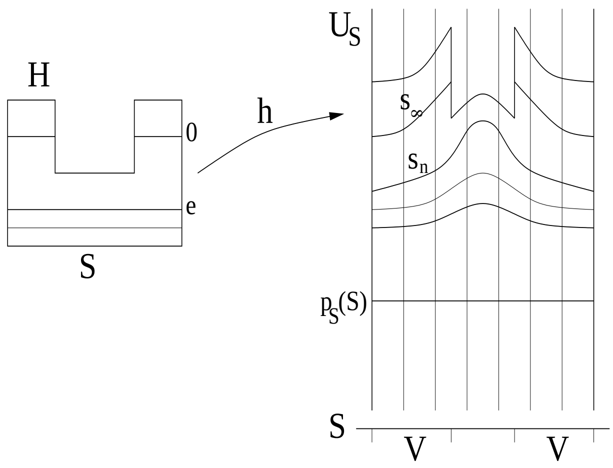
The map g can be lifted to U_S around ∂W and around W_e, thanks to properties (i) and (ii). In this way, and up to a reparametrization, we obtain an embedding

$$h : H \to U_S,$$

where $H = \{(z,w) \in S \times \mathbb{D} \mid z \in V \text{ or } |w-e| < \varepsilon\}$ (for some $\varepsilon > 0$ small), such that:

(i) $h(\cdot, w)$ is a section of U_S over S (if $|w-e| < \varepsilon$) or over V (if $|w-e| \geq \varepsilon$);
(ii) $h(z,e) = s_n(z)$ for every $z \in S$;
(iii) $h(z,0) = s_\infty(z)$ for every $z \in V$.

(note, however, that generally speaking the section $h(\cdot, w)$ has not boundary values in some torus Γ_t, when $w \neq 0, e$). In some sense, we are in a situation similar to the one already encountered in the construction of covering tubes in Section 4, but rotated by 90 degrees.



Consider now the meromorphic immersion $\Pi_S \circ h : H \dashrightarrow X$. By [Iv1], this map can be meromorphically extended to the envelope $S \times \mathbb{D}$, and clearly this extension is still a meromorphic immersion. Each vertical fiber $\{z\} \times \mathbb{D}$ is sent to a disc tangent to the foliation $\mathcal{F}$, and possibly passing through $Sing(\mathcal{F})$. But for every $z \in V$ we already have, by construction, that such a disc can be lifted to U_S. By our definition and construction of U_S, it then follows that the same holds for every $z \in S$: every intersection point with $Sing(\mathcal{F})$ is a vanishing end. Hence the full family $S \times \mathbb{D}$ can be lifted to U_S, or in other words the embedding $h : H \to U_S$ can be extended to $\widehat{h} : S \times \mathbb{D} \to U_S$.

Take now $\widehat{h}(\cdot, 0)$: it is a section over S which extends s_∞. Thus, the section s_∞ from Lemma 5.4 can be extended from V to S, and the sequence of discs $D_n \subset U_S$ uniformly converges to $D_\infty = s_\infty(S)$.

5.4 Construction of the limit Levi-flat hypersurface

Let us resume. We are assuming that our Levi-flat extension exists over $S \times \mathbb{D}(t_0)$, providing an embedded real analytic family of Levi-flat hypersurfaces $M_t \subset U_S$ with boundaries Γ_t, $t < t_0$. Given any sequence of holomorphic discs $D_{t_n}^{\theta_n} \subset M_{t_n}$, $t_n \to t_0$, we have proved that (up to subsequencing) $D_{t_n}^{\theta_n}$ converges uniformly to some disc D_∞ with $\partial D_\infty \subset \Gamma_{t_0}$. Given any point $p \in \Gamma_{t_0}$, we may choose the sequence $D_{t_n}^{\theta_n}$ so that ∂D_∞ will contain p. It remains to check that all the discs so constructed glue together in a real analytic way, giving M_{t_0}, and that this M_{t_0} glues to M_t, $t < t_0$, in a real analytic way, giving the Levi-flat extension over $S \times \overline{\mathbb{D}(t_0)}$.

This can be seen using a Lemma from [BeG, §5]. It says that if D is an embedded disc in a complex surface Y with boundary in a real analytic totally real surface $\Gamma \subset Y$, and if the winding number (Maslov index) of Γ along ∂D is zero, then D belongs to a unique embedded real analytic family of discs D^ε, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $D^0 = D$, with boundaries in Γ (incidentally, in our real analytic context this can be easily proved by the doubling argument used in Lemma 5.5, which reduces the statement to the well known fact that a smooth rational curve of zero selfintersection belongs to a unique local fibration by smooth rational curves). Moreover, if Γ is moved in a real analytic way, then the family D^ε also moves in a real analytic way.

For our discs $D_t^\theta \subset M_t$, $t < t_0$, the winding number of Γ_t along ∂D_t^θ is zero. By continuity of this index, if D_∞ is a limit disc then the winding number of Γ_{t_0} along ∂D_∞ is also zero. Thus, D_∞ belongs to a unique embedded real analytic family D_∞^ε, with $\partial D_\infty^\varepsilon \subset \Gamma_{t_0}$. This family can be deformed, real analytically, to a family D_t^ε with $\partial D_t^\varepsilon \subset \Gamma_t$, for every t close to t_0. When $t = t_n$, such a family $D_{t_n}^\varepsilon$ necessarily contains $D_{t_n}^{\theta_n}$, and thus coincides with $D_{t_n}^\theta$ for θ in a suitable interval around θ_n. Hence, for every $t < t_0$ the family D_t^ε coincides with D_t^θ, for θ in a suitable interval.

In this way, for every limit disc D_∞ we have constructed a piece

$$\bigcup_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)} D_\infty^\varepsilon$$

of our limit M_{t_0}, this piece is real analytic and glues to M_t, $t < t_0$, in a real analytic way.

Because each $p \in \Gamma_{t_0}$ belongs to some limit disc D_∞, we have completed in this way our construction of the Levi-flat hypersurface M_{t_0}, and the proof of Theorem 5.1 is finished.

6 Hyperbolic foliations

We can now draw the first consequences of the convexity of covering tubes given by Theorem 5.1, still following [Br2] and [Br3].

As in the previous Section, let X be a compact Kähler manifold of dimension n, equipped with a foliation by curves $\mathcal{F}$ which is not a rational quasi-fibration. Let $T \subset X^0$ be local transversal to $\mathcal{F}^0$. We firstly need to discuss the pertinence of hypotheses (a) and (b) that we made at the beginning of Section 5.

Concerning (a), let us simply observe that $Indet(\Pi_T)$ is an analytic subset of codimension at least two in U_T, and therefore its projection to T by P_T is a countable union of locally analytic subsets of positive codimension in T (a thin subset of T). Hypothesis (a) means that the closed disc $S \subset T$ is chosen so that it is not contained in that projection, and its boundary ∂S is disjoint from that projection.

Concerning (b), let us set

$$R = \{z \in T \mid \text{area}(P_T^{-1}(z)) < +\infty\}.$$

Lemma 6.1. *Either R is a countable union of analytic subsets of T of positive codimension, or $R = T$. In this second case, U_T is isomorphic to $T \times \mathbb{C}$.*

Proof. If $z \in R$, then $\widetilde{L_z}$ has finite area and, a fortiori, L_z^0 has finite area. In particular, L_z^0 is properly embedded in X^0: otherwise, L_z^0 should cut some foliated chart, where $\mathcal{F}^0$ is trivialized, along infinitely many plaques, and so L_z^0 would have infinite area. Because $X \setminus X^0$ is an analytic subset of X, the fact that $L_z^0 \subset X^0$ is properly embedded and with finite area implies that its closure $\overline{L_z^0}$ in X is a complex compact curve, by Bishop extension theorem [Siu] [Chi]. This closure coincides with $\overline{L_z}$, the closure of L_z.

The finiteness of the area of $\widetilde{L_z}$ implies also that the covering $\widetilde{L_z} \to L_z$ has finite order, i.e. the orbifold fundamental group of L_z is finite. By the previous paragraph, L_z can be compactified (as a complex curve) by adding a finite set. This excludes the case $\widetilde{L_z} = \mathbb{D}$: a finite quotient of the disc does not enjoy such a property. Also, the case $\widetilde{L_z} = \mathbb{P}$ is excluded by our standing assumptions. Therefore $\widetilde{L_z} = \mathbb{C}$. Moreover, again the finiteness of the orbifold fundamental group implies that L_z is equal to $\mathbb{C}$ with at most one multiple point. The closure $\overline{L_z}$ is a rational curve in X.

Now, by general principles of analytic geometry [CaP], rational curves in X (Kähler) constitute an analytic space with countable base, each irreducible component of which can be compactified by adding points corresponding to trees of rational curves. It follows easily from this fact that the subset

$$R' = \{z \in T \mid \overline{L_z} \text{ is rational}\}$$

is either a countable union of analytic subsets of T of positive codimension, or it is equal to the full T. Moreover, if A' is a component of R' then we can find a meromorphic map $A' \times \mathbb{P} \dashrightarrow X$ sending $\{z\} \times \mathbb{P}$ to $\overline{L_z}$, for every $z \in A'$ (compare with the arguments used at the beginning of the proof of Theorem 3.1).

Not every $z \in A'$, however, belongs to R: a point $z \in A'$ belongs to R if and only if among the points of $\{z\} \times \mathbb{P}$ sent to $Sing(\mathcal{F})$ only one does not correspond to a vanishing end of L_z^0, and at most one corresponds to a vanishing end of order $m \geq 2$. By a simple semicontinuity argument, $A' \cap R = A$ is an analytic subset of A'. Hence R also satisfies the above dichotomy.

