Non-isothermal flow of molten glass: mathematical challenges and industrial questions

Andro Mikelić

Institut Camille Jordan, UFR Mathématiques,

Université Claude Bernard Lyon 1, Lyon, FRANCE

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It is a great pleasure for me to be here, in Montecatini Terme, and lecture during this week. I wish to thank the organizers of the workshop for inviting me to give a talk, and to Antonio Fasano for the hospitality.

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The results are obtained in collaboration with A. Farina and A. Fasano (Dipartimento di Matematica, Universita degli Studi di Firenze, Italy).

The numerical results on the model, by Thierry Clopeau (UFR Maths, Lyon) and Javier Olaiz (Ezus, Lyon) will not be presented now.

References:

Detailed mathematical proof corresponding to the talks could be found in

[1] A. Farina, A. Fasano, A. Mikelić: On the equations governing the flow of mechanically incompressible, but thermally expansible, viscous fluids, to appear in *M^3AS : Math. Models Methods Appl. Sci.,* Vol. 18 (2008), no. 6, p. 813-858.

[2] T. Clopeau, A. Farina, A. Fasano, A. Mikelić: Asymptotic equations for the terminal phase of glass fiber drawing and their analysis, under revision in *Nonlinear Analysis TMA: Real World Applications,* 2008.

[3] E. Feiereisl, Ph. Laurençot, A. Mikelić: Global solutions to the Matovich-Pearson equations. in preparation.

Heat and glass 1

- Glass is an *amorphous* material. Its molecules are not arranged in a regular, specific pattern, like those of a crystalline material, but are random in their configuration.
- $\Rightarrow$ glass reacts to heat differently than do other materials. Whereas metals on heating instantaneously change from solid to liquid once they reach a specific temperature (called the melting point), glass goes through a very gradual transformation from a material which behaves like a solid to a material which behaves like a liquid.
- It is this unique characteristic of glass which allows it to be blown, or to be worked in the myriad ways which we call *kilnforming*.

Heat and glass 2

- Because of its amorphous molecular configuration glass at room temperature is sometimes referred to as a *supercooled liquid*. Even in its solid form its molecular structure is that of a stiff liquid.
- As it is heated, glass gradually begins to behave more and more like a liquid until at temperatures above 1093°C it will flow easily with a consistency similar to honey.
- The temperatures at which glass is worked in a kiln are usually between 593°C and 927°C. Within this range a wide variety of effects may be achieved by a variety of processes.

Drawing of continuous glass fibers 1

- The drawing of continuous glass fibers is a widely used procedure. Continuous glass fibers are produced in a range of diameters and to tight tolerances for different end uses (reinforcing plastics, cement, paper etc). They could be woven into industrial fabrics.
- Industrial glass fibers are manufactured using a bushing with more than a thousand nozzles. A glass melting furnace supplies it with molten glass with a temperature between 1100 and 1500°C.
- The hot glass melt is then drawn down into a fiber by a drawing force. The result is production in parallel of many fibers, which are cooled and collected on the rotating drum.

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Drawing of continuous glass fibers 3

