DYNAMICS OF ENTIRE FUNCTIONS

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ABSTRACT. We discuss the dynamics of entire functions and its Newton maps.

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1. Polynomials and Entire Functions

Throughout this text, f will always denote an entire function $f\colon \mathbb{C} \to \mathbb{C}$.

Definition 1.1 (Fatou and Julia Sets).
The Fatou set $F(f)$ *is the set of all* $z \in \mathbb{C}$ *that have a neighborhood* U *on which the family of iterates* $f^{\circ n}$ *forms a normal family (in the sense of Montel). The* Julia set $J(f)$ *is the complement of the Fatou set:* $J(f) := \mathbb{C} \setminus F(f)$*. A connected component of the Fatou set is called a* Fatou component*.*

Definition 1.2 (Local Fixed Point Theory).
Consider a local holomorphic function $g\colon U \to \mathbb{C}$ *with* $U \subset \mathbb{C}$ *open, and a fixed point* $p \in U$ *with derivative* $\mu := g'(p)$*. Then the fixed point* p *is called*

- **attracting:** *if* $|\mu| < 1$ *(and* superattracting *if* $\mu = 0$*);*
- **repelling:** *if* $|\mu| > 1$*;*
- **indifferent:** *if* $|\mu| = 1$*; in particular*
- **rationally indifferent (or parabolic):** *if* μ *is a root of unity;*
- **irrationally indifferent:** *if* $|\mu|$ *is indifferent but not a root of unity.*

Theorem 1.3 (Local Fixed Point Theory).
Local holomorphic maps have the following normal forms near fixed points:

- *in a neighborhood of a superattracting fixed point,* g *is conformally conjugate to* $z \mapsto z^d$ *near* 0*, for a unique* $d \geq 2$*;*
- *in a neighborhood of an attracting or repelling fixed point,* g *is conformally conjugate to the linear map* $z \mapsto \mu z$ *near* 0 *("attracting and repelling fixed points are locally linearizable");*
- *local normal forms in a neighborhood of parabolic fixed points are complicated; but within attracting petals,* g *is conformally conjugate to* $z \mapsto z + 1$ *in a right half plane;*
- *irrationally indifferent fixed points may or may not be linearizable.*

Remark. If an irrationally indifferent fixed point is linearizable, then a maximal linearizable neighborhood is called a *Siegel disk.* The Yoccoz condition is a sufficient condition for linearizability of any local holomorphic map, in particular for every entire function. Non-linearizable irrationally indifferent fixed points are called *Cremer points*; they are not associated to any type of Fatou component. (As a result, if the Julia set equals $\mathbb{C}$, then all fixed points are repelling or Cremer.)

Remark. The same classification applies of course to periodic points: these are fixed points of appropriate iterates.

Theorem 1.4 (Classification of Fatou Components).
Any Fatou component has exactly one of the following types:

- *a periodic component in which the dynamics converges to an attracting cycle*
- *a periodic component in which the dynamics converges to a parabolic cycle*
- *a periodic component in which the dynamics is conformally conjugate to an irrational rotation (a Siegel disk)*
- *a periodic component in which the dynamics converges to ∞ (a Baker domain)*
- *a preperiodic component that eventually maps to a periodic component in one of the types above;*
- *a non-periodic component for which all forward iterates are disjoint (a wandering domain)*

Remark. The difference to the polynomial case is the existence of Baker domains (these are similar to parabolic domains, except that the parabolic boundary point is replaced by the essential singularity at ∞) and of wandering domains. Rational and meromorphic maps may also have Arnol'd-Herman rings, but these do not exist for entire maps.

Theorem 1.5 (Connectivity of Fatou Components).
Every periodic Fatou component is simply connected. Wandering domains may have any connectivity, finite or infinite.

Theorem 1.6 (Periodic Points).
Every entire function has periodic points of all periods, with at most one exception. The Julia set is non-empty and equals the closure of the set of repelling periodic points. The Fatou set may or may not be empty.