Finally, if $R = T$ then we have a map $T \times \mathbb{P} \dashrightarrow X$ sending each fiber $\{z\} \times \mathbb{P}$ to $\overline{L_z}$ and (z, ∞) to the unique nonvanishing end of L_z^0. It follows that $U_T = T \times \mathbb{C}$. □

Let now $U \subset X$ be an open connected subset where $\mathcal{F}$ is generated by a holomorphic vector field $v \in \Theta(U)$, vanishing precisely on $Sing(\mathcal{F}) \cap U$. Set $U^0 = U \setminus (Sing(\mathcal{F}) \cap U)$, and consider the real function

$$F : U^0 \to [-\infty, +\infty)$$

$$F(q) = \log \|v(q)\|_{Poin}$$

where, as usual, $\|v(q)\|_{Poin}$ is the norm of $v(q)$ measured with the Poincaré metric on L_q. Recall that this "metric" is identically zero when L_q is parabolic, so that F is equal to $-\infty$ on the intersection of U^0 with parabolic leaves.

Proposition 6.2. *The function F above is either plurisubharmonic or identically $-\infty$.*

Proof. Let $T \subset U^0$ be a transversal to $\mathcal{F}^0$, and let U_T be the corresponding covering tube. Put on the fibers of U_T their Poincaré metric. The vector field v induces a nonsingular vertical vector field on U_T along $p_T(T)$, which we denote again by v. Due to the arbitrariness of T, and by a connectivity argument, we need just to verify that the function on T defined by

$$F(z) = \log \|v(p_T(z)\|_{Poin}$$

is either plurisubharmonic or identically $-\infty$. That is, the fiberwise Poincaré metric on U_T has a plurisubharmonic variation.

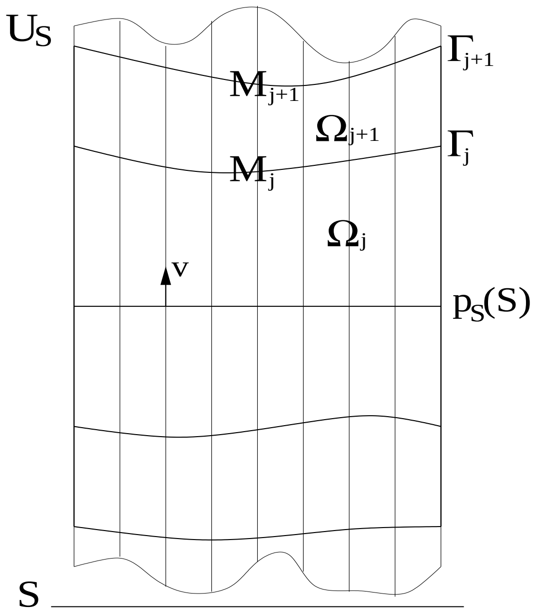
The upper semicontinuity of F being evident (see e.g. [Suz, §3] or [Kiz]), let us consider the submean inequality over discs in T.

Take a closed disc $S \subset T$ as in Theorem 5.1, i.e. satisfying hypotheses (a) and (b) of Section 5. By that Theorem, and by choosing an increasing sequence of compact subsets K_j in ∂U_S, we can find a sequence of relatively compact domains $\Omega_j \subset U_S$, $j \in \mathbb{N}$, such that:

(i) the relative boundary of Ω_j in U_S is a real analytic Levi-flat hypersurface $M_j \subset U_S$, with boundary $\Gamma_j \subset \partial U_S$, filled by a $\mathbb{S}^1$-family of graphs of holomorphic sections of U_S with boundary values in Γ_j;

(ii) for every $z \in S$, the fiber $\Omega_j(z) = \Omega_j \cap P_S^{-1}(z)$ is a disc, centered at $p_S(z)$; moreover, for $z \in \partial S$ we have $\cup_{j=1}^{+\infty} \Omega_j(z) = P_S^{-1}(z)$.

Note that one cannot hope that the exhaustive property in (ii) holds also for z in the interior of S.



We may apply to Ω_j, whose boundary is Levi-flat and hence pseudoconvex, the result of Yamaguchi discussed in Section 2, more precisely Proposition 2.2. It says that the function on S

$$F_j(z) = \log \|v(p_S(z)\|_{Poin(j)},$$

where $\|v(p_S(z)\|_{Poin(j)}$ is the norm with respect to the Poincaré metric on the disc $\Omega_j(z)$, is plurisubharmonic. Hence we have at the center 0 of $S \simeq \overline{\mathbb{D}}$ the submean inequality:

$$F_j(0) \leq \frac{1}{2\pi} \int_0^{2\pi} F_j(e^{i\theta}) d\theta.$$

We now pass to the limit $j \to +\infty$. For every $z \in \partial S$ we have $F_j(z) \to F(z)$, by the exhaustive property in (ii) above. Moreover, we may assume that $\Omega_j(z)$ is an increasing sequence for every $z \in \partial S$ (and in fact for every $z \in S$, but this is not important), so that $F_j(z)$ converges to $F(z)$ in a decreasing way, by the monotonicity property of the Poincaré metric. It follows that the boundary integral in the submean inequality above converges, as $j \to +\infty$, to $\frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) d\theta$ (which may be $-\infty$, of course).

Concerning $F_j(0)$, it is sufficient to observe that, obviously, $F(0) \leq F_j(0)$, because $\Omega_j(0) \subset P_S^{-1}(0)$, and so $F(0) \leq \liminf_{j\to+\infty} F_j(0)$. In fact, and because $\Omega_j(0)$ is increasing, $F_j(0)$ converges to some value c in $[-\infty, +\infty)$, but we may have the strict inequality $F(0) < c$ if $\Omega_j(0)$ do not exhaust $P_S^{-1}(0)$. Therefore the above submean inequality gives, at the limit,

$$F(0) \leq \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) d\theta$$

that is, the submean inequality for F on S.

Take now an arbitrary closed disc $S \subset T$, centered at some point $p \in T$. By Lemma 6.1 and the remarks before it, we may approximate S by a sequence of closed discs S_j with the same center p and satisfying moreover hypotheses (a) and (b) before Theorem 5.1 (unless $R = T$, but in that case $U_T = T \times \mathbb{C}$ and $F \equiv -\infty$). More precisely, if $\varphi : \overline{\mathbb{D}} \to T$ is a parametrization of S, $\varphi(0) = p$, then we may uniformly approximate φ by a sequence of embeddings $\varphi_j : \overline{\mathbb{D}} \to T$, $\varphi_j(0) = p$, such that $S_j = \varphi_j(\overline{\mathbb{D}})$ satisfies the assumptions of Theorem 5.1. Hence we have, by the previous arguments and for every j,

$$F(p) \leq \frac{1}{2\pi} \int_0^{2\pi} F(\varphi_j(e^{i\theta})) d\theta$$

and passing to the limit, using Fatou Lemma, and taking into account the upper semicontinuity of F, we finally obtain

$$F(p) \leq \limsup_{j\to+\infty} \frac{1}{2\pi} \int_0^{2\pi} F(\varphi_j(e^{i\theta})) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \limsup_{j\to+\infty} F(\varphi_j(e^{i\theta})) d\theta \leq$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} F(\varphi(e^{i\theta})) d\theta.$$

This is the submean inequality on an arbitrary disc $S \subset T$, and so F is, if not identically $-\infty$, plurisubharmonic. □

Because $U \setminus U^0$ is an analytic subset of codimension at least two, the above plurisubharmonic function F on U^0 admits a (unique) plurisubharmonic extension to the full U, given explicitely by

$$F(q) = \limsup_{p \in U^0,\ p \to q} F(p) \ , \quad q \in U \setminus U^0.$$

Proposition 6.3. *We have $F(q) = -\infty$ for every $q \in U \setminus U^0$.*

Proof. The vector field v on U has a local flow: a holomorphic map

$$\Phi : \mathcal{D} \to U$$

defined on a domain of the form

$$\mathcal{D} = \{(p,t) \in U \times \mathbb{C} \mid |t| < \rho(p)\}$$

for a suitable lower semicontinuous function $\rho : U \to (0, +\infty]$, such that $\Phi(p,0) = p$, $\frac{\partial\Phi}{\partial t}(p,0) = v(p)$, and $\Phi(p, t_1 + t_2) = \Phi(\Phi(p,t_1), t_2)$ whenever it makes sense. Standard results on ordinary differential equations show that we may choose the function ρ so that $\rho \equiv +\infty$ on $U \setminus U^0 =$ the zero set of v.

Take $q \in U \setminus U^0$ and $p \in U^0$ close to it. Then $\Phi(p, \cdot)$ sends the large disc $\mathbb{D}(\rho(p))$ into $L_p^0 \cap U^0$, and consequently into L_p, with derivative at 0 equal to $v(p)$. It follows, by monotonicity of the Poincaré metric, that the Poincaré norm of $v(p)$ is bounded from above by something like $\frac{1}{\rho(p)}$, which tends to 0 as $p \to q$. We therefore obtain that $\log \|v(p)\|_{Poin}$ tends to $-\infty$ as $p \to q$. $\square$

The functions $F : U \to [-\infty, +\infty)$ so constructed can be seen [Dem] as local weights of a (singular) hermitian metric on the tangent bundle $T_{\mathcal{F}}$ of $\mathcal{F}$, and by duality on the canonical bundle $K_{\mathcal{F}} = T^*_{\mathcal{F}}$. Indeed, if $v_j \in \Theta(U_j)$ are local generators of $\mathcal{F}$, for some covering $\{U_j\}$ of X, with $v_j = g_{jk} v_k$ for a multiplicative cocycle g_{jk} generating $K_{\mathcal{F}}$, then the functions $F_j = \log \|v_j\|_{Poin}$ are related by $F_j - F_k = \log |g_{jk}|$. The curvature of this metric on $K_{\mathcal{F}}$ is the current on X, of bidegree $(1,1)$, locally defined by $\frac{i}{\pi}\partial\bar{\partial} F_j$. Hence Propositions 6.2 and 6.3 can be restated in the following more intrinsic form, where we set

$$Parab(\mathcal{F}) = \{p \in X^0 \mid \widetilde{L_p} = \mathbb{C}\}.$$

Theorem 6.4. *Let X be a compact connected Kähler manifold and let $\mathcal{F}$ be a foliation by curves on X. Suppose that $\mathcal{F}$ has at least one hyperbolic leaf. Then the Poincaré metric on the leaves of $\mathcal{F}$ induces a hermitian metric on the canonical bundle $K_{\mathcal{F}}$ whose curvature is positive, in the sense of currents. Moreover, the polar set of this metric coincides with $Sing(\mathcal{F}) \cup Parab(\mathcal{F})$.*

A foliation with at least one hyperbolic leaf will be called **hyperbolic foliation**. The existence of a hyperbolic leaf (and the connectedness of X) implies that $\mathcal{F}$ is not a rational quasi-fibration, and all the local weights F introduced above are plurisubharmonic, and not identically $-\infty$.

