Flow of the glass melt through a die: Stationary flow with fixed boundaries of mechanically incompressible, but thermally expansible, viscous fluids

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Thanks:

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Detailed proofs are in

A. Farina, A. Fasano, A. Mikelić : On the equations governing the flow of mechanically incompressible, but thermally expansible, viscous fluids, to appear in *M^3AS : Math. Models Methods Appl. Sci.*, Vol. 18, no. 6 (2008).

Introduction

In this talk, I'll present the study of the system

$$\text{div } \vec{v} = \frac{\alpha}{\rho(\vartheta)} \vec{v} \nabla \vartheta, \quad \rho(\vartheta) = 1 - \alpha \vartheta \tag{1}$$

$$\rho(\vartheta)(\vec{v}\nabla)\vec{v} = -\textbf{Ar}\, \vartheta \vec{e}_g - \nabla P + \frac{1}{\textbf{Re}} \text{ Div } \{2\mu(\vartheta) D(\vec{v})\} - \frac{2}{3\textbf{Re}} \nabla \big(\mu(\vartheta) \text{ div } \vec{v}\big) \tag{2}$$

$$\rho(\vartheta) c_{p1}(\vartheta) \vec{v} \nabla \vartheta = \frac{1}{\textbf{Pe}} \text{ div } \big(\lambda(\vartheta) \nabla \vartheta\big), \tag{3}$$

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$$\rho(\vartheta)(\vec{v}\nabla)\vec{v} = -\textbf{Ar}\, \vartheta \vec{e}_g - \nabla P + \frac{1}{\textbf{Re}} \text{ Div } \left\{ 2\mu(\vartheta) D(\vec{v}) \right\}$$

$$-\frac{2}{3\textbf{Re}} \nabla \left(\mu(\vartheta) \text{ div } \vec{v} \right) \tag{5}$$

$$\rho(\vartheta) c_{p1}(\vartheta) \vec{v} \nabla \vartheta = \frac{1}{\textbf{Pe}} \text{ div } \left(\lambda(\vartheta) \nabla \vartheta \right), \tag{6}$$

in a cylindrical domain
$\Omega = \{ r < R(x_3, \phi) \leq 1 \} \times [0, 2\pi] \times (0, 1)$
in $I\!R^3$. $R : [0,1] \times [0, 2\pi] \to (0, 1]$ is a C^∞-map.

I1

The boundary of Ω contains 3 distinct parts: the lateral boundary $\Gamma_{lat} = \{r = R(x_3, \phi\}$, the inlet boundary $\Gamma_{in} = \{x_3 = 1$ and $r \leq R(1, \phi)$, $\phi \in [0, 2\pi]\}$ and the outlet boundary $\Gamma_{out} = \{x_3 = 0$ and $r \leq R(0, \phi)$, $\phi \in [0, 2\pi]\}$. $\alpha \geq 0$ is a parameter (the "**expansivity**").