Definition 1.7 (Singular Value).
A value $w \in \mathbb{C}$ is a regular value *if w has a neighborhood U so that $f\colon f^{-1}(U) \to U$ is a covering; otherwise w is called a* singular value. *A* critical value *is a point $w = f(z)$ with $f'(z) = 0$; the point z is a critical point. An* asymptotic value *is a point $w \in \mathbb{C}$ such that there exists a curve $\gamma\colon [0,\infty) \to \mathbb{C}$ so that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to w$ as $t \to \infty$.*

There are entire functions in which every $a \in \mathbb{C}$ is an asymptotic value. On the other hand, according to the Gross Star Theorem, every entire function f has the property that for almost every $a \in \mathbb{C}$ and for almost every direction, the ray at a in this direction can be lifted under f along all branches.

The description of a singularity $a \in \overline{\mathbb{C}}$ begins as follows. For $r > 0$, let U_r be a component of $f^{-1}(D_\chi(a,r))$ (where $D_\chi(a,r)$ is the r-neighborhood of a with respect to the spherical metric), chosen so that $U_{r'} \subset U_r$ for $r' < r$. The either $\bigcap_r U_r = \{z\}$ for a unique $z \in \mathbb{C}$, or $\bigcap_r U_r = \emptyset$. If $\bigcap_r U_r = \{z\}$, then $f(z) = a$, and a is a regular value for this choice of $\{U_r\}U_r$ if $f'(z) \neq 0$, and a is a critical value if $f'(z) = 0$. If $\bigcap_r U_r = \emptyset$, then a is an asymptotic value for this choice of $\{U_r\}$. Of course, for the same function f, the same point $a \in \mathbb{C}$ can be a regular or singular value of different types for different choices of the $\{U_r\}$.

Critical values are called *algebraic singularities*, and asymptotic values are called *transcendental singularities* (for a particular choice of the $\{U_r\}$). Any choice $\{U_r\}$ with $\bigcap U_r = \emptyset$ is called an *asymptotic tract*.

An asymptotic value is a *direct singularity* for a choice of $\{U_r\}$ if there is an $r > 0$ so that $f(U_r) \not\ni a$, and an *indirect singularity* otherwise. A direct singularity is a *logarithmic singularity* if there is an U_r so that $f\colon U_r \to D_\chi(a,r) \setminus \{a\}$ is a universal covering.

Theorem 1.8 (Singular Values)**.**
The set of singular values is the closure of the set of critical and asymptotic values.

Definition 1.9 (The Escaping Set)**.**
The set $I(f)$ is the set of points $z \in \mathbb{C}$ with $f^{\circ n}(z) \to \infty$.

Conjecture 1.10 (Eremenko)**.**

- **Weak version:** *Every component of $I(f)$ is unbounded.*
- **Strong version:** *Every $z \in I(f)$ can be connected to ∞ within $I(f)$.*

Example 1.11 (A Baker Domain)**.**
The map $f(z) = z + 1 + e^{-z}$ has a Baker domain containing the right half plane.

Example 1.12 (A Simply Connected Wandering Domain)**.**
The map $f(z) = z + \sin z + 2\pi$ has a simply connected wandering domain.

Example 1.13 (A Baker Wandering Domain)**.**
For appropriate values of $c > 0$ and $r_n > 0$, the map

$$f(z) = cz^2 \prod_{n \geq 1} (1 + z/r_n)$$

has a multiply connected wandering domain.

Example 1.14 (Construction of Multiply Connected Wandering Domains)**.**
Kisaka and Shishikura have examples of wandering domains of arbitrary connectivity, and Bergweiler has shown that these may coexist with simply connected wandering domains.

REMARK. Meromorphic functions with finitely many poles often behave similarly as entire functions.

Lemma 1.15 (The Zalcman Lemma)**.**
Let $\mathcal{F}$ be a non-normal family of holomorphic functions in $\mathbb{D}$. Then there exist a non-constant holomorphic map $g\colon \mathbb{D} \to \mathbb{C}$ and sequences $f_n \in \mathcal{F}$, $z_n \in \mathbb{D}$ and $\rho_n \to 0$ so that $z_n \to z \in \mathbb{D}$ and

$$g_n(\zeta) := f_n(z_n + \rho_n \zeta) \longrightarrow g(\zeta)$$

uniformly on compacts in $\mathbb{D}$.