Let us state two evident but important Corollaries.

Corollary 6.5. *The canonical bundle $K_{\mathcal{F}}$ of a hyperbolic foliation $\mathcal{F}$ is pseudoeffective.*

Corollary 6.6. *Given a hyperbolic foliation $\mathcal{F}$, the subset*

$$Sing(\mathcal{F}) \cup Parab(\mathcal{F})$$

is complete pluripolar in X.

We think that the conclusion of this last Corollary could be strengthened. The most optimistic conjecture is that $Sing(\mathcal{F}) \cup Parab(\mathcal{F})$ is even an *analytic subset* of X. At the moment, however, we are very far from proving such a fact (except when $\dim X = 2$, where special techniques are available, see [MQ1] and [Br1]). Even the *closedness* of $Sing(\mathcal{F}) \cup Parab(\mathcal{F})$ seems an open problem! This is related to the more general problem of the continuity of the leafwise Poincaré metric (which would give, in particular, the closedness of its polar set). Let us prove a partial result in this direction, following a rather standard hyperbolic argument [Ghy] [Br2]. Recall that a complex compact analytic space Z is *hyperbolic* if every holomorphic map of $\mathbb{C}$ into Z is constant [Lan].

Theorem 6.7. *Let $\mathcal{F}$ be a foliation by curves on a compact connected Kähler manifold M. Suppose that:*

(i) every leaf is hyperbolic;
(ii) $Sing(\mathcal{F})$ is hyperbolic.

Then the leafwise Poincaré metric is continuous.

Proof. Let us consider the function

$$F : U^0 \to \mathbb{R} \quad , \quad F(q) = \log \|v(q)\|_{Poin}$$

introduced just before Proposition 6.2. We have to prove that F is continuous (the continuity on the full U is then a consequence of Proposition 6.3). We have already observed, during the proof of Proposition 6.2, that F is upper semicontinuous, hence let us consider its lower semicontinuity.

Take $q_\infty \in U^0$ and take a sequence $\{q_n\} \subset U^0$ converging to q_∞. For every n, let $\varphi_n : \mathbb{D} \to X$ be a holomorphic map into $L_{q_n} \subset X$, sending $0 \in \mathbb{D}$ to $q_n \in L_{q_n}$. For every compact subset $K \subset \mathbb{D}$, consider

$$I_K = \{\|\varphi_n'(t)\| \mid t \in K, n \in \mathbb{N}\} \subset \mathbb{R}$$

(the norm of φ_n' is here computed with the Kähler metric on X).

Claim: I_K is a bounded subset of $\mathbb{R}$.

Indeed, in the opposite case we may find a subsequence $\{n_j\} \subset \mathbb{N}$ and a sequence $\{t_j\} \subset K$ such that $\|\varphi_{n_j}'(t_j)\| \to +\infty$ as $j \to +\infty$. By Brody's Reparametrization Lemma [Lan, Ch. III], we may reparametrize these discs so that they converge to an entire curve: there exists maps $h_j : \mathbb{D}(r_j) \to \mathbb{D}$, with $r_j \to +\infty$, such that the maps

$$\psi_j = \varphi_j \circ h_j : \mathbb{D}(r_j) \to X$$

converge, uniformly on compact subsets, to a nonconstant map

$$\psi : \mathbb{C} \to X.$$

It is clear that ψ is tangent to $\mathcal{F}$, more precisely $\psi'(t) \in T_{\psi(t)}\mathcal{F}$ whenever $\psi(t) \notin Sing(\mathcal{F})$, because each ψ_j has the same property. Moreover, by hypothesis (ii) we have that the image of ψ is not contained in $Sing(\mathcal{F})$. Therefore, $S = \psi^{-1}(Sing(\mathcal{F}))$ is a discrete subset of $\mathbb{C}$, and $\psi(\mathbb{C} \setminus S)$ is contained in some leaf L^0 of $\mathcal{F}^0$.

Take now $t_0 \in S$. It corresponds to a parabolic end of L^0. On a small compact disc B centered at t_0, $\psi|_B$ is uniform limit of $\psi_j|_B : B \to X$, which are maps into leaves of $\mathcal{F}$. If U_T is a covering tube associated to some transversal T cutting L^0, then the maps $\psi_j|_B$ can be lifted to U_T, in such a way that they converge on ∂B to some map which lifts $\psi|_{\partial B}$. The structure of U_T (absence of vanishing cycles) implies that, in fact, we have convergence on the full B, to a map which lifts $\psi|_B$. By doing so at every $t_0 \in S$, we see that $\psi : \mathbb{C} \to X$ can be fully lifted to U_T, i.e. $\psi(\mathbb{C})$ is contained in the leaf L of $\mathcal{F}$ obtained by completion of L^0. But this contradicts hypothesis (i), and proves the Claim.

The Claim implies now that, up to subsequencing, the maps $\varphi_n : \mathbb{D} \to X$ converge, uniformly on compact subsets, to some $\varphi_\infty : \mathbb{D} \to X$, with $\varphi_\infty(0) = q_\infty$. As before, we obtain $\varphi_\infty(\mathbb{D}) \subset L_{q_\infty}$.

Recall now the extremal propery of the Poincaré metric: if we write $\varphi_n'(0) = \lambda_n \cdot v(q_n)$, then $\|v(q_n)\|_{Poin} \leq \frac{1}{|\lambda_n|}$, and equality is atteined if φ_n is a uniformization of L_{q_n}. Hence, with this choice of $\{\varphi_n\}$, we see that

$$\|v(q_\infty)\|_{Poin} \leq \frac{1}{|\lambda_\infty|} = \lim_{n\to+\infty} \frac{1}{|\lambda_n|} = \lim_{n\to+\infty} \|v(q_n)\|_{Poin}$$

i.e. $F(q_\infty) \leq \lim_{n\to+\infty} F(q_n)$. Due to the arbitrariness of the initial sequence $\{q_n\}$, this gives the lower semicontinuity of F. $\square$

Of course, due to hypothesis (i) such a result says nothing about the possible closedness of $Sing(\mathcal{F}) \cup Parab(\mathcal{F})$, when $Parab(\mathcal{F})$ is not empty, but at least it leaves some hope. The above proof breaks down when there are parabolic leaves, because Brody's lemma does not allow to control where the limit entire curve ψ is located: even if each ψ_j passes through q_{n_j}, it is still possible that ψ does not pass through q_∞, because the points in $\psi_j^{-1}(q_{n_j})$ could exit from every compact subset of $\mathbb{C}$. Hence, the only hypothesis "L_{q_∞} is hyperbolic" (instead of "all the leaves are hyperbolic") is not sufficient to get a contradiction and prove the Claim. In other words, the (parabolic) leaf L appearing in the Claim above could be "far" from q_∞, but still could have some influence on the possible discontinuity of the leafwise Poincaré metric at q_∞.

The subset $Sing(\mathcal{F}) \cup Parab(\mathcal{F})$ being complete pluripolar, a natural question concerns the computation of its Lelong numbers. For instance, if these Lelong numbers were positive, then, by Siu Theorem [Dem], we should get that $Sing(\mathcal{F}) \cup Parab(\mathcal{F})$ is a countable union of analytic subsets, a substantial step toward the conjecture above. However, we generally expect that these Lelong numbers are zero, even when $Sing(\mathcal{F}) \cup Parab(\mathcal{F})$ is analytic.

Example 6.8. Let E be an elliptic curve and let $X = \mathbb{P} \times E$. Let $\alpha = f(z)dz$ be a meromorphic 1-form on $\mathbb{P}$, with poles $P = \{z_1, \ldots, z_k\}$ of orders $\{\nu_1, \ldots, \nu_k\}$. Consider the (nonsingular) foliation $\mathcal{F}$ on X defined by the (saturated) Kernel of the meromorphic 1-form $\beta = f(z)dz - dw$, i.e. by the differential equation $\frac{dw}{dz} = f(z)$. Then each fiber $\{z_j\} \times E$, $z_j \in P$, is a leaf of $\mathcal{F}$, whereas each other fiber $\{z\} \times E$, $z \notin P$, is everywhere transverse to $\mathcal{F}$. In [Br1] such a foliation is called *turbulent*. Outside the elliptic leaves $P \times E$, every leaf is a regular covering of $\mathbb{P} \setminus P$, by the projection $X \to \mathbb{P}$. Hence, if $k \geq 3$ then these leaves are hyperbolic, and their Poincaré metric coincides with the pull-back of the Poincaré metric on $\mathbb{P} \setminus P$.

Take a point $(z_j, w) \in P \times E = Parab(\mathcal{F})$. Around it, the foliation is generated by the holomorphic and nonvanishing vector field $v = f(z)^{-1}\frac{\partial}{\partial z} + \frac{\partial}{\partial w}$, whose z-component has at $z = z_j$ a zero of order ν_j. The weight $F = \log \|v\|_{Poin}$ is nothing but than the pull-back of $\log \|f(z)^{-1}\frac{\partial}{\partial z}\|_{Poin}$, where the norm is measured in the Poincaré metric of $\mathbb{P} \setminus P$. Recalling that the Poincaré metric of the punctured disc $\mathbb{D}^*$ is $\frac{idz \wedge d\bar{z}}{|z|^2(\log |z|^2)^2}$, we see that F is something like

$$\log |z - z_j|^{\nu_j - 1} - \log \big| \log |z - z_j|^2 \big|.$$

Hence the Lelong number along $\{z_j\} \times E$ is positive if and only if $\nu_j \geq 2$, which can be considered as an "exceptional" case; in the "generic" case $\nu_j = 1$ the pole of F along $\{z_j\} \times E$ is a weak one, with vanishing Lelong number.