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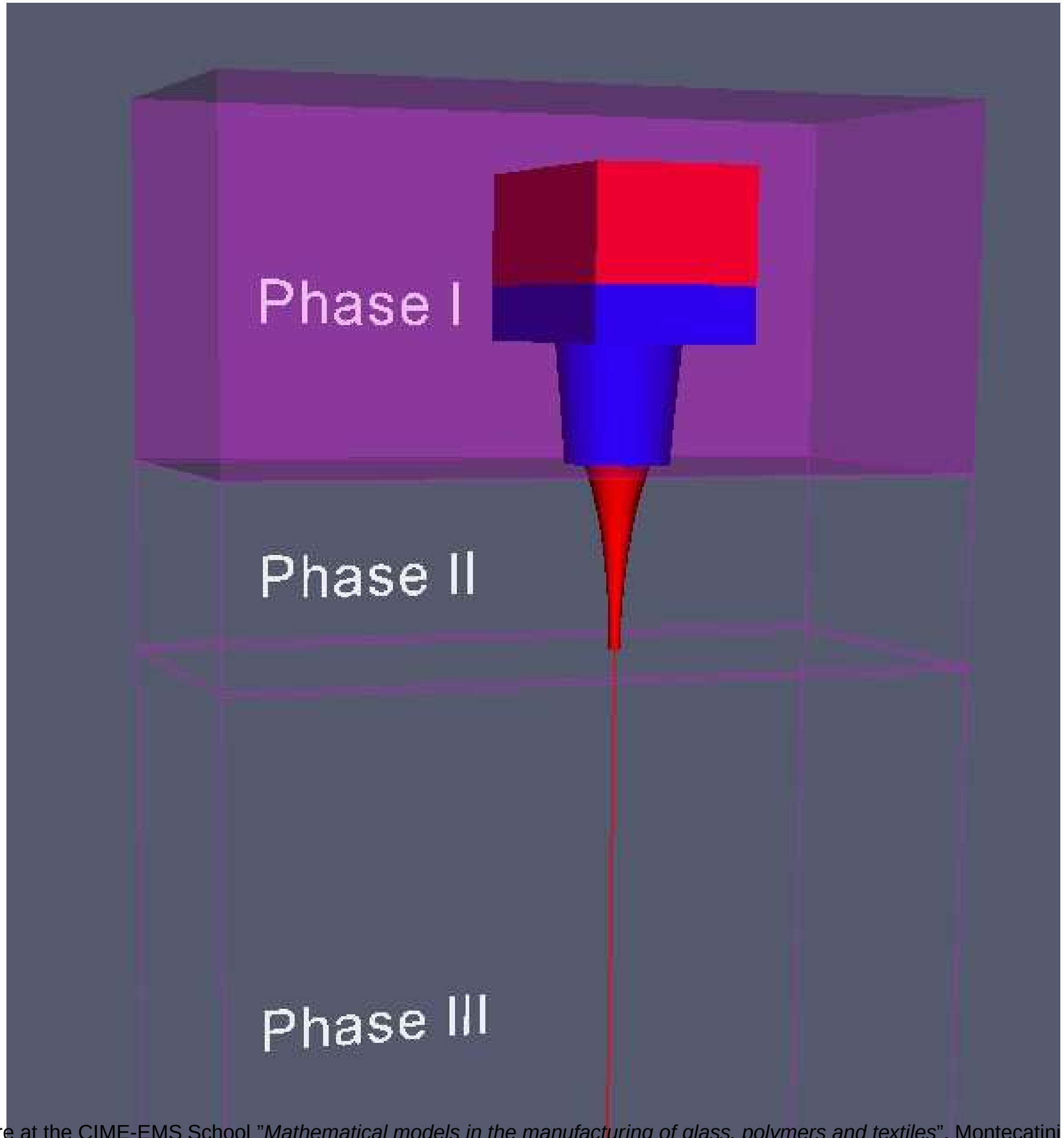
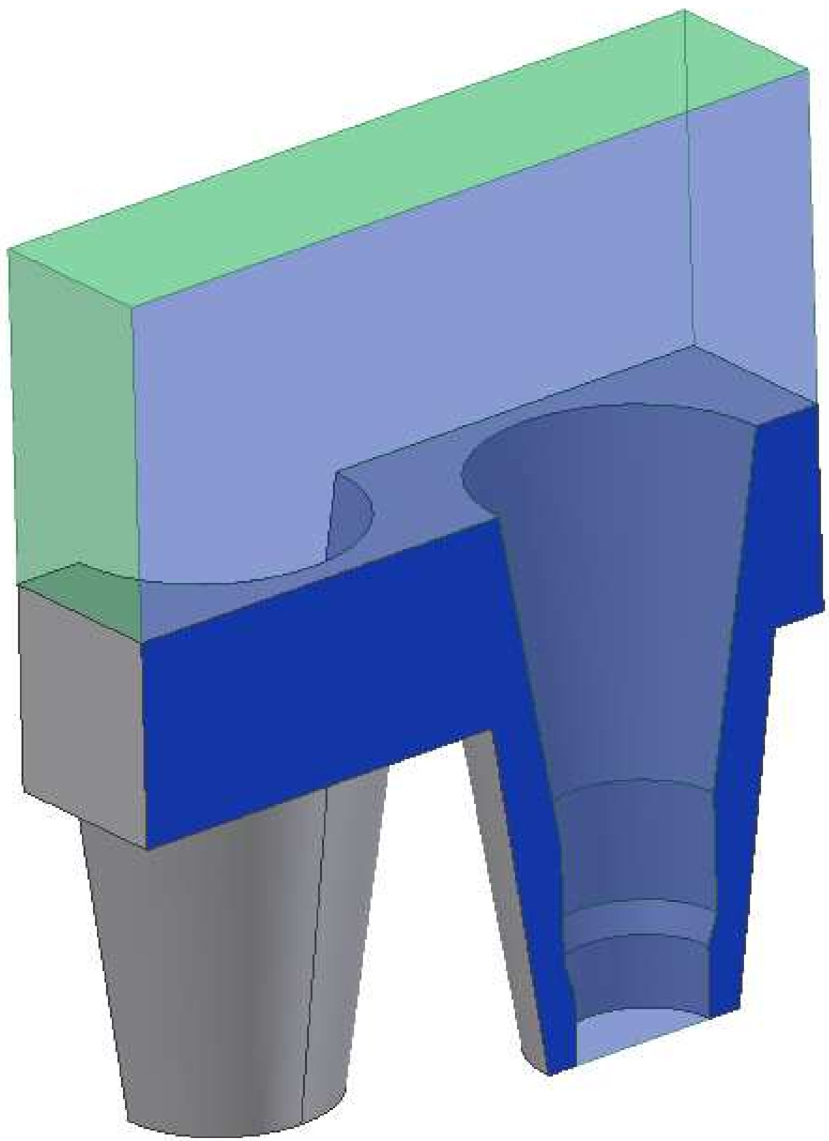
System (1)-(3) describes a dimensionless model for the stationary motion of a **mechanically incompressible, but thermally expansible viscous fluid.** It is widely used in industrial simulations of flows of hot melted glasses, polymers etc.

We'll consider the system (10)-(12) in the realistic situation, when the parameter $\alpha = -K_\rho(T_w - 1)$ (the " **expansivity** " or the "**thermal expansion coefficient**") is small.