2. Entire Functions of Finite Type and of Bounded Type

Definition 2.1 (Special Classes of Entire Functions)**.**
The class S ("Speiser class") consists of those entire functions that have only finitely many singular values. The class B ("the Eremenko-Lyubich class of functions of bounded type") consists of those entire functions for which all singular values are contained in a bounded set in $\mathbb{C}$.

An entire function f has finitely many critical points *and finitely many asymptotic tracts if and only if $f(z) = c + \int P(z') \exp(Q(z'))\, dz'$ for polynomials P, Q.*

Definition 2.2 (Order of an Entire Function)**.**
The order of an entire function f is defined as

$$\limsup_{z\to\infty} \log\log|f(z)|/\log|z| \in [0,\infty] \,.$$

This value may be finite or infinite.

Remark. The prototypical example of a function of finite order $s \in \mathbb{N}$ is $z \mapsto \exp(z^s)$.

Remark. An entire (or meromorphic) function f has polynomial Schwarzian derivative $S(f) = (f'/f'')' - (f'/f'')^2/2$ if and only if it has only finitely many asymptotic values and no critical values (and, if meromorphic, only simple poles, so it is locally injective).

Remark. For a function of class B, all transcendental singularities are logarithmic.

Theorem 2.3 (Fatou Components for Class S and Class B)**.**
A function of class S does not have Baker domains or wandering domains. A function of class B does not have Baker domains or wandering domains in which the dynamics converges to ∞.

(Proof uses logarithmic coordinates for class B, and the Sullivan method for class S.)

Theorem 2.4 (The Fatou-Shishikura-Inequality)**.**
A function of class B has no more non-repelling periodic points than infinite critical orbits.

Definition 2.5 (Logarithmic Coordinates)**.**
For a function $f \in B$, a corresponding function in logarithmic coordinates is a function $F\colon H_x \to \mathbb{C}$, where $H_x = \{z \in \mathbb{C}\colon \operatorname{Re} z > x\}$ is a right half plane and $\exp \circ F = f \circ \exp$. A tract *is a simply connected domain $T \subset \mathbb{C}$ so that $F\colon T \to H_x$ is a conformal isomorphism.*

Theorem 2.6 (Existence of Logarithmic Coordinates). *Logarithmic coordinates exist for every entire function of class B, outside of any disk containing all singular values.*

Theorem 2.7 (Existence of Dynamic Rays in Special Cases). *For exponential maps and certain maps with "hyperbolic tracts", the set $I(f)$ contains a Cantor bouquet of curves.*

Definition 2.8 (Dynamic Ray or "Hair"). *A* dynamic ray tail *is an injective curve $\gamma\colon (\tau,\infty) \to I(f)$ so that*

- *$f^{\circ n}(\gamma(t)) \to \infty$ as $t \to \infty$ for every $n \geq 0$, and*
- *$f^{\circ n}(\gamma(\tau,\infty)) \to \infty$ as $n \to \infty$ uniformly in t.*

A dynamic ray *is a maximal injective curve $\gamma\colon (0,\infty) \to I(f)$ so that for every $\tau > 0$, the restriction $\gamma|_{(\tau,\infty)}$ is a dynamic ray tail.*

Theorem 2.9 (Dynamic Rays for Bounded Type and Finite Order). *For functions of bounded type and finite order, or compositions thereof, I consists entirely of rays (the strong version of Eremenko's conjecture holds).*

Theorem 2.10 (Non-Existence of Dynamic Rays). *There are entire functions of bounded type for which every path component of $I(f)$ is bounded (or even a point).*

3. The Escaping Set

Theorem 3.1 (The Escaping Set)**.**
Every transcendental entire function f has the following properties:

- $I(f) \cap J(f) \neq \emptyset$;
- $J(f) = \partial I(f)$;
- *all components of* $\overline{I(f)}$ *are unbounded.*

There are several reasons why the escaping set is interesting:

- it is a non-empty invariant set (and unlike the Julia set, it is always a proper subset of $\mathbb{C}$)
- the set $I(f)$ often has useful structure, such as the union of curves to ∞. It thus gives structure to the dynamic plane.
- it is a subset of the Julia set that is often easy to control; for instance, it allows to give lower bounds on the Hausdorff dimension of the Julia set.