Remark 6.9. We used the convexity property stated by Theorem 5.1 as a substitute of the Stein property required by the results of Nishino, Yamaguchi, Kizuka discussed in Section 2. One could ask if, after all, such a convexity property can be used to prove the Steinness of U_T, when T is Stein. If the ambient manifold X is Stein, instead of Kähler compact, Il'yashenko proved in [Il1] and [Il2] (see Section 2) that indeed U_T is Stein, using Cartan-Thullen-Oka convexity theory over Stein manifolds. See also [Suz] for a similar approach to V_T, [Br6] for some result in the case of projective manifolds, close in spirit to [Il2], and [Nap] and [Ohs] for related results in the case of proper fibrations by curves.

For instance, suppose that all the fibers of U_T are hyperbolic, and that the fiberwise Poincaré metric is of class C^2. Then we can take the function $\psi : U_T \to \mathbb{R}$ defined by $\psi = \psi_0 + \varphi \circ P_T$, where $\psi_0(q)$ is the squared hyperbolic distance (in the fiber) between q and the basepoint $p_T(P_T(q))$, and $\varphi : T \to \mathbb{R}$ is a strictly plurisubharmonic exhaustion of T. A computation shows that ψ is strictly plurisubharmonic (thanks to the plurisubharmonic variation of the fiberwise Poincaré metric on U_T), and being also exhaustive we deduce that U_T is Stein. Probably, this can be done also if the fiberwise Poincaré metric is less regular, say C^0. But when there are parabolic fibers such a simple argument cannot work, because ψ is no more exhaustive (one can try perhaps to use a renormalization argument like the one used in the proof of Theorem 2.3). However, if *all* the fibers are parabolic then we shall see later that U_T is a product $T \times \mathbb{C}$ (if T is small), and hence it is Stein.

A related problem concerns the existence on U_T of holomorphic functions which are not constant on the fibers. By Corollary 6.5, $K_{\mathcal{F}}$ is pseudoeffective, if $\mathcal{F}$ is hyperbolic. Let us assume a little more, namely that it is effective. Then any nontrivial section of $K_{\mathcal{F}}$ over X can be lifted to U_T, giving a holomorphic section of the relative canonical bundle of the fibration. As in Lemmata 2.5 and 2.6, this section can be integrated along the (simply connected and pointed) fibers, giving a holomorphic function on U_T not constant on generic fibers.

7 Extension of meromorphic maps from line bundles

In order to generalize Corollary 6.5 to cover (most) parabolic foliations, we need an extension theorem for certain meromorphic maps. This is done in the present Section, following [Br5].

7.1 Volume estimates

Let us firstly recall some results of Dingoyan [Din], in a slightly simplified form due to our future use.

Let V be a connected complex manifold, of dimension n, and let ω be a smooth closed semipositive $(1,1)$-form on V (e.g., the pull-back of a Kähler form by some holomorphic map from V). Let $U \subset V$ be an open subset, with boundary ∂U compact in V. Suppose that the mass of ω^n on U is finite: $\int_U \omega^n < +\infty$. We look for some condition ensuring that also the mass on V is finite: $\int_V \omega^n < +\infty$. In other words, we look for the boundedness of the ω^n-volume of the ends $V \setminus U$.

Set

$$P_\omega(V,U) = \{\varphi : V \to [-\infty, +\infty) \text{ u.s.c.} \mid dd^c\varphi + \omega \geq 0,\ \varphi|_U \leq 0\}$$

where u.s.c. means upper semicontinuous, and the first inequality is in the sense of currents. This first inequality defines the so-called *ω-plurisubharmonic functions*. Note that locally the space of ω-plurisubharmonic functions can be identified with a translation of the space of the usual plurisubharmonic functions: locally the form ω admits a smooth potential ϕ ($\omega = dd^c\phi$), and so φ is ω-plurisubharmonic is and only if $\varphi + \phi$ is plurisubharmonic. In this way, most local problems on ω-plurisubharmonic functions can be reduced to more familiar problems on plurisubharmonic functions.

Remark that the space $P_\omega(V,U)$ is not empty, for it contains at least all the constant nonpositive functions on V.

Suppose that $P_\omega(V,U)$ satisfies the following condition:

(A) the functions in $P_\omega(V,U)$ are locally uniformly bounded from above: for every $z \in V$ there exists a neighbourhood $V_z \subset V$ of z and a constant c_z such that $\varphi|_{V_z} \leq c_z$ for every $\varphi \in P_\omega(V,U)$.

Then we can introduce the upper envelope

$$\Phi(z) = \sup_{\varphi \in P_\omega(V,U)} \varphi(z) \qquad \forall z \in V$$

and its upper semicontinuous regularization

$$\Phi^*(z) = \limsup_{w \to z} \Phi(w) \qquad \forall z \in V.$$

The function

$$\Phi^* : V \to [0, +\infty)$$

is identically zero on U, upper semicontinuous, and ω-plurisubharmonic (Brelot-Cartan [Kli]), hence it belongs to the space $P_\omega(V,U)$. Moreover, by results of Bedford and Taylor [BeT] [Kli] the wedge product $(dd^c\Phi^* + \omega)^n$ is well defined, as a locally finite measure on V, and it is identically zero outside $\overline{U}$:

$$(dd^c\Phi^* + \omega)^n \equiv 0 \qquad \text{on } V \setminus \overline{U}.$$

Indeed, let $B \subset V \setminus \overline{U}$ be a ball around which ω has a potential. Let $P_\omega(B, \Phi^*)$ be the space of ω-plurisubharmonic functions ψ on B such that $\limsup_{z \to w} \psi(z) \leq \Phi^*(w)$ for every $w \in \partial B$. Let Ψ^* be the regularized upper envelope of the family $P_\omega(B, \Phi^*)$ (which is bounded from above by the maximum principle). Remark that $\Phi^*|_B$ belongs to $P_\omega(B, \Phi^*)$, and so $\Psi^* \geq \Phi^*$ on B. By [BeT], Ψ^* satisfies the homogeneous Monge-Ampère equation $(dd^c\Psi^* + \omega)^n = 0$ on B, with Dirichlet boundary condition $\limsup_{z \to w} \Psi^*(z) = \Phi^*(w)$, $w \in \partial B$ ("balayage"). Then the function $\widetilde{\Phi}^*$ on V, which is equal to Ψ^* on B and equal to Φ^* on $V \setminus \overline{B}$, still belongs to $P_\omega(V,U)$, and it is everywhere not smaller than Φ^*. Hence, by definition of Φ^*, we must have $\widetilde{\Phi}^* = \Phi^*$, i.e. $\Phi^* = \Psi^*$ on B and so Φ^* satisfies the homogeneous Monge-Ampère equation on B.

Suppose now that the following condition is also satisfied:

(B) $\Phi^* : V \to [0, +\infty)$ is exhaustive on $V \setminus U$: for every $c > 0$, the subset $\{\Phi^* < c\} \setminus U$ is relatively compact in $V \setminus U$.

Roughly speaking, this means that the function Φ^* solves on $V \setminus \overline{U}$ the homogeneous Monge-Ampère equation, with boundary conditions 0 on ∂U and $+\infty$ on the "boundary at infinity" of $V \setminus U$.

Theorem 7.1. [Din] *Under assumptions (A) and (B), the ω^n-volume of V is finite:*

$$\int_V \omega^n < +\infty.$$

Proof. The idea is that, using Φ^*, we can push all the mass of ω^n on $V \setminus U$ to the compact set ∂U. Note that we certainly have

$$\int_V (dd^c\Phi^* + \omega)^n < +\infty$$

because, after decomposing $V = U \cup (V \setminus U) \cup \partial U$, we have:

(i) on U, $\Phi^* \equiv 0$ and $\int_U \omega^n$ is finite by standing assumptions;

(ii) on $V \setminus U$, $(dd^c\Phi^* + \omega)^n \equiv 0$;
(iii) ∂U is compact (but, generally speaking, ∂U is charged by the measure $(dd^c\Phi^* + \omega)^n$).

Hence the theorem follows from the next inequality.

Lemma 7.2. [Din, Lemma 4]

$$\int_V \omega^n \leq \int_V (dd^c\Phi^* + \omega)^n.$$

Proof. More generally, we shall prove that for every $k = 0, \ldots, n-1$:

$$\int_V (dd^c\Phi^* + \omega)^{k+1} \wedge \omega^{n-k-1} \geq \int_V (dd^c\Phi^* + \omega)^k \wedge \omega^{n-k},$$

so that the desired inequality follows by concatenation. We can decompose the integral on the left hand side as

$$\int_V (dd^c\Phi^* + \omega)^k \wedge \omega^{n-k} + \int_V dd^c\Phi^* \wedge (dd^c\Phi^* + \omega)^k \wedge \omega^{n-k-1}$$

and so we need to prove that, setting $\eta = (dd^c\Phi^* + \omega)^k \wedge \omega^{n-k-1}$, we have

$$I = \int_V dd^c\Phi^* \wedge \eta \geq 0.$$

Here all the wedge products are well defined, because Φ^* is locally bounded, and moreover η is a closed positive current of bidegree $(n-1, n-1)$ [Kli].