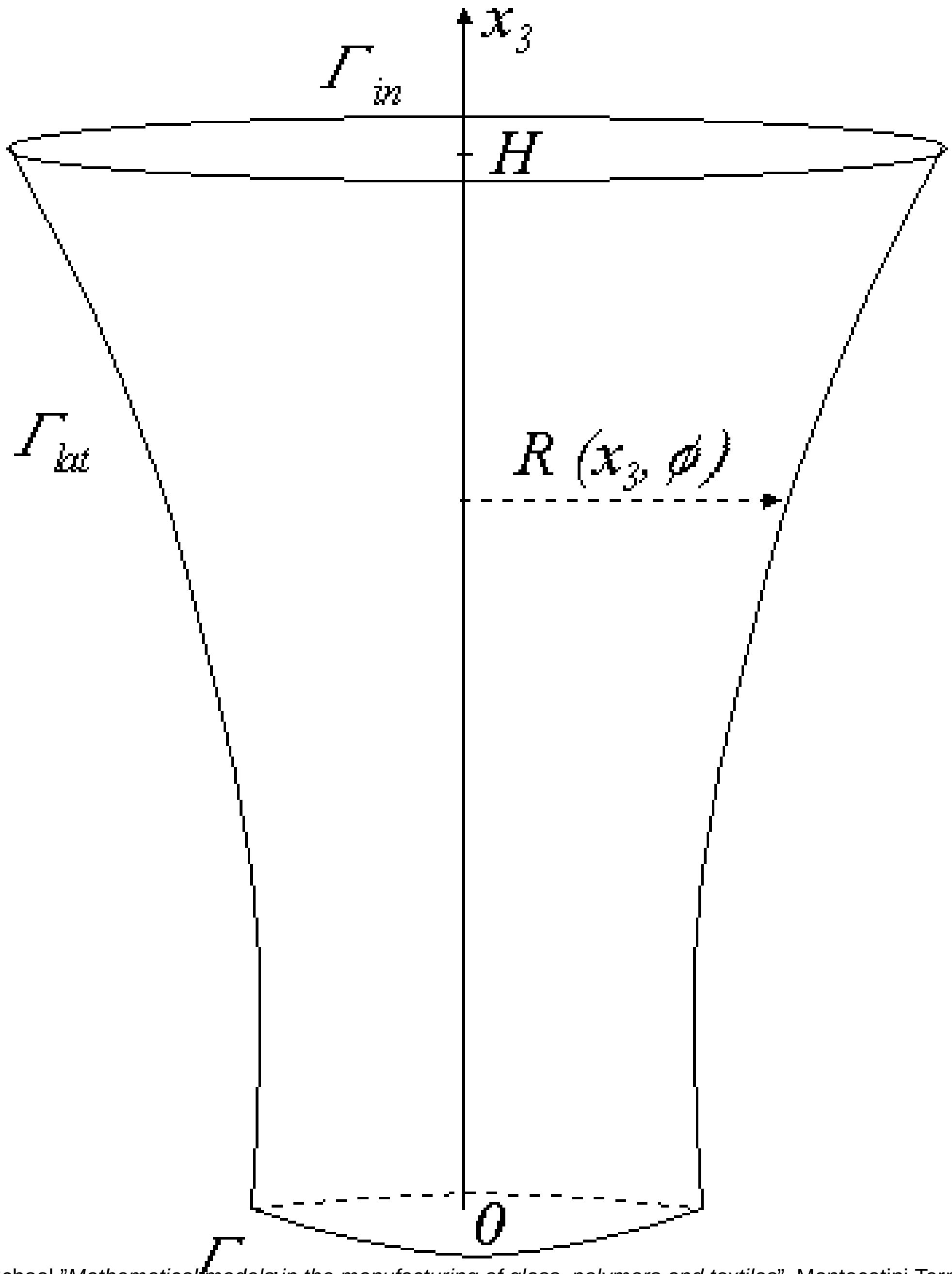
I1a



M10



M10



M11

Our plan is the following:

- We prove that the system (1) – (3), with suitable boundary conditions, has a solution.
- We establish uniqueness of the solution for small data.
- We prove that, in the limit $\alpha \to 0$, our system reduces to the Oberbeck-Boussinesq system. Furthermore, we calculate the first correction term in this asymptotic limit.

We recall the Oberbeck-Boussinesq system:

$$\text{div } \vec{v}^{OB} = 0 \tag{7}$$

$$\frac{D\vec{v}^{OB}}{Dt} = -\mathbf{Ar}\ \vartheta^{OB} \vec{e}_g - \nabla p_{mot}^{OB} + \frac{1}{\mathbf{Re}} \text{ Div } \{2\mu(\vartheta^{OB}) D(\vec{v}^{OB})\} \tag{8}$$

M12

$$c_{p1}(\vartheta^{OB})\frac{D\vartheta^{OB}}{Dt} = \frac{1}{\mathbf{Pe}}\,\mathrm{div}\,\big(\lambda(\vartheta^{OB})\nabla\vartheta^{OB}\big). \tag{9}$$

In the situation which is of interest for us, the quantities from (**??**) are small but non-zero. Our goal is to study a model which generalizes the Oberbeck-Boussinesq approximation and reduces to it in the limit.

M12

$$c_{p1}(\vartheta^{OB})\frac{D\vartheta^{OB}}{Dt} = \frac{1}{\textbf{Pe}}\ \text{div}\ \big(\lambda(\vartheta^{OB})\nabla\vartheta^{OB}\big). \tag{13}$$

In the situation which is of interest for us, the quantities from (**??**) are small but non-zero. Our goal is to study a model which generalizes the Oberbeck-Boussinesq approximation and reduces to it in the limit.

NB: The full non-stationary system:

$$\text{div}\ \vec{v} = -\frac{K_\rho}{\rho(\vartheta)}(T_w - 1)\frac{D\vartheta}{Dt} \tag{14}$$

$$\rho(\vartheta)\frac{D\vec{v}}{Dt} = -\textbf{Ar}\ \vartheta\vec{e}_g - \nabla(P + \frac{2}{3\textbf{Re}}\mu(\vartheta)\ \text{div}\ \vec{v}) + \frac{1}{\textbf{Re}}\ \text{Div}\ \big\{2\mu(\vartheta)D(\vec{v}) \tag{15}$$

$$\rho(\vartheta)c_{p1}(\vartheta)\frac{D\vartheta}{Dt} = \frac{1}{\textbf{Pe}}\ \text{div}\ \big(\lambda(\vartheta)\nabla\vartheta\big), \tag{16}$$