Definition 3.2 (Explosion Set)**.**
A set $X \subset \mathbb{C}$ is called an explosion set *if $X \cup \{\infty\}$ is connected in $\overline{\mathbb{C}}$, but X is totally disconnected.*

Theorem 3.3 (Explosion Set)**.**
For every exponential map $z \mapsto \lambda e^z$ with an attracting fixed point, the landing points of dynamic rays form an explosion set in $\mathbb{C}$.

Definition 3.4 (Non-Separable Continua)**.**
A continuum X is called indecomposable *if it cannot be written as the union of two proper subcontinua.*

Theorem 3.5 (Indecomposable Continua)**.**
For every postsingularly finite exponential map $z \mapsto \lambda e^z$, there are dynamic rays that accumulate at indecomposable continua.

Definition 3.6 (The Sets $A(f)$, $L(f)$ and $Z(f)$)**.**
For an entire function f, the set $A(f)$ is defined as follows:

$$A(f) := \{z \in \mathbb{C}\colon \exists k \geq 0 \ \ \forall n \geq 0\colon |f^{\circ(n+k)}(z)| \geq M(R, f^{\circ n})\}$$

where $M(R,g) = \max_{|z|=R} |g(z)|$.

The set $L(f)$ is defined as

$$L(f) := \left\{ z \in I(f)\colon \ \limsup \frac{\log |f^{\circ n}(z)|}{n} < \infty \right\} .$$

The set $Z(f)$ is defined as

$$Z(f) := \left\{ z \in \mathbb{C}\colon \ \frac{\ln \ln |f^{\circ n}(z)|}{n} \to \infty \ \text{as}\ n \to \infty \right\} .$$

The set $A(f)$ describes essentially those points that escape to ∞ as fast as possible. (This definition does not depend on R provided R is large enough so that $J(f) \cap \{z \in \mathbb{C}\colon |z| < R\} \neq \emptyset$; smaller values of R may require larger values of k.) The set $L(f)$ describes slow escape, while $Z(f)$ describes those points that "zip" to ∞.

Theorem 3.7 (Speed of Escape)**.**
The sets $A(f)$ and $Z(f)$ satisfy the following:
- *$A(f) \subset Z(f)$, and both sets are completely invariant;*
- *$J(f) \cap A(f) \neq \emptyset$;*
- *every Fatou component is either in $Z(f)$ or disjoint from $Z(f)$; every Fatou component is either in $A(f)$ or disjoint from $A(f)$;*
- *$J(f) = \partial Z(f)$; moreover, if f has no wandering domains, then $J(f) = \overline{Z(f)}$;*
- *$J(f) = \partial A(f) = \partial L(f)$;*
- *every (periodic) Baker domain lies in $L(f)$;*
- *every Baker wandering domain is in $A(f)$;*
- *every component of $A(f)$ is unbounded.*
- *every component of $\overline{L(f)}$ is unbounded.*

Wandering domains may escape slow enough for $L(f)$ and fast enough for $A(f)$.

Theorem 3.8 (Slow Escape Possible)**.**
For every real sequence $K_n \to \infty$, there is a $z \in I(f) \cap J(f)$ so that $|f^{\circ n}(z)| < K_n$ for all sufficiently large n.

Theorem 3.9 (In Case of Baker Wandering Domains)**.**
If f has no Baker Wandering Domain, then every component of $A(f) \cap J(f)$ is unbounded. If f has a Baker Wandering Domain, then $A(f)$ and $I(f)$ are connected.