Take a sequence of smooth functions $\chi_n : \mathbb{R} \to [0,1]$, $n \in \mathbb{N}$, such that $\chi_n(t) = 1$ for $t \leq n$, $\chi_n(t) = 0$ for $t \geq n+1$, and $\chi_n'(t) \leq 0$ for every t. Thus, for every $z \in V$ we have $(\chi_n \circ \Phi^*)(z) = 0$ for $n \leq \Phi^*(z) - 1$ and $(\chi_n \circ \Phi^*)(z) = 1$ for $n \geq \Phi^*(z)$. Hence it is sufficient to prove that

$$I_n = \int_V (\chi_n \circ \Phi^*) \cdot dd^c\Phi^* \wedge \eta \geq 0$$

for every n. By assumption (B), the support of $\chi_n \circ \Phi^*$ intersects $V \setminus U$ along a compact subset. Moreover, Φ^* is identically zero on U. Thus, the integrand above has compact support in V, as well as $(\chi_n \circ \Phi^*) \cdot d^c\Phi^* \wedge \eta$. Hence, by Stokes formula,

$$I_n = -\int_V d(\chi_n \circ \Phi^*) \wedge d^c\Phi^* \wedge \eta = -\int_V (\chi_n' \circ \Phi^*) \cdot d\Phi^* \wedge d^c\Phi^* \wedge \eta.$$

Now, $d\Phi^* \wedge d^c\Phi^*$ is a positive current, and its product with η is a positive measure. From $\chi_n' \leq 0$ we obtain $I_n \geq 0$, for every n. □

This inequality completes the proof of the theorem. □

7.2 Extension of meromorphic maps

As in [Din, §6], we shall use the volume estimate of Theorem 7.1 to get an extension theorem for certain meromorphic maps into Kähler manifolds.

Consider the following situation. It is given a compact connected Kähler manifold X, of dimension n, and a line bundle L on X. Denote by E the total space of L, and by $\Sigma \subset E$ the graph of the null section of L. Let $U_\Sigma \subset E$ be a connected (tubular) neighbourhood of Σ, and let Y be another compact Kähler manifold, of dimension m.

Theorem 7.3. [Br5] *Suppose that L is not pseudoeffective. Then any meromorphic map*

$$f : U_\Sigma \setminus \Sigma \dashrightarrow Y$$

extends to a meromorphic map

$$\bar{f} : U_\Sigma \dashrightarrow Y.$$

Before the proof, let us make a link with [BDP]. In the special case where X is projective, and not only Kähler, the non pseudoeffectivity of L translates into the existence of a covering family of curves $\{C_t\}_{t\in B}$ on X such that $L|_{C_t}$ has negative degree for every $t \in B$ [BDP]. This means that the normal bundle of Σ in E has negative degree on every $C_t \subset \Sigma \simeq X$. Hence the restriction of E over C_t is a surface E_t which contains a compact curve Σ_t whose selfintersection is negative, and thus contractible to a normal singularity. By known results [Siu] [Iv1], every meromorphic map from $U_t \setminus \Sigma_t$ (U_t being a neighbourhood of Σ_t in E_t) into a compact Kähler manifold can be meromorphically extended to U_t. Because the curves C_t cover the full X, this is sufficient to extends from $U_\Sigma \setminus \Sigma$ to U_Σ.

Of course, if X is only Kähler then such a covering family of curves could not exist, and we need a more global approach, which avoids any restriction to curves. Even in the projective case, this seems a more natural approach than evoking [BDP].

Proof. We begin with a simple criterion for pseudoeffectivity, analogous to the well known fact that a line bundle is ample if and only if its dual bundle has strongly pseudoconvex neighbourhoods of the null section. Recall that an open subset W of a complex manifold E is *locally pseudoconvex in E* if for every $w \in \partial W$ there exists a neighbourhood $U_w \subset E$ of w such that $W \cap U_w$ is Stein.

Lemma 7.4. *Let X be a compact connected complex manifold and let L be a line bundle on X. The following two properties are equivalent:*

(i) L is pseudoeffective;
(ii) in the total space E^* of the dual line bundle L^* there exists a neighbourhood $W \neq E^*$ of the null section Σ^* which is locally pseudoconvex in E^*.

Proof. The implication (i) $\Rightarrow$ (ii) is quite evident. If h is a (singular) hermitian metric on L with positive curvature [Dem], then in a local trivialization $E|_{U_j} \simeq U_j \times \mathbb{C}$ the unit ball is expressed by $\{(z,t) \mid |t| < e^{h_j(z)}\}$, where $h_j : U_j \to [-\infty, +\infty)$ is the plurisubharmonic weight of h. In the dual local trivialization $E^*|_{U_j} \simeq U_j \times \mathbb{C}$, the unit ball of the dual metric is expressed by $\{(z,s) \mid |s| < e^{-h_j(z)}\}$. The plurisubharmonicity of h_j gives (and is equivalent to) the Steinness of such an open subset of $U_j \times \mathbb{C}$ (recall Hartogs Theorem on Hartogs Tubes mentioned in Section 2). Hence we get (ii), with W equal to the unit ball in E^*.

The implication (ii) $\Rightarrow$ (i) is not more difficult. Let $W \subset E^*$ be as in (ii). On E^* we have a natural $\mathbb{S}^1$-action, which fixes Σ^* and rotates each fiber. For every $\vartheta \in \mathbb{S}^1$, let W_ϑ be the image of W by the action of ϑ. Then

$$W' = \cap_{\vartheta \in \mathbb{S}^1} W_\vartheta$$

is still a nontrivial locally pseudoconvex neighbourhood of Σ^*, for local pseudoconvexity is stable by intersections. For every $z \in X$, W' intersects the fiber E^*_z along an open subset which is $\mathbb{S}^1$-invariant, a connected component of which is therefore a disc W^0_z centered at the origin (possibly $W^0_z = E^*_z$ for certain z, but not for all); the other components are annuli around the origin. Using the local pseudoconvexity of W', i.e. its Steinness in local trivializations $E^*|_{U_j} \simeq U_j \times \mathbb{C}$, it is easy to see that these annuli and discs cannot merge when z moves in X. In other words,

$$W'' = \cup_{z \in X} W^0_z$$

is a connected component of W', and of course it is still a nontrivial pseudoconvex neighbourhood of Σ^*. We may use W'' as unit ball for a metric on L^*. As in the first part of the proof, the corresponding dual metric on L has positive curvature, in the sense of currents. □

Consider now, in the space $U_\Sigma \times Y$, the graph Γ_f of the meromorphic map $f : U^0_\Sigma = U_\Sigma \setminus \Sigma \dashrightarrow Y$. By definition of meromorphicity, Γ_f is an irreducible analytic subset of $U^0_\Sigma \times Y \subset U_\Sigma \times Y$, whose projection to U^0_Σ is proper and generically bijective. It may be singular, and in that case we replace it by a resolution of its singularities, still denoted by Γ_f. The (new) projection

$$\pi : \Gamma_f \to U^0_\Sigma$$

is a proper map, and it realizes an isomorphism between $\Gamma_f \setminus Z$ and $U^0_\Sigma \setminus B$, for suitable analytic subsets $Z \subset \Gamma_f$ and $B \subset U^0_\Sigma$, with B of codimension at least two.

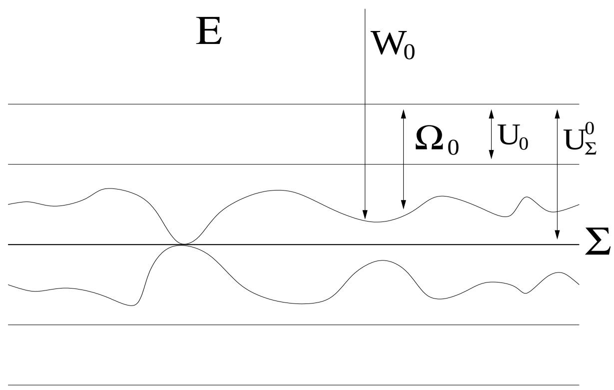
The manifold $U_\Sigma \times Y$ is Kähler. The Kähler form restricted to the graph of f and pulled-back to its resolution gives a smooth, semipositive, closed $(1,1)$-form ω on Γ_f. Fix a smaller (tubular) neighbourhood U'_Σ of Σ, and set $U_0 = U_\Sigma \setminus \overline{U'_\Sigma}$, $U = \pi^{-1}(U_0) \subset \Gamma_f$. Up to restricting a little the initial U_Σ, we may assume that the $\omega^{n'}$-volume of the shell U is finite ($n' = n+1 = \dim \Gamma_f$). Our aim is to prove that

$$\int_{\Gamma_f} \omega^{n'} < +\infty.$$

Indeed, this is the volume of the graph of f. Its finiteness, together with the analyticity of the graph in $U_\Sigma^0 \times Y$, imply that the closure of that graph in $U_\Sigma \times Y$ is still an analytic subset of dimension n', by Bishop's extension theorem [Siu] [Chi]. This closure, then, is the graph of the desired meromorphic extension $\bar{f} : U_\Sigma \dashrightarrow Y$.

We shall apply Theorem 7.1. Hence, consider the space $P_\omega(\Gamma_f, U)$ of ω-plurisubharmonic functions on Γ_f, nonpositive on U, and let us check conditions (A) and (B) above, at the beginning of this Section.

Consider the open subset $\Omega \subset \Gamma_f$ where the functions of $P_\omega(\Gamma_f, U)$ are locally uniformly bounded from above. It contains U, and it is a general fact that it is locally pseudoconvex in Γ_f [Din, §3]. Therefore $\Omega' = \Omega \cap (\Gamma_f \setminus Z)$ is locally pseudoconvex in $\Gamma_f \setminus Z$. Its isomorphic projection $\pi(\Omega')$ is therefore locally pseudoconvex in $U_\Sigma^0 \setminus B$. Classical characterizations of pseudoconvexity [Ran, II.5] show that $\Omega_0 = \text{interior}\{\pi(\Omega') \cup B\}$ is locally pseudoconvex in U_Σ^0. From $\Omega \supset U$ we also have $\Omega_0 \supset U_0$.



Take now in E the neighbourhood of infinity $W_0 = \Omega_0 \cup (E \setminus U_\Sigma)$. Because $E \setminus \Sigma$ is naturally isomorphic to $E^* \setminus \Sigma^*$, the isomorphism exchanging null sections and sections at infinity, we can see W_0 as an open subset of E^*, so that $W = W_0 \cup \Sigma^*$ is a neighbourhood of Σ^*, locally pseudoconvex in E^*. Because L is not pseudoeffective by assumption, Lemma 7.4 says that $W = E^*$. That is, $\Omega_0 = U_\Sigma^0$.