Existence for the stationary problem

Problem A : We consider the system (1) – (3) in Ω, with the boundary conditions

$$\vec{v} = v_1 \vec{e}_g, \ \vartheta = 1 \quad \text{on} \ \Gamma_{in} \tag{17}$$

$$\vec{v} = v_2 \vec{e}_g, \ \vartheta = 0 \quad \text{on} \ \Gamma_{out} \tag{18}$$

$$\vec{v} = 0, \ -\frac{1}{\textbf{Pe}} \lambda(\vartheta) \nabla \vartheta \cdot \vec{n} = q_0 \vartheta + \mathcal{S} \quad \text{on} \ \Gamma_{lat} \tag{19}$$

$$v_1 \in C_0^1(\Gamma_{in}) \cap C^\infty(\bar{\Gamma}_{in}), \ v_2 \in C_0^1(\Gamma_{out}) \cap C^\infty(\bar{\Gamma}_{out}),$$

$$\mathcal{S} \in C^\infty(\bar{\Gamma}_{lat}), \ \mathcal{S} \geq 0 \tag{20}$$

$$\int_{\Gamma_{in}} \rho(1) v_1 \ r dr d\phi = \int_{\Gamma_{in}} (1-\alpha) v_1 \ r dr d\phi = \int_{\Gamma_{out}} v_2 \rho(0) \ r dr d\phi \tag{21}$$

E1

We start by studying the energy equation for a given $\vec{w} = \rho\vec{v} \in \mathcal{H}(\Omega)$, where $\mathcal{H}(\Omega) = \{\vec{z} \in L^3(\Omega)^3 \mid$ div $\vec{z} = 0$ in $\Omega,\ \vec{z}\cdot\vec{n} = 0$ on $\Gamma_{lat},\ \vec{z}\cdot\vec{n}|_{\Gamma_{in}} = \rho(1)v_1,\ \vec{z}\cdot\vec{n}|_{\Gamma_{out}} = \rho(0)v_2\}$.
Our nonlinearities c_{p1} and λ are defined only for ϑ such that the density is not negative. We extend it on $I\!R$ by setting

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Our nonlinearities c_{p1} and λ are defined only for ϑ such that the density is not negative. We extend it on $I\!R$ by setting

$$c_{p1}(\vartheta) = \begin{cases} c_{p1}(1)/\vartheta^2 \ \text{ for } \ \vartheta > 1 \\ \left(\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0\vartheta}\right)^2 c_{p1}\left(-\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\right) \ \text{ for } \ \vartheta < -\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}, \end{cases} \tag{24}$$

$$\lambda(\vartheta) = \begin{cases} \lambda(1) \ \text{ for } \ \vartheta > 1 \\ \lambda\left(-\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\right) \ \text{ for } \ \vartheta < -\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}. \end{cases} \tag{25}$$

E2

We will prove that $\vartheta \in [-\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}, 1]$ Now for a given

$$\begin{cases} \vec{w} \in \mathcal{H}(\Omega),\ \mathcal{S} \in L^\infty(\Gamma_{lat}),\ \mathcal{S} \geq 0 \\ \lambda, c_{p1} \in W^{1,\infty}(I\!R),\ \lambda \geq \lambda_0 \text{ and constants } q_0,\ \text{Pe } \geq 0, \end{cases} \tag{26}$$

we consider the problem

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we consider the problem

Problem Θ: Find $\vartheta \in H^1(\Omega) \cap L^\infty(\Omega)$ such that

$$c_{p1}(\vartheta)\vec{w}\nabla\vartheta = \frac{1}{\textbf{Pe}} \operatorname{div} \big(\lambda(\vartheta)\nabla\vartheta\big), \tag{31}$$

$$\vartheta = 1 \ \text{ on } \ \Gamma_{in} \quad \text{and} \quad \vartheta = 0 \ \text{ on } \ \Gamma_{out} \tag{32}$$

$$-\frac{1}{\textbf{Pe}}\lambda(\vartheta)\nabla\vartheta \cdot \vec{n} = q_0\vartheta + \mathcal{S} \quad \text{ on } \ \Gamma_{lat} \tag{33}$$

E3

Proposition 2 Under the stated assumptions, **Problem** Θ has at least one variational solution in $H^1(\Omega)$, satisfying the estimate

$$\frac{\lambda_0}{2\mathbf{Pe}}\|\nabla\vartheta\|^2_{L^2(\Omega)^3} + \frac{q_0}{4}\|\vartheta\|^2_{L^2(\Gamma_{lat})} \leq A_0 + B_0\|\vec{w}\|_{L^1(\Omega)^3}, \tag{34}$$

where

$$B_0 = \|c_{p1}\|_\infty \max\left\{1, \frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\right\},$$

$$\begin{aligned} A_0 \quad &= \quad \frac{\sqrt{q_o}+4}{4q_0}\|\mathcal{S}\|_{L^2(\Gamma_{lat})} + 2\sqrt{q_0}\,|\Gamma_{lat}| + \\ &+ \frac{\|\lambda\|_\infty}{2\mathbf{Pe}\lambda_0}|\Omega| + \frac{3}{2}\|\rho\|_\infty\,\|c_{p1}\|_\infty\|v_1\|_{L^2(\Gamma_{in})}. \end{aligned}$$

E4

Corollary 1 Under the assumptions of Proposition 2, we have the following estimate

$$\|\vartheta\|^2_{L^2(\Omega)} \leq 8 \max\left\{\frac{\mathbf{Pe}}{\lambda_o}, \frac{1}{q_o}\|\partial_{x_3} R\|_\infty\right\}\left(A_0 + B_0\|\vec{w}\|_{L^1(\Omega)^3}\right). \tag{35}$$

Lemma 1 Any variational solution $\vartheta \in H^1(\Omega)$ to **Problem** Θ, satisfying the a priori estimate (34), satisfies also

$$-\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0} \leq \vartheta \leq 1 \quad \text{a.e. on } \Omega. \tag{36}$$