Relation between $A(f)$, its components, closure, etc. The set $Z(f)$ etc., connected continua in I and $\overline{I}$.

4. Hausdorff Dimension

Theorem 4.1 (Hausdorff Dimension 2).
For every map $z \mapsto \lambda e^z$, the Julia set (and even the escaping set) has Hausdorff dimension 2.

Theorem 4.2 (Julia Set of Positive Measure).
For every map $z \mapsto ae^z + be^{-z}$ with $ab \neq 0$, the Julia set (and even the escaping set) has positive 2-dimensional Lebesgue measure.

Theorem 4.3 (Dynamic Rays of Dimension One).
For every exponential or cosine map, the set of dynamic rays has Hausdorff dimension 1. The same holds for significantly larger classes of maps, including those of bounded type and finite order which have an attracting fixed point that contains all singular values in its immediate basin.

Theorem 4.4 (The Dimension Paradox for Cosine Maps).
Every postcritically finite cosine map $C(z) = ae^z + be^{-z}$ has the following properties:

- *the Julia set equals $\mathbb{C}$;*
- *every dynamic ray lands at a unique point in $\mathbb{C}$;*
- *every point in $\mathbb{C}$ is either on a dynamic ray, or it is the landing point of one, two, or four dynamic rays;*
- *the set of dynamic rays has Hausdorff dimension 1;*
- *the landing points of these rays are the complement of the one-dimensiona set of rays.*

Theorem 4.5 (Unbounded Fatou Component of Finite Area).
For the map $z \mapsto \sin z$ (which has an attracting fixed point), the Fatou set intersects every vertical strip of width 2π in an unbounded set of finite planar Lebesgue measure.

Lemma 4.6 (The Parabola Lemma).
For every map $z \mapsto \lambda e^z$ or $z \mapsto ae^z + be^{-z}$, the set of escaping parameters that escape within the parabola $\{z \in \mathbb{C} \colon |\operatorname{Im} z| < |\operatorname{Re} z|^{1/q}\}$ has Hausdorff dimension at most $1 + 1/q$.

Theorem 4.7 (Hausdorff Dimension Greater than 1).
For every entire map of bounded type, the Hausdorff dimension of $J(f)$ strictly exceeds 1.

It is still an open question whether there exists an entire function the Julia set of which has Hausdorff dimension 1.

Theorem 4.8 (Hausdorff Dimension 2)**.**
For every entire map of bounded type and finite order, the Julia set has Hausdorff dimension 2*. More generally, if* f *has bounded type and for every* $\varepsilon > 0$ *there is a* $r_\varepsilon > 0$ *so that*

$$\log\log|f(z)| \leq (\log|z|)^{q+\varepsilon}$$

for $|z| > r_\varepsilon$*, then the Julia set has Hausdorff dimension at least* $1+1/q$*.*

(Note that functions of finite order satisfy the condition for $q = 1$, so the second statement generalizes the first.)

Theorem 4.9 (Explicit Values of Hausdorff Dimension)**.**
For every $p \in (1, 2]$*, there is an explicit example of an entire function for which the Julia set has Hausdorff dimension* p*.*

Karpinska: dimension of escaping points for exponentials.

5. Parameter Spaces

Definition 5.1 (Quasiconformally Equivalent Entire Functions).
Two functions f, g of bounded type are called quasiconformally equivalent near ∞ *if there are quasiconformal homeomorphisms $\varphi, \psi\colon \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ f = g \circ \psi$ near ∞.*

Theorem 5.2. *Let f, g be two entire functions of bounded type that are quasiconformally equivalent near ∞. Then there exist $R > 0$ and a quasiconformal homeomorphism $\vartheta\colon \mathbb{C} \to \mathbb{C}$ so that $\vartheta \circ f = g \circ \vartheta$ on*

$$A_R := \{z \in \mathbb{C}\colon |f^{\circ n}(z)| > R \text{ for all } n \geq 1\} .$$

Furthermore ϑ has zero dilatation on $\{z \in A_R\colon |f^{\circ n}(z)| \to \infty\}$.