This implies that the original $\Omega \subset \Gamma_f$ contains, at least, $\Gamma_f \setminus Z$. But, by the maximum principle, a family of ω-plurisubharmonic functions locally bounded outside an analytic subset is automatically bounded also on the same analytic subset. Therefore $\Omega = \Gamma_f$, and condition (A) of Theorem 7.1 is fulfilled.

Condition (B) is simpler [Din, §4]. We just have to exhibit a ω-plurisubharmonic function on Γ_f which is nonpositive on U and exhaustive on $\Gamma_f \setminus U$. On U^0_Σ we take the function

$$\psi(z) = -\log dist(z, \Sigma)$$

where $dist(\cdot, \Sigma)$ is the distance function from Σ, with respect to the Kähler metric ω_0 on U_Σ. Classical estimates (Takeuchi) give $dd^c\psi \geq -C \cdot \omega_0$, for some positive constant C. Thus

$$dd^c(\psi \circ \pi) \geq -C \cdot \pi^*(\omega_0) \geq -C \cdot \omega$$

because $\omega \geq \pi^*(\omega_0)$. Hence $\frac{1}{C}(\psi \circ \pi)$ is ω-plurisubharmonic on Γ_f. For a sufficiently large $C' > 0$, $\frac{1}{C}(\psi \circ \pi) - C'$ is moreover negative on U, and it is exhaustive on $\Gamma_f \setminus U$. Thus condition (B) is fulfilled.

Finally we can apply Theorem 7.1, obtain the finiteness of the volume of the graph of f, and conclude the proof of the theorem. □

Remark 7.5. We think that Theorem 7.3 should be generalizable to the following "nonlinear" statement: if U_Σ is any Kähler manifold and $\Sigma \subset U_\Sigma$ is a compact hypersurface whose normal bundle is not pseudoeffective, then any meromorphic map $f : U_\Sigma \setminus \Sigma \dashrightarrow Y$ (Y Kähler compact) extends to $\bar{f} : U_\Sigma \dashrightarrow Y$. The difficulty is to show that a locally pseudoconvex subset of $U^0_\Sigma = U_\Sigma \setminus \Sigma$ like Ω_0 in the proof above can be "lifted" in the total space of the normal bundle of Σ, preserving the local pseudoconvexity.

8 Parabolic foliations

We can now return to foliations.

As usual, let X be a compact connected Kähler manifold, $\dim X = n$, and let $\mathcal{F}$ be a foliation by curves on X different from a rational quasi-fibration. Let us start with some general remarks, still following [Br5].

8.1 Global tubes

The construction of holonomy tubes and covering tubes given in Section 4 can be easily modified by replacing the transversal $T \subset X^0$ with the full X^0. That is, all the holonomy coverings $\widehat{L_p}$ and universal coverings $\widetilde{L_p}$, $p \in X^0$, can be glued together, without the restriction $p \in T$. The results are complex manifolds $V_\mathcal{F}$ and $U_\mathcal{F}$, of dimension $n+1$, equipped with submersions

$$Q_\mathcal{F} : V_\mathcal{F} \to X^0 \quad , \qquad P_\mathcal{F} : U_\mathcal{F} \to X^0$$

sections

$$q_\mathcal{F} : X^0 \to V_\mathcal{F} \quad , \qquad p_\mathcal{F} : X^0 \to U_\mathcal{F}$$

and meromorphic maps

$$\pi_{\mathcal{F}} : V_{\mathcal{F}} \dashrightarrow X \quad , \qquad \Pi_{\mathcal{F}} : U_{\mathcal{F}} \dashrightarrow X$$

such that, for any transversal $T \subset X^0$, we have $Q_{\mathcal{F}}^{-1}(T) = V_T$, $q_{\mathcal{F}}|_T = q_T$, $\pi_{\mathcal{F}}|_{Q_{\mathcal{F}}^{-1}(T)} = \pi_T$, etc.

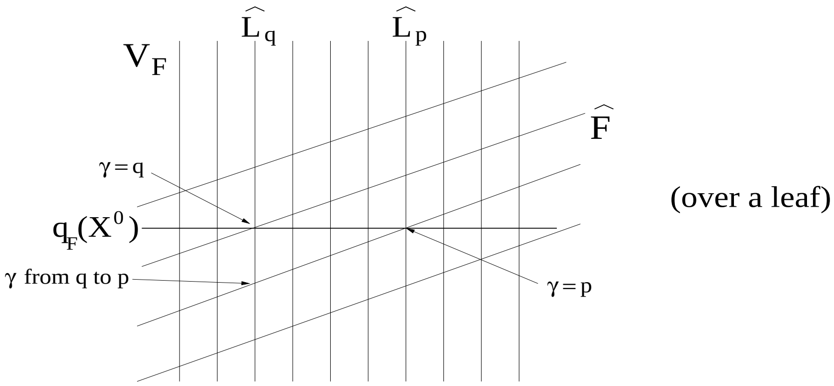
Remark that if $D \subset X^0$ is a small disc contained in some leaf L_p of $\mathcal{F}$, $p \in D$, then $Q_{\mathcal{F}}^{-1}(D)$ is naturally isomorphic to the product $\widehat{L_p} \times D$: for every $q \in D$, $\widehat{L_q}$ is the same as $\widehat{L_p}$, but with a different basepoint. More precisely, thinking to points of $\widehat{L_q}$ as equivalence classes of paths starting at q, we see that for every $q \in D$ the isomorphism between $\widehat{L_q}$ and $\widehat{L_p}$ is completely canonical, once D is fixed and because D is contractible. This means that D can be lifted, in a canonical way, to a foliation by discs in $Q_{\mathcal{F}}^{-1}(D)$, transverse to the fibers. In this way, by varying D in X^0, we get in the full space $V_{\mathcal{F}}$ a nonsingular foliation by curves $\widehat{\mathcal{F}}$, which projects by $Q_{\mathcal{F}}$ to $\mathcal{F}^0$.

If $\gamma : [0,1] \to X^0$ is a loop in a leaf, $\gamma(0) = \gamma(1) = p$, then this foliation $\widehat{\mathcal{F}}$ permits to define a *monodromy map* of the fiber $\widehat{L_p}$ into itself. This monodromy map is just the covering transformation of $\widehat{L_p}$ corresponding to γ (which may be trivial, if the holonomy of the foliation along γ is trivial).

In a similar way, in the space $U_{\mathcal{F}}$ we get a canonically defined nonsingular foliation by curves $\widetilde{\mathcal{F}}$, which projects by $P_{\mathcal{F}}$ to $\mathcal{F}^0$. And we have a fiberwise covering

$$F_{\mathcal{F}} : U_{\mathcal{F}} \to V_{\mathcal{F}}$$

which is a local diffeomorphism, sending $\widetilde{\mathcal{F}}$ to $\widehat{\mathcal{F}}$.



In the spaces $U_{\mathcal{F}}$ and $V_{\mathcal{F}}$ we also have the graphs of the sections $p_{\mathcal{F}}$ and $q_{\mathcal{F}}$. They are *not* invariant by the foliations $\widetilde{\mathcal{F}}$ and $\widehat{\mathcal{F}}$: in the notation above, with D in a leaf and $p, q \in D$, the basepoint $q_{\mathcal{F}}(q) \in \widehat{L_q}$ corresponds to the constant path $\gamma(t) \equiv q$, whereas the point of $\widehat{L_q}$ in the same leaf (of $\widehat{\mathcal{F}}$) of

$q_{\mathcal{F}}(p) \in \widehat{L_p}$ corresponds to the class of a path in D from q to p. In fact, the graphs $p_{\mathcal{F}}(X^0) \subset U_{\mathcal{F}}$ and $q_{\mathcal{F}}(X^0) \subset V_{\mathcal{F}}$ are hypersurfaces everywhere transverse to $\widetilde{\mathcal{F}}$ and $\widehat{\mathcal{F}}$.

A moment of reflection shows also the following fact: the normal bundle of the hypersurface $p_{\mathcal{F}}(X^0)$ in $U_{\mathcal{F}}$ (or $q_{\mathcal{F}}(X^0)$ in $V_{\mathcal{F}}$) is naturally isomorphic to $T_{\mathcal{F}}|_{X^0}$, the tangent bundle of the foliation restricted to X^0. That is, the manifold $U_{\mathcal{F}}$ (resp. $V_{\mathcal{F}}$) can be thought as an "integrated form" of the (total space of the) tangent bundle of the foliation, in which tangent lines to the foliation are replaced by universal coverings (resp. holonomy coverings) of the corresponding leaves. From this perspective, which will be useful below, the map $\Pi_{\mathcal{F}} : U_{\mathcal{F}} \dashrightarrow X$ is a sort of "skew flow" associated to $\mathcal{F}$, in which the "time" varies not in $\mathbb{C}$ but in the universal covering of the leaf.

Let us conclude this discussion with a trivial but illustrative example.

Example 8.1. Suppose $n = 1$, i.e. X is a compact connected curve and $\mathcal{F}$ is the foliation with only one leaf, X itself. The manifold $V_{\mathcal{F}}$ is composed by equivalence classes of paths in X, where two paths are equivalent if they have the same starting point and the same ending point (here holonomy is trivial!). Clearly, $V_{\mathcal{F}}$ is the product $X \times X$, $Q_{\mathcal{F}}$ is the projection to the first factor, $q_{\mathcal{F}}$ is the diagonal embedding of X into $X \times X$, and $\pi_{\mathcal{F}}$ is the projection to the second factor. Note that the normal bundle of the diagonal $\Delta \subset X \times X$ is naturally isomorphic to TX. The foliation $\widehat{\mathcal{F}}$ is the horizontal foliation, and note that its monodromy is trivial, corresponding to the fact that the holonomy of the foliation is trivial. The manifold $U_{\mathcal{F}}$ is the fiberwise universal covering of $V_{\mathcal{F}}$, with basepoints on the diagonal. It is *not* the product of X with the universal covering $\widetilde{X}$ (unless $X = \mathbb{P}$, of course). It is only a locally trivial $\widetilde{X}$-bundle over X. The foliation $\widetilde{\mathcal{F}}$ has nontrivial monodromy: if $\gamma : [0,1] \to X$ is a loop based at p, then the monodromy of $\widetilde{\mathcal{F}}$ along γ is the covering transformation of the fiber over p (i.e. the universal covering of X with basepoint p) associated to γ. The foliation $\widetilde{\mathcal{F}}$ can be described as the suspension of the natural representation $\pi_1(X) \to Aut(\widetilde{X})$ [CLN].