E4A

Remark 4 Under the condition that the interior angle between the lateral boundary $r = R(x_3, \phi) \in C^2$ and the upper and lower surfaces is $\pi/2$, we can apply the elliptic regularity. This can be seen directly by extending the equation for $x_3 > 1$ and $x_3 < 0$. Let $\ell_3(\Omega)$ be the elliptic regularity constant in estimating the $W^{1,3}(\Omega)-$ norm using elliptic potentials, i.e. the solution θ for the mixed problem

$$\text{div } (\nabla\theta - \vec{f}) = 0 \quad \text{in } \Omega \tag{37}$$

$$\nabla\theta \cdot \vec{n} = g \quad \text{on } \Gamma_{lat} \quad \text{and} \quad \theta = 0 \text{ on } \Gamma_{out} \cup \Gamma_{in} \tag{38}$$

is estimated as

$$\|\nabla\theta\|_{L^3(\Omega)^3} \leq \ell_3(\Omega) \Big\{ \|g\|_{L^3(\Gamma_{lat})} + \|\vec{f}\|_{L^3(\Omega)^3} \Big\}. \tag{39}$$

E5

Then we have

$$\|\nabla\vartheta\|_{L^3(\Omega)^3} \leq E_{00} + E_{01}\|\vec{w}\|_{L^3(\Omega)}, \qquad (40)$$

where

$$E_{00} = \frac{|\Gamma_{lat}|^{1/3}\,\ell_3(\Omega)\,\mathbf{Pe}}{\lambda_0}\left(\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})} + q_o \max\left\{1, \frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\right\}\right.$$
$$\left. + \frac{\|\lambda\|_\infty}{\lambda_0}\left(|\Omega|^{1/3} + \ell_3(\Omega)|\Gamma_{lat}|^{1/3}\right)\right),$$

and

E5

Then we have

$$\|\nabla\vartheta\|_{L^3(\Omega)^3} \leq E_{00} + E_{01}\|\vec{w}\|_{L^3(\Omega)}, \quad (43)$$

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$$\left. + \frac{\|\lambda\|_\infty}{\lambda_0}\left(|\Omega|^{1/3} + \ell_3(\Omega)|\Gamma_{lat}|^{1/3}\right)\right.,$$

and

$$E_{01} = \frac{\ell_3(\Omega)}{\lambda_0}\mathbf{Pe}\,\|c_{p1}\|_\infty\,\max\left\{1, \frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\right\}. \quad (45)$$

E6

Now we turn to the continuity and momentum equations, determining the velocity field $\vec{w} = \rho\vec{v}$ and the pressure p.
Our system reads
For given ϑ determine $\{\vec{w}, p\}$ satisfying

$$\text{div}\ \vec{w} = 0 \quad \text{in} \quad \Omega \tag{46}$$

$$(\vec{w}\nabla)\frac{\vec{w}}{\rho} = -\mathbf{Ar}\vartheta\vec{e}_g - \nabla(p + \frac{2\mu}{3\mathbf{Re}}\ \text{div}\ \frac{\vec{w}}{\rho}) + \frac{1}{\mathbf{Re}}\ \text{Div}\ \big(2\mu(\vartheta)D(\frac{\vec{w}}{\rho})\big) \quad \text{in} \quad \Omega \tag{47}$$

$$\vec{w} = 0 \ \text{ on } \ \Gamma_{lat}, \ \vec{w} = (1-\alpha)v_1\vec{e}_g \ \text{ on } \ \Gamma_{in} \text{ and } \vec{w} = v_2\vec{e}_g \ \text{ on } \ \Gamma_{out}, \tag{48}$$

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Our system reads
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$$\text{div}\,\vec{w} = 0 \quad \text{in} \quad \Omega \tag{49}$$

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$$\vec{w} = 0 \;\text{ on }\; \Gamma_{lat},\; \vec{w} = (1-\alpha)v_1\vec{e}_g \;\text{ on }\; \Gamma_{in} \text{ and } \vec{w} = v_2\vec{e}_g \;\text{ on }\; \Gamma_{out}, \tag{51}$$

Next, there exists

E7

$$\vec{\zeta} \in H^2(\Omega)^3, \nabla\vec{\zeta} \in L^3(\Omega)^9, \quad \text{with curl } \vec{\zeta} \text{ satisfying } (48) \tag{52}$$

Now we adapt the well-known Hopf construction to our non-standard nonlinearities. we have

E7

$$\vec{\zeta} \in H^2(\Omega)^3, \nabla\vec{\zeta} \in L^3(\Omega)^9, \quad \text{with curl } \vec{\zeta} \text{ satisfying } (48) \qquad (55)$$

Now we adapt the well-known Hopf construction to our non-standard nonlinearities. we have

Proposition 2 Let us suppose that $1/\rho \in L^\infty(\Omega)$, $1/\rho \leq 1/\rho_{min}$. Then, for every $\gamma > 0$ there is a $\vec{\xi}$, depending on γ and ρ_{min}, such that

$$\vec{\xi} \in H^1(\Omega)^3, \text{ div } \vec{\xi} = 0 \text{ in } \Omega, \ \vec{\xi} = \vec{\zeta} \text{ on } \partial\Omega \text{ and} \qquad (56)$$

$$|\int_\Omega \frac{1}{\rho}(\vec{\phi}\nabla)\vec{\phi} \cdot \vec{\xi}\, dx| \leq \gamma \|\vec{\phi}\|^2_{H^1(\Omega)^3}, \quad \forall \vec{\phi} \in H^1_0(\Omega)^3 \qquad (57)$$

Let us introduce now some useful constants:

E8

$$\begin{cases} \gamma = \dfrac{2\mu_{min}}{\rho_{max}\textbf{Re}}, \quad A_{00} = 4\dfrac{\sqrt{6}}{\rho_{min}^2}, \; B_{00} = \dfrac{\mu_{min}}{\rho_{max}}, \; C_{01} = 2 \cdot 6^{1/6}|\Omega|^{5/6} \\ B_{01} = 2 \cdot 6^{1/3}\dfrac{1}{\rho_{min}^2}\|\vec{\xi}\|_{L^6(\Omega)^3}, \quad B_{02} = \dfrac{4\mu_{max}6^{1/6}}{\rho_{min}^3} \\ C_{00} = \dfrac{1}{\rho_{min}}\Big(\|\vec{\xi}\|_{L^6(\Omega)^3}^2 + \dfrac{2\mu_{max}}{\textbf{Re}}\|D(\vec{\xi})\|_{L^2(\Omega)^9}\Big). \end{cases} \tag{58}$$