Corollary 5.3 (No Invariant Line Fields).
Entire functions of bounded type do not support invariant line fields on their Julia sets.

Theorem 5.4 (Exponential Parameter Space).
The parameter space of exponential maps $z \mapsto \lambda e^z$ has the following properties:

- *there is a unique hyperbolic component W of period 1; it is conformally parametrized by a conformal isomorphism $\mu\colon \mathbb{D}^* \to W$, $\mu \mapsto \mu \exp(-\mu)$, so that the map E_λ with $\lambda = \mu \exp(-\mu)$ has an attracting fixed point with multiplier μ;*
- *for every period $n \geq 2$, there are countably many hyperbolic components of period n; on each component, the multiplier map $\mu\colon W \to \mathbb{D}^*$ is a universal covering;*
- *for every hyperbolic component, the multiplier map extends continuously and locally injectively to the boundary;*
- *there is an explicit canonical classification of hyperbolic components and hyperbolic parameters, as well as parameters with indifferent orbits;*
- *escaping parameters (those for which the singular value escapes to ∞) are organized in the form of* parameter rays, *together with landing points of certain parameter rays; this yields an explicit classification of all escaping parameters;*
- *the Hausdorff dimension of the parameters is 1, while the Hausdorff dimension of all escaping parameters (parameter rays and some of their landing points) is 2;*
- *exponential parameter space fails to be locally connected at any point on a parameter ray;*
- *there is an explicit classification of all parameters for which the singular orbit is finite (i.e., strictly preperiodic).*

Conjecture 5.5 (Exponential Parameter Space)**.**

- *Hyperbolicity is dense in the space of exponential maps.*
- *Fibers in exponential parameter space are trivial.*

Remark. Fibers in exponential parameter space are defined in analogy as for the Mandelbrot set. The second conjecture says that all non-hyperbolic exponential maps are combinatorially rigid (their landing patterns of periodic dynamic rays differ); it is the analog to the famous conjecture that the Mandelbrot set is locally connected. The second conjecture implies the first.

6. Newton Maps of Entire Functions

In this section, f denotes an entire transcendental function and $N_f := \mathrm{id} - f/f'$ denotes the associated Newton map.

Definition 6.1 (Basin and Immediate Basin).
For a root α of f, we define its basin *as $U_\alpha := \{z \in \mathbb{C} \colon N_f^{\circ n}(z) \to \alpha\}$ as $n \to \infty$. The* immediate basin *is the connected component of U_α containing α.*

Theorem 6.2 (Immediate Basins Simply Connected).
Every root of f has simply connected immediate basin.

It is an open question whether every Fatou component of N_f is simply connected. In particular, the question is open whether Baker domains are always simply connected.

Theorem 6.3 (Wandering Newton Domains Simply Connected).
If a Newton map has a wandering domain, then it is simply connected.

Definition 6.4 (Virtual Immediate Basin).
A Virtual immediate basin *is a maximal subset of $\mathbb{C}$ (with respect to inclusion) among all connected open subsets of $\mathbb{C}$ in which the dynamics converges to ∞ locally uniformly and which have an absorbing set. (An* absorbing set *in a domain V is a subset A such that $N_f(\overline{A}) \subset A$ and every compact $K \subset V$ has a $n \geq 0$ so that $N_f^{\circ n}(K) \subset A$.)*

Theorem 6.5 (Virtual Immediate Basins Simply Connected).
Every virtual immediate basin is simply connected.

Every virtual immediate basin is contained in a Baker domain; it is an open question whether this basin equals a Baker domain (this is true for simply connected Baker domains).

Theorem 6.6 (Two Accesses Enclose Basin).
Let f be an entire function (polynomial or transcendental) and let U_α be the immediate basin of α for N_α. Let $\Gamma_1, \Gamma_2 \subset U_\alpha$ represent two curves representing different invariant accesses to ∞, and let W be an unbounded component of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$. Then W contains the an immediate basin of a root of f or a virtual immediate basin, provided the following finiteness condition holds: $N_f^{-1}(z) \cap W$ is finite for all $z \in \mathbb{C}$.