8.2 Parabolic foliations

After these preliminaries, let us concentrate on the class of **parabolic foliations**, i.e. let us assume that all the leaves of $\mathcal{F}$ are uniformised by $\mathbb{C}$. In this case, the Poincaré metric on the leaves is identically zero, hence quite useless. But our convexity result Theorem 5.1 still gives a precious information on covering tubes.

Theorem 8.2. *Let X be a compact connected Kähler manifold and let $\mathcal{F}$ be a parabolic foliation on X. Then the global covering tube $U_{\mathcal{F}}$ is a locally trivial $\mathbb{C}$-fibration over X^0, isomorphic to the total space of $T_{\mathcal{F}}$ over X^0, by an isomorphism sending $p_{\mathcal{F}}(X^0)$ to the null section.*

Proof. By the discussion above (local triviality of $U_{\mathcal{F}}$ along the leaves), the first statement is equivalent to say that, if $T \subset X^0$ is a small transversal (say, isomorphic to $\mathbb{D}^{n-1}$), then $U_T \simeq T \times \mathbb{C}$.

We use for this Theorem 2.3 of Section 2. We may assume that there exists an embedding $T \times \mathbb{D} \xrightarrow{j} U_T$ sending fibers to fibers and $T \times \{0\}$ to $p_T(T)$. Then we set

$$U_T^\varepsilon = U_T \setminus \{j(T \times \overline{\mathbb{D}(\varepsilon)})\}.$$

In order to apply Theorem 2.3, we need to prove that the fiberwise Poincaré metric on U_T^ε has a plurisubharmonic variation, for every $\varepsilon > 0$ small.

But this follows from Theorem 5.1 in exactly the same way as we did in Proposition 6.2 of Section 6. We just replace, in that proof, the open subsets $\Omega_j \subset U_S$ (for $S \subset T$ a generic disc) with

$$\Omega_j^\varepsilon = \Omega_j \setminus \{j(S \times \overline{\mathbb{D}(\varepsilon)})\}.$$

Then the fibration $\Omega_j^\varepsilon \to S$ is, for j large, a fibration by annuli, and its boundary in U_S has two components: one is the Levi flat M_j, and the other one is the Levi-flat $j(S \times \partial\mathbb{D}(\varepsilon))$. Then Theorem 2.1 of Section 2, or more simply the annular generalization of Proposition 2.2, gives the desired plurisubharmonic variation on Ω_j^ε, and then on U_S^ε by passing to the limit, and finally on U_T^ε by varying S.

Hence $U_T \simeq T \times \mathbb{C}$ and $U_{\mathcal{F}}$ is a locally trivial $\mathbb{C}$-fibration over X^0.

Let us now define explicitely the isomorphism between $U_{\mathcal{F}}$ and the total space $E_{\mathcal{F}}$ of $T_{\mathcal{F}}$ over X^0.

Take $p \in X^0$ and let $v_p \in E_{\mathcal{F}}$ be a point over p. Then v_p is a tangent vector to L_p at p, and it can be lifted to $\widetilde{L_p}$ as a tangent vector $\widetilde{v_p}$ at p. Suppose $\widetilde{v_p} \neq 0$. Then, because $\widetilde{L_p} \simeq \mathbb{C}$, $\widetilde{v_p}$ can be extended, in a uniquely defined way, to a complete holomorphic and nowhere vanishing vector field $\widetilde{v}$ on $\widetilde{L_p}$. If, instead, we have $\widetilde{v_p} = 0$, then we set $\widetilde{v} \equiv 0$. Take $q \in \widetilde{L_p}$ equal to the image of p by the time-one flow of $\widetilde{v}$. We have in this way defined a map $(E_{\mathcal{F}})_p \to \widetilde{L_p}$, $v_p \mapsto q$, which obviously is an isomorphism, sending the origin of $(E_{\mathcal{F}})_p$ to the basepoint of $\widetilde{L_p}$. In other words: because L_p is parabolic, we have a canonically defined isomorphism between $(T_pL_p, 0)$ and $(\widetilde{L_p}, p)$.

By varying p in X^0, we thus have a bijective map

$$E_{\mathcal{F}}|_{X^0} \to U_{\mathcal{F}}$$

sending the null section to $p_{\mathcal{F}}(X^0)$, and we need to verify that this map is *holomorphic*. This follows from the fact that $U_{\mathcal{F}}$ (and $E_{\mathcal{F}}$ also, of course) is a locally trivial fibration. In terms of the previous construction, we take a local transversal $T \subset X^0$ and a nowhere vanishing holomorphic section v_p, $p \in T$, of $E_{\mathcal{F}}$ over T. The previous construction gives a vertical vector field $\widetilde{v}$ on U_T, which is, on every fiber, complete holomorphic and nowhere vanishing, and moreover it is holomorphic along $p_T(T) \subset U_T$. After a trivialization

$U_T \simeq T \times \mathbb{C}$, sending $p_T(T)$ to $\{w = 0\}$, this vertical vector field $\widetilde{v}$ becomes something like $F(z,w)\frac{\partial}{\partial w}$, with F nowhere vanishing, $F(z,\cdot)$ holomorphic for every fixed z, and $F(\cdot,0)$ also holomorphic. The completeness on fibers gives that F is in fact *constant* on every fiber, i.e. $F = F(z)$, and so F is in fact fully holomorphic. Thus $\widetilde{v}$ is fully holomorphic on the tube. This means precisely that the above map $E_{\mathcal{F}}|_{X^0} \to U_{\mathcal{F}}$ is holomorphic. □

Example 8.3. Consider a foliation $\mathcal{F}$ generated by a global holomorphic vector field $v \in \Theta(X)$, vanishing precisely on $Sing(\mathcal{F})$. This means that $T_{\mathcal{F}}$ is the trivial line bundle, and $E_{\mathcal{F}} = X \times \mathbb{C}$. The compactness of X permits to define the *flow* of v

$$\Phi : X \times \mathbb{C} \to X$$

which sends $\{p\} \times \mathbb{C}$ to the orbit of v through p, that is to L_p^0 if $p \in X^0$ or to $\{p\}$ if $p \in Sing(\mathcal{F})$. Recalling that $L_p = L_p^0$ for a generic leaf, and observing that every L_p^0 is obviously parabolic, we see that $\mathcal{F}$ is a parabolic foliation. It is also not difficult to see that, in fact, $L_p = L_p^0$ for every leaf, i.e. there are no vanishing ends, and so the map

$$\Pi_{\mathcal{F}} : U_{\mathcal{F}} \to X^0$$

is everywhere holomorphic, with values in X^0. We have $U_{\mathcal{F}} = X^0 \times \mathbb{C}$ (by Theorem 8.2, which is however quite trivial in this special case), and the map $\Pi_{\mathcal{F}} : X^0 \times \mathbb{C} \to X^0$ can be identified with the restricted flow $\Phi : X^0 \times \mathbb{C} \to X^0$.

Remark 8.4. It is a general fact [Br3] that vanishing ends of a foliation $\mathcal{F}$ produce rational curves in X over which the canonical bundle $K_{\mathcal{F}}$ has negative degree. In particular, if $K_{\mathcal{F}}$ is algebraically nef (i.e. $K_{\mathcal{F}} \cdot C \geq 0$ for every compact curve $C \subset X$) then $\mathcal{F}$ has no vanishing end.

8.3 Foliations by rational curves

We shall say that a foliation by curves $\mathcal{F}$ is a **foliation by rational curves** if for every $p \in X^0$ there exists a rational curve $R_p \subset X$ passing through p and tangent to $\mathcal{F}$. This class of foliations should not be confused with the smaller class of rational quasi-fibrations: certainly a rational quasi-fibration is a foliation by rational curves, but the converse is in general false, because the above rational curves R_p can pass through $Sing(\mathcal{F})$ and so L_p (which is equal to R_p minus those points of $R_p \cap Sing(\mathcal{F})$ not corresponding to vanishing ends) can be parabolic or even hyperbolic. Thus the class of foliations by rational curves is transversal to our fundamental trichotomy rational quasi-fibrations / parabolic foliations / hyperbolic foliations.

A typical example is the radial foliation in the projective space $\mathbb{C}P^n$, i.e. the foliation generated (in an affine chart) by the radial vector field $\sum z_j \frac{\partial}{\partial z_j}$: it is a foliation by rational curves, but it is parabolic. By applying a birational map of the projective space, we can get also a hyperbolic foliation by rational curves.On the other hand, it is a standard exercise in bimeromorphic geometry to see that any foliation by rational curves can be transformed, by a bimeromorphic map, into a rational quasi-fibration. For instance, the radial foliation above can be transformed into a rational quasi-fibration, and even into a $\mathbb{P}$-bundle, by blowing-up the origin.

We have seen in Section 6 that the canonical bundle $K_{\mathcal{F}}$ of a hyperbolic foliation is always pseudoeffective. At the opposite side, for a rational quasi-fibration $K_{\mathcal{F}}$ is never pseudoeffective: its degree on a generic leaf (a smooth rational curve disjoint from $Sing(\mathcal{F})$) is equal to -2, and this prevents pseudo-effectivity. For parabolic foliations, the situation is mixed: the radial foliation in $\mathbb{C}P^n$ has canonical bundle equal to $\mathcal{O}(-1)$, which is not pseudoeffective; a foliation like in Example 8.3 has trivial canonical bundle, which is pseudoeffective. One can also easily find examples of parabolic foliations with ample canonical bundle, for instance most foliations arising from complete polynomial vector fields in $\mathbb{C}^n$ [Br4].

The following result, which combines Theorem 8.2 and Theorem 7.3, shows that most parabolic foliations have pseudoeffective canonical bundle.