E8

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Theorem 1 Let us suppose that

$$\Delta_B = \frac{1}{\textbf{Re}}\Big(B_{00} - \alpha B_{02}\|\nabla\vartheta\|_{L^3(\Omega)^3}\Big) - \alpha B_{01}\|\nabla\vartheta\|_{L^2(\Omega)^3} > 0 \tag{62}$$

and $\Delta_{det} = \Delta_B^2 - 4A_{00}\alpha\|\nabla\vartheta\|_{L^2(\Omega)^3}(C_{00} + \textbf{Ar}C_{01}\|\nabla\vartheta\|_{L^2(\Omega)^3}) > 0$ (63)

E9

Then there is a solution $\vec{w} \in H^1(\Omega)^3$ for the problem (46)-(48), satisfying the estimate

$$\|D(\vec{w})\|_{L^2(\Omega)^9} \leq \|D(\vec{\xi})\|_{L^2(\Omega)^9} + \frac{C_{00} + \mathbf{Ar} C_{01} \|\nabla \vartheta\|_{L^2(\Omega)^3}}{\sqrt{\Delta_{det}}} \quad (64)$$

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Then there is a solution $\vec{w} \in H^1(\Omega)^3$ for the problem (46)-(48), satisfying the estimate

$$\|D(\vec{w})\|_{L^2(\Omega)^9} \leq \|D(\vec{\xi})\|_{L^2(\Omega)^9} + \frac{C_{00} + \textbf{Ar} C_{01} \|\nabla \vartheta\|_{L^2(\Omega)^3}}{\sqrt{\Delta_{det}}} \quad (65)$$

Next we define our iterative procedure:

Let $\gamma = \frac{\mu_{min}}{\rho_{max} \textbf{Re}}$ and $\vec{\xi}$ be the corresponding vector valued function from Hopf's construction.

For a given $\vec{w}^m = \vec{W}^m + \xi$, such that $\vec{W}^m \in B_R = \{\vec{z} \in H_0^1(\Omega)^3 : \ \text{div } \vec{z} = 0$ in Ω and $\|D(\vec{W}^m)\|_{L^2(\Omega)^9} \leq R\}$, we calculate ϑ^m, a solution to (27)-(29).

E9A

Next, with this ϑ^m, we determine a solution $\vec{w}^{m+1} = \vec{W}^{m+1} + \xi$ for the problem (46)-(48), satisfying the estimate (64).

The natural question arising in the iterative process is **if** $\vec{W}^{m+1}$ **remains in** B_R ?

We have the following result

E9A

Next, with this ϑ^m, we determine a solution $\vec{w}^{m+1} = \vec{W}^{m+1} + \xi$ for the problem (46)-(48), satisfying the estimate (64).
The natural question arising in the iterative process is **if** $\vec{W}^{m+1}$ **remains in** B_R ?
We have the following result

Proposition 3 Let the constants A_0 and B_0 be given as in Proposition 1 and let E_{00} and E_{01} be given by Lemma 2. Let the constants $B_{00}, B_{01}, B_{02}, A_{00}, C_{00}$ and C_{01} be given by formula (58). Let $\vec{\xi}$ be generalized Hopf's lift, given by Proposition 2 and corresponding to γ. Let R be given by

E10

$$R = \sqrt{2}\mathbf{Re}\frac{\rho_{max}}{\mu_{min}}\Bigg(2\mathbf{Ar}|\Omega|^{5/6}6^{1/6}\max\{1, \frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\} + \frac{1}{\rho_{min}}\big(\|\vec{\xi}\|^2_{L^6(\Omega)^3} + \frac{2\mu_{max}}{\mathbf{Re}}\|D(\vec{\xi})\|_{L^2(\Omega)^9}\big)\Bigg). \quad (66)$$

Then for all $\alpha > 0$ such that

$$\Delta_1 = \frac{1}{\mathbf{Re}}\big(B_{00} - \alpha B_{02}\Big(E_{00} + E_{01}(\|\vec{\xi}\|_{L^3(\Omega)^3} + 48^{1/12}|\Omega|^{1/6}R)\Big)\big) - \alpha B_{01}\sqrt{\frac{\mathbf{Pe}}{\lambda_0}}\big(\sqrt{2A_0} + \sqrt{2B_0}\big(\|\vec{\xi}\|^{1/2}_{L^1(\Omega)^3} + |\Omega|^{1/4}\sqrt{H}R^{1/2}\big)\big) > 0 \text{ and} \quad (67)$$

E11

$$\Delta_2 = \Delta_1^2 - 4A_{00}\alpha\sqrt{\frac{2\,\mathbf{Pe}}{\lambda_0}}\Big[\sqrt{A_0} + \sqrt{B_0}\Big(\|\vec{\xi}\|^{1/2}_{L^1(\Omega)^3} + |\Omega|^{1/4}\sqrt{R}\,\Big)\Big] \cdot$$

$$\Big[C_{00} + \mathbf{Ar}\,C_{01}\max\{1, \frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\}\Big] > \frac{B_{00}^2}{2\mathbf{Re}^2}, \tag{68}$$

$\vec{W}^m \in B_R$ implies $\vec{W}^{m+1} \in B_R$.