Remark. In the case of a polynomial f, the finiteness condition always holds, and there is no virtual immediate basin. The result thus says that any two accesses of any immediate basins enclose another immediate basin.

Theorem 6.7 (Rational Newton Map).
The Newton map N_f of f is rational if and only if $f = pe^q$ for polynomials p and q. In this case, ∞ is a parabolic fixed point with multiplier 1 *and multiplicity* $\deg(q) + 1$.

Theorem 6.8 (Area of Immediate Basins).
For $f = pe^q$ with polynomials p and q and $\deg q \geq 3$, *every immediate basin has finite Lebesgue are in the plane.*

Virtual immediate basins may be thought of as basins of a root at ∞. The prototypical case is $f(z) = \exp(z)$, $N_f(z) = z - 1$: there is no root, all points converge to $-\infty$ under N_f, and indeed f converges to 0 along these orbits. Douady thus asked whether there was a relation between asymptotic values 0 of f and virtual immediate basins. This is indeed often the case.

Theorem 6.9 (Logarithmic Singularity Implies Virtual Immediate Basin).
If f has an asymptotic value 0 *which is a logarithmic singularity, then N_f has a virtual immediate basin.*

There is a partial converse as follows.

Theorem 6.10 (Virtual Immediate Basin Implies Asymptotic Value).

Let V be a virtual immediate basin with absorbing set A. If the quotient A/N_f has sufficiently large modulus, then f has an asymptotic value 0.

Remark. There are indeed counterexamples when A/N_f has small modulus.

We conclude with the following conjecture on root finding of the Riemann ζ function by Newton's method. Let ξ be the entire function the roots of which are the non-trivial roots of ζ (so that $\xi(s) = \xi(1-s)$).

Conjecture 6.11 (The Riemann ζ function).
There are constant $c, s > 0$ with the following property. If there is a root α of ξ whose immediate basin does not contain one of the points $c_n^{\pm} := \pm 2 + cn/\log|n|$, then the Riemann hypothesis is false and there is a root α' off the critical line with $|\alpha' - \alpha| < s$ and the immediate basin of α' contains a point $c_n^{\pm}$.

Remark. The preceding conjecture says that the points c_n form an efficient set of starting points for finding all roots of f, so that the the first N starting points find at least $c''N$ distinct roots of ξ; and if they do not find all roots, then close to a missed root these starting points find a root that violates the Riemann hypothesis.

7. A Few Open Questions

Question 7.1. *Is there an entire functions whose Julia set has Hausdorff dimension* 1*?*

Question 7.2. *Does every Newton map of an entire function always have connected Julia set?*

Question 7.3. *Suppose for an exponential map* $z \mapsto \lambda \exp(z)$*, the singular value does not converge to* ∞*. Is every repelling periodic point the landing point of a periodic dynamic ray? Is there a generalization to larger classes of maps (of bounded type and finite order)?*

Question 7.4. *Give, for various classes of entire functions, a necessary and sufficient condition for an irrationally indifferent periodic point to have a Siegel disk.*

Question 7.5. *Show that the following map does not have a wandering domain (or at least not one that intersects the real axis):*

$$f(z) = z/2 + (1 - \cos \pi z)(z + 1/2)/2 + ((1/2 - \cos \pi z) \sin \pi z)/\pi$$

Remark. The relevance of the last question is the following: it is known that $\mathbb{Z}$ is in the Fatou set for this function f. Moreover, on $\mathbb{Z}$, the function f coincides with the well-known $3n + 1$ problem: $f(n) = n/2$ if n even and $f(n) = (3n + 1)/2$ if n is odd. Therefore, solving this question would prove that the $3n + 1$ problem does not have an orbit tending to ∞ (it would still be possible for f to have periodic integer orbits other than the cycle $1 \mapsto 2 \mapsto 1$; such as the fixed point -1 or $-5 \mapsto -7 \mapsto -10 \mapsto -5$).

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