Theorem 8.5. *Let $\mathcal{F}$ be a parabolic foliation on a compact connected Kähler manifold X. Suppose that its canonical bundle $K_{\mathcal{F}}$ is not pseudoeffective. Then $\mathcal{F}$ is a foliation by rational curves.*

Proof. Consider the meromorphic map

$$\Pi_{\mathcal{F}} : E_{\mathcal{F}}|_{X^0} \simeq U_{\mathcal{F}} \dashrightarrow X$$

given by Theorem 8.2. Because $Sing(\mathcal{F}) = X \setminus X^0$ has codimension at least two, such a map meromorphically extends [Siu] to the full space $E_{\mathcal{F}}$:

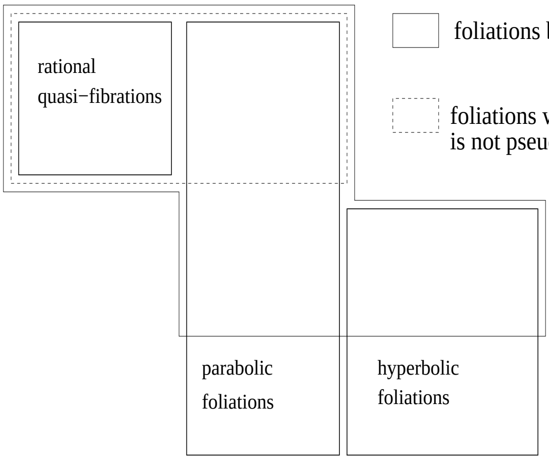
$$\Pi_{\mathcal{F}} : E_{\mathcal{F}} \dashrightarrow X.$$

The section at infinity of $E_{\mathcal{F}}$ is the same as the null section of $E^*_{\mathcal{F}}$, the total space of $K_{\mathcal{F}}$. If $K_{\mathcal{F}}$ is not pseudoeffective, then by Theorem 7.3 $\Pi_{\mathcal{F}}$ extends to the full $\overline{E_{\mathcal{F}}} = E_{\mathcal{F}} \cup \{$ section at $\infty\}$, as a meromorphic map

$$\overline{\Pi_{\mathcal{F}}} : \overline{E_{\mathcal{F}}} \dashrightarrow X.$$

By construction, $\overline{\Pi_{\mathcal{F}}}$ sends the rational fibers of $\overline{E_{\mathcal{F}}}$ to rational curves in X tangent to $\mathcal{F}$, which is therefore a foliation by rational curves. $\square$

Note that the converse to this theorem is not always true: for instance, a parabolic foliation like in Example 8.3 has trivial (pseudoeffective) canonical bundle, yet it can be a foliation by rational curves, for some special v. A parabolic foliation is a foliation by rational curves if and only if the meromorphic map $\Pi_{\mathcal{F}} : E_{\mathcal{F}} \dashrightarrow X$ introduced in the proof above extends to the section at infinity, and this can possibly occur even if $K_{\mathcal{F}}$ is pseudoeffective, or even ample.



We have now completed our analysis of positivity properties of the canonical bundle of a foliation, and their relation to uniformisation. We may resume the various inclusions of the various classes of foliations in the diagram above.

Let us discuss the classical case of fibrations.

Example 8.6. Suppose that $\mathcal{F}$ is a fibration over some base B, i.e. there exists a holomorphic map $f : X \to B$ whose generic fiber is a leaf of $\mathcal{F}$ (but there may be singular fibers, and even some higher dimensional fibers). Let g be the genus of a generic fiber, and suppose that $g \geq 1$. The relative canonical bundle of f is defined as

$$K_f = K_X \otimes f^*(K_B^{-1}).$$

It is related to the canonical bundle $K_{\mathcal{F}}$ of $\mathcal{F}$ by the relation

$$K_f = K_{\mathcal{F}} \otimes \mathcal{O}_X(D)$$

where D is an *effective* divisor which takes into account the possible ramifications of f along nongeneric fibers. Indeed, by adjunction along the leaves, we have $K_X = K_{\mathcal{F}} \otimes N^*_{\mathcal{F}}$, where $N^*_{\mathcal{F}}$ denotes the determinant conormal bundle of $\mathcal{F}$. If ω is a local generator of K_B, then $f^*(\omega)$ is a local section of $N^*_{\mathcal{F}}$ which vanishes along the ramification divisor D of f, hence $f^*(K_B) = N^*_{\mathcal{F}} \otimes \mathcal{O}_X(-D)$, whence the relation above.

Because $\mathcal{F}$ is not a foliation by rational curves, we have, by the Theorems above, that $K_{\mathcal{F}}$ is pseudoeffective, and therefore also K_f is pseudoeffective. In particular, $f_*(K_{\mathcal{F}})$ and $f_*(K_f)$ are "pseudoeffective sheaves" on B, in the sense that their degrees with respect to Kähler metrics on B are nonnegative. This must be compared with Arakelov's positivity theorem [Ara] [BPV, Ch. III]. But, as in Arakelov's results, something more can be said. Suppose that B is a curve (or restrict the fibration f over some curve in B) and let us distinguish between the hyperbolic and the parabolic case.

• $g \geq 2$. Then the pseudoeffectivity of $K_{\mathcal{F}}$ is realized by the leafwise Poincaré metric (Theorem 6.4). A subtle computation [Br2] [Br1] shows that this leafwise (or fiberwise) Poincaré metric has a *strictly* plurisubharmonic variation, unless the fibration is isotrivial. This means that if f is *not* isotrivial then the degree of $f_*(K_{\mathcal{F}})$ (and, a fortiori, the degree of $f_*(K_f)$) is *strictly positive*.

• $g = 1$. We put on every smooth elliptic leaf of $\mathcal{F}$ the (unique) flat metric with total area 1. It is shown in [Br4] (using Theorem 8.2 above) that this leafwise metric extends to a metric on $K_{\mathcal{F}}$ with positive curvature. In other words, the pseudoeffectivity of $K_{\mathcal{F}}$ is realized by a leafwise flat metric. Moreover, still in [Br4] it is observed that if the fibration is not isotrivial then the curvature of such a metric on $K_{\mathcal{F}}$ is strictly positive on directions transverse to the fibration. We thus get the same conclusion as in the hyperbolic case: if f is *not* isotrivial then the degree of $f_*(K_{\mathcal{F}})$ (and, a fortiori, the degree of $f_*(K_f)$) is *strictly positive*.

Let us conclude with several remarks around the pseudoeffectivity of $K_{\mathcal{F}}$.

Remark 8.7. In the case of hyperbolic foliations, Theorem 6.4 is very efficient: not only $K_{\mathcal{F}}$ is pseudoeffective, but even this pseudoeffectivity is realized by an explicit metric, induced by the leafwise Poincaré metric. This gives further useful properties. For instance, we have seen that the polar set of the metric is filled by singularities and parabolic leaves. Hence, for example, if all the leaves are hyperbolic and the singularities are isolated, then $K_{\mathcal{F}}$ is not only pseudoeffective but even nef (numerically eventually free [Dem]). This efficiency is unfortunately lost in the case of parabolic foliations, because in Theorem 8.5 the pseudoeffectivity of $K_{\mathcal{F}}$ is obtained via a more abstract argument. In particular, we do not know how to control the polar set of the metric. See, however, [Br4] for some special cases in which a distinguished metric on $K_{\mathcal{F}}$ can be constructed even in the parabolic case, besides the case of elliptic fibrations discussed in Example 8.6 above.

Remark 8.8. According to general principles [BDP], once we know that $K_{\mathcal{F}}$ is pseudoeffective we should try to understand its discrepancy from being nef. There is on X a unique maximal countable collection of compact analytic subsets $\{Y_j\}$ such that $K_{\mathcal{F}}|_{Y_j}$ is *not* pseudoeffective. It seems reasonable to try to develop the above theory in a "relative" context, by replacing X with Y_j, and then to prove something like this: every Y_j is $\mathcal{F}$-invariant, and the restriction of $\mathcal{F}$ to Y_j is a foliation by rational curves. Note, however, that the restriction of a foliation to an invariant analytic subspace Y is a dangerous operation. Usually, we like to work with "saturated" foliations, i.e. with a singular set of codimension at least two (see, e.g., the beginning of the proof of Theorem 8.5 for the usefulness of this condition). If $Z = Sing(\mathcal{F}) \cap Y$ has codimension one in Y, this means that our "restriction of $\mathcal{F}$ to Y" is not really $\mathcal{F}|_Y$, but rather its saturation. Consequently, the canonical bundle of that restriction is not really $K_{\mathcal{F}}|_Y$, but rather $K_{\mathcal{F}}|_Y \otimes \mathcal{O}_Y(-\mathcal{Z})$, where $\mathcal{Z}$ is an effective divisor supported in Z. If $Z = Sing(\mathcal{F}) \cap Y$ has codimension zero in Y (i.e., $Y \subset Sing(\mathcal{F})$), the situation is even worst, because then there is not a really well defined notion of restriction to Y.

Remark 8.9. The previous remark is evidently related to the problem of constructing minimal models of foliations by curves, i.e. birational models (on possibly singular varieties) for which the canonical bundle is nef. In the projective context, results in this direction have been obtained by McQuillan and Bogomolov [BMQ] [MQ2]. From this birational point of view, however, we rapidly meet another open and difficult problem: the resolution of the singularities of $\mathcal{F}$. A related problem is the construction of birational models for which there are no vanishing ends in the leaves, compare with Remark 8.4 above.

Remark 8.10. Finally, the pseudoeffectivity of $K_{\mathcal{F}}$ may be measured by finer invariants, like Kodaira dimension or numerical Kodaira dimension. When $\dim X = 2$ then the picture is rather clear and complete [MQ1] [Br1]. When $\dim X > 2$ then almost everything seems open (see, however, the case of fibrations discussed above). Note that already in dimension two the so-called "abundance" does not hold: there are foliations (Hilbert Modular Foliations [MQ1] [Br1]) whose canonical bundle is pseudoeffective, yet its Kodaira dimension is $-\infty$. The classification of these exceptional foliations was the first motivation for the plurisubharmonicity result of [Br2].

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