E11

$$\Delta_2 = \Delta_1^2 - 4A_{00}\alpha\sqrt{\frac{2\,\mathbf{Pe}}{\lambda_0}}\Big[\sqrt{A_0} + \sqrt{B_0}\big(\|\vec{\xi}\|_{L^1(\Omega)^3}^{1/2} + |\Omega|^{1/4}\sqrt{R}\,\big)\Big]\cdot$$

$$\Big[C_{00} + \mathbf{Ar}\,C_{01}\max\{1, \frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0}\}\Big] > \frac{B_{00}^2}{2\mathbf{Re}^2}, \qquad (70)$$

$\vec{W}^m \in B_R$ implies $\vec{W}^{m+1} \in B_R$.

Theorem 2 There is a weak solution $\{\vartheta, \vec{v}\} \in W^{1,3}(\Omega) \times H^1(\Omega)^3$ for the **Problem A** , such that

$$\begin{cases} \|\nabla\vartheta\|_{L^2(\Omega)^3} \leq \sqrt{\frac{2\mathbf{Pe}}{\lambda_0}}\left(\sqrt{A_0 + B_0\|\vec{\xi}\|_{L^1(\Omega)}} + \sqrt{B_0|\Omega|^{1/2}R}\right), \\ -\frac{\|\mathcal{S}\|_{L^\infty(\Gamma_{lat})}}{q_0} \leq \vartheta \leq 1, \quad \text{and} \quad \|D(\vec{v} - \vec{\xi})\|_{L^2(\Omega)^9} \leq R, \end{cases} \qquad (71)$$

E12

where $\vec{\xi}$ is given by Proposition 2 with $\gamma = 2\mu_{min}/(\rho_{max}\textbf{Re})$.

Now we are in position to pass to the limit when the expansivity parameter α tends to zero.

First we remark that the *a priori* estimates from the previous section are *independent* of α, $|\alpha| \leq \alpha_0$, where α_0 is the maximal positive α satisfying (67)-(68). Consequently we have

E13

Theorem 3 Let $\{\vartheta(\alpha), \vec{v}(\alpha)\}$, $\alpha \in (0, \alpha_0)$, be a sequence of weak solutions to Problem **A**, satisfying the bounds (69). Then there exists $\{\vartheta^{OB}, \vec{v}^{OB}\} \in W^{1,3}(\Omega) \times H^1(\Omega)^3$ and a subsequence $\{\vartheta(\alpha_k), \vec{v}(\alpha_k)\}$ such that

$$\begin{cases} \vartheta(\alpha_k) \rightarrow \vartheta^{OB}, & \text{uniformly on } \bar{\Omega} \\ \vartheta(\alpha_k) \rightharpoonup \vartheta^{OB}, & \text{weakly in } W^{1,3}(\Omega) \\ \vec{v}(\alpha) \rightharpoonup \vec{v}^{OB}, & \text{weakly in } H^1(\Omega)^3. \end{cases}$$

Furthermore, $\{\vartheta^{OB}, \vec{v}^{OB}\}$ is a weak solution for the equations (7) – (9) , satisfying the boundary conditions (17)-(21) and the bounds (69).

Uniqueness

Uniqueness

The uniqueness

Quite technical. For small data there is a unique weak solution $\{\vec{v}, \vartheta\} \in H^1(\Omega)^3 \times (W^{1,3}(\Omega) \cap C(\bar{\Omega}))$ for **Problem A,** satisfying the bounds (69) .

The regularity of solutions

Lemma 22 Let c_{p0} and $\lambda \in C^\infty(\mathbb{R})$. Furthermore let $\Gamma_{lat} \in C^\infty$ and $\mathcal{S} \in C^\infty(\bar{\Gamma}_{lat})$. Then $\vartheta \in W^{2,6}(\Omega) \subseteq C^{1,1/2}(\bar{\Omega})$.

Lemma 23 Let $\vartheta \in W^{2,6}(\Omega)$, let $\Gamma_{lat} \in C^\infty$ and let $v_j \in C^\infty$, $j = 1, 2$ satisfy (20). Then $\{\vec{w}, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $\forall q < +\infty$.

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Lemma 23 Let $\vartheta \in W^{2,6}(\Omega)$, let $\Gamma_{lat} \in C^\infty$ and let $v_j \in C^\infty$, $j = 1, 2$ satisfy (20). Then $\{\vec{w}, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $\forall q < +\infty$.

Theorem 24 Let the assumptions on the data from Lemmas 22 and 23 hold true. Then every weak solution $\{\vec{v}, p, \vartheta\} \in H^1(\Omega)^3 \times L^2(\Omega) \times (H^1(\Omega) \cap L^\infty(\Omega))$ for **Problem A** is an element of $W^{2,q}(\Omega)^3 \times W^{1,q}(\Omega) \times W^{2,q}(\Omega)$, $\forall q < \infty$. Furthermore $\{\vec{v}, p, \vartheta\} \in C^\infty(\Omega)^5$.

The Boussinesq limit

In this section we reconsider the limit when the expansivity parameter α tends to zero. We saw at the end of the section on existence of a weak solution that the obtained *a priori* estimates allow passing to the Boussinesq limit.
Having justified the Oberbeck-Boussinesq system as the limit equations when the expansivity parameter α tends to zero, the next question is: **What is the accuracy of the approximation ?**.
The answer relies on the uniqueness and regularity results from the previous sections.

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The answer relies on the uniqueness and regularity results from the previous sections.
We start by studying the equations for the derivatives with respect to α. For simplicity we suppose that

$$V_1,\, v_2 \;\text{ are independent of }\; \alpha \;\text{ and } v_1 = V_1/(1-\alpha). \tag{73}$$

BL1

Proposition 25 Let us suppose that the conditions (67)-(68), and (smallness of data), ensuring existence, uniqueness and regularity of a solution lying inside the ball defined by the bounds (69). Furthermore let the solution $\{\vec{v}, p, \vartheta\}$ satisfies the inequalities

$$\mathcal{N} = \frac{\lambda_0}{\mathbf{Pe}} - \frac{C_6(\Omega)}{2}(1 + \frac{H}{\sqrt{2}})\Bigg(\|c_{p1}\|_{L^\infty(I\!R)} \|\vec{v}\|_{L^3(\Omega)^3} \rho_{max} + \frac{\|\lambda'\|_{L^\infty(I\!R)}}{\mathbf{Pe}} \|\nabla\vartheta\|_{L^3(\Omega)^3} \Bigg) > 0 \tag{74}$$

$$\frac{2}{\mathbf{Re}} \frac{\mu_{min}}{\rho_{max}} > 4(\frac{3}{2})^{1/4}\sqrt{H}\|D(\vec{v})\|_{L^2(\Omega)^9} + H\mathcal{L}_\Theta\Big\{ \frac{2\alpha}{\mathbf{Re}} \frac{\mu_{max}}{\rho_{min}} \|\vec{v}\|_{L^\infty(\Omega)^3} + \mathbf{Ar}\frac{H^2}{\sqrt{2}} + \frac{2}{\mathbf{Re}}\|\mu'\|_{L^\infty(I\!R)}\|D(\vec{v})\|_{L^3(\Omega)^9} C_6(\Omega) + \alpha H \|\vec{v}\|^2_{L^\infty(\Omega)^3} \Big\}, \tag{75}$$

BL2

where

$$\mathcal{L}_\Theta = \frac{H \|c_{p1}\|_{L^\infty(I\!R)} \|\vartheta\|_{L^\infty(\Omega)}}{\sqrt{2}\mathcal{N}} \tag{76}$$

Then derivatives of the solution, with respect to α, exist at all orders as continuous functions of α.

BL2

where

$$\mathcal{L}_\Theta = \frac{H\|c_{p1}\|_{L^\infty(I\!R)}\|\vartheta\|_{L^\infty(\Omega)}}{\sqrt{2}\mathcal{N}} \tag{77}$$

Then derivatives of the solution, with respect to α, exist at all orders as continuous functions of α.

With this result, we are ready to state the error estimate for Boussinesq's limit.
First, we write the 1st order correction, i.e. the system defining the first derivatives $\{\vec{w}^0, \tilde{\pi}^0, \theta^0\} = \frac{d}{d\alpha}\{\vec{v}, \tilde{p}, \vartheta\}|_{\alpha=0}$:

BL3

$$\text{div}\,\left\{\vec{w}^0 - \vartheta^{OB}\vec{v}^{OB}\right\} = 0 \quad \text{in} \quad \Omega \tag{78}$$

$$-\vartheta^{OB}(\vec{v}^{OB}\nabla)\vec{v}^{OB} + \left\{(\vec{w}^0\nabla)\vec{v}^{OB} + (\vec{v}^{OB}\nabla)\vec{w}^0\right\} = -\mathbf{Ar}\,\theta^0\vec{e}_g - \nabla\tilde{\pi}^0$$
$$+\frac{2}{\mathbf{Re}}\,\text{Div}\,\left\{\mu(\vartheta^{OB})D(\vec{w}^0) + \mu'(\vartheta^{OB})\theta^0 D(\vec{v}^{OB})\right\} \;\text{in}\; \Omega \tag{79}$$

$$\text{div}\,\left\{-\frac{\lambda(\vartheta^{OB})}{\mathbf{Pe}}\nabla\theta^0 + (\vec{v}^{OB}c_p(\vartheta^{OB}) - \frac{\lambda'(\vartheta^{OB})}{\mathbf{Pe}}\nabla\vartheta^{OB})\theta^0 - \right.$$
$$\left.\vartheta^{OB}\vec{v}^{OB}C_p(\vartheta^{OB}) + \vec{w}^0 C_p(\vartheta^{OB})\right\} = 0 \quad \text{in} \quad \Omega \tag{80}$$

$$\theta^0 = 0, \vec{w}^0 = 0 \;\text{on}\; \Gamma_{out}; \quad \theta^0 = 0, \vec{w}^0 = V_2\vec{e}_g \;\text{on}\; \Gamma_{in} \tag{81}$$

$$\vec{w}^0 = 0 \;\text{and}\; -\frac{1}{\mathbf{Pe}}(\lambda(\vartheta^{OB})\nabla\theta^0 +$$
$$\lambda'(\vartheta^{OB})\theta^0\nabla\vartheta^{OB})\cdot\vec{n} = q_0\theta^0 \;\text{on}\; \Gamma_{lat}, \tag{82}$$

BL4

Under the conditions of the preceding Proposition, with $\alpha = 0$, the system (78)-(82) has a unique smooth solution. Hence we have established rigorously the $\mathcal{O}(\alpha^2)$ approximation for **Problem A.** Clearly, one could continue to any order.
The result is given by the following theorem, which is a straightforward corollary of Proposition 25.

BL4

Under the conditions of the preceding Proposition, with $\alpha = 0$, the system (78)-(82) has a unique smooth solution. Hence we have established rigorously the $\mathcal{O}(\alpha^2)$ approximation for **Problem A**. Clearly, one could continue to any order.
The result is given by the following theorem, which is a straightforward corollary of Proposition 25.
Theorem 26 Let us suppose the assumptions of Proposition 25. Then we have

$$\|\vec{v} - \vec{v}^{OB} - \alpha \vec{w}^0\|_{W^{1,\infty}(\Omega)^3} + \|\vartheta - \vartheta^{OB} - \alpha\theta^0\|_{W^{1,\infty}(\Omega)} \leq C\alpha^2 \tag{85}$$

$$\inf_{\mathcal{C}\in\mathbb{R}} \|p - p^{OB} - \alpha p^0 + \mathcal{C}\|_{L^\infty(\Omega)^3} \leq C\alpha^2 \tag{86}$$

where $p^0 = \tilde{\pi}^0 - 2\mu(\vartheta^{OB})$ div $\vec{w}^0/(3$ **Re**$)$.

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