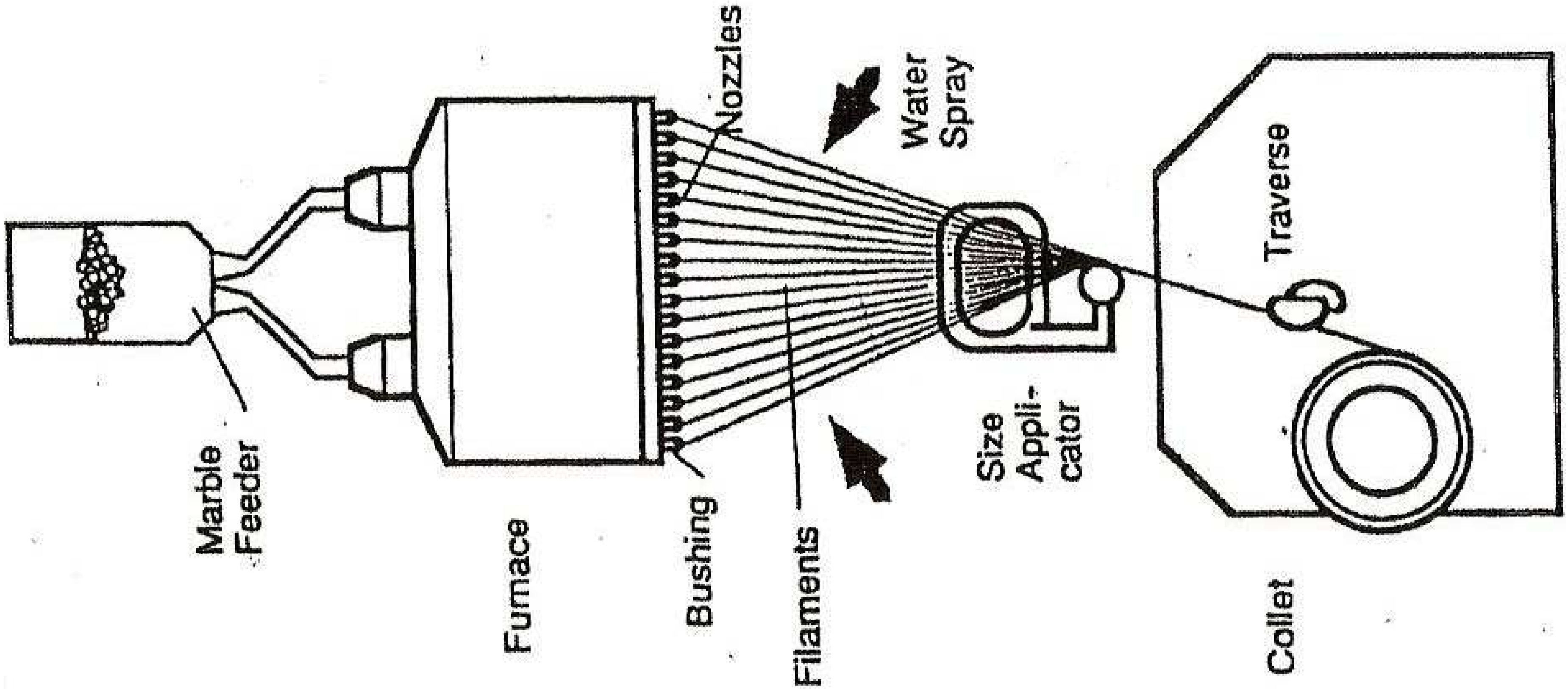


Figure 1: *Industrial drawing of continuous glass fibers*

Drawing of a single glass fiber 1

- In order to understand the glass fiber forming process, it is important to study the drawing of a single glass fiber.
- It is an important simplification, since we can disregard interaction between fibers and between fibers and the surrounding air.
- For a single glass fiber, hot glass melt is drawn from a dye into air. After leaving the dye, the molten glass forms a free liquid jet. The jet is cooled and attenuated as it proceeds through the air. Finally, the cold fiber is collected on a rotating drum.
- A mathematical model of the manufacturing procedure contains necessarily both free boundary hydrodynamics and thermal processes. Their coupling comes from the temperature dependent viscosity, density and surface tension.

Drawing of a single glass fiber 2

We may distinguish four stages (see Fig. 2):

(a) The flow of molten glass at high temperature in the reservoir, feeding the fiber production system.

(b) The non-isothermal flow through the die, with rigid lateral boundaries.

(c) The viscous jet flow with rapidly changing physical parameters, owing to the fast cooling, up to the formation of a "fiber" (high viscosity, small variation of the axial velocity and very small radial velocity).

(d) The motion of the glass fiber, drawn down by a device called spinner (or spool).

From (b) to (d) axial velocity changes by several orders of magnitude.

Drawing of a single glass fiber 4

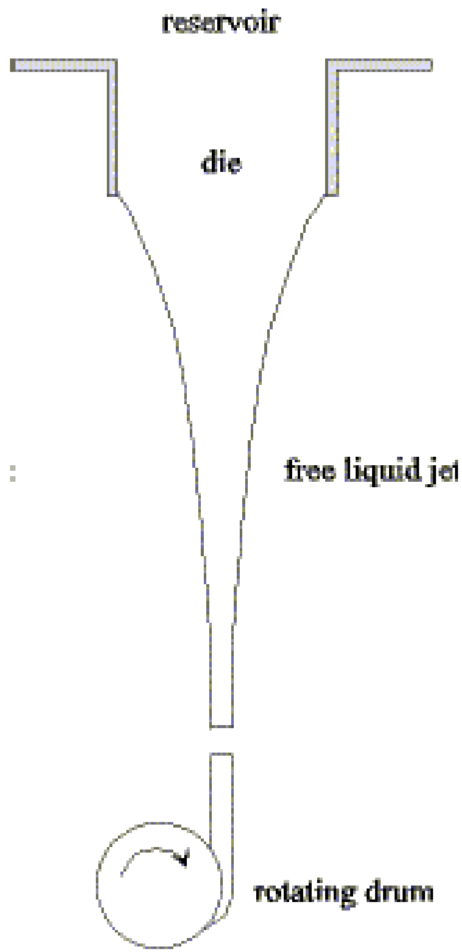


Figure 2: *A schematic of the four stages in the glass fibers manufacturing process.*

Molten glass 1

- We suppose that molten glass is a Newtonian fluid.
- Industrial unknowns are the following:
  - velocity is $\mathbf{v} = v_z \mathbf{e}_z + v_r \mathbf{e}_r$;
  - hydrodynamic pressure is p;
  - temperature is T;
  - fiber radius (being the distance from the symmetry axis) is $R = R(t, z)$;
  - Coefficients measured for the industrial aplications are the following:
    - specific heat is $c_p = c_p(T)$;
    - density is $\rho = \rho(T)$;
    - viscosity is $\mu = \mu(T)$;
    - surface tension is $\sigma = \sigma(T)$;
    - thermal conductivity is $\lambda = \lambda(T)$.

Molten glass II

- Concerning the shear viscosity μ we assume the well known Vogel-Fulcher-Tamman's (VFT) formula

$$\log \mu(\vartheta) = -C_\mu + \frac{A_\mu}{T_R - B_\mu + T_R\left(\widetilde{T}_w - 1\right)\vartheta}, \quad A_\mu, B_\mu, C_\mu > 0. \tag{1}$$

μ is monotonically decreasing with ϑ. We note that the temperature in the problems we are considering is such that the denominator in (1) is always positive. ϑ is the **rescaled dimensionless temperature**, that is

$$\vartheta = \frac{T - T_R}{T_w - T_R}, \quad \Leftrightarrow \quad T = T_R\left[1 + \vartheta\left(\widetilde{T}_w - 1\right)\right], \;\; \text{with} \;\; \widetilde{T}_w = \frac{T}{T} \tag{2}$$

where T_w, $T_w > T_R$, is another characteristic temperature (the temperature at the inlet boundary).

Molten glass III

The characteristic parameter values are the following:

- constant axial velocity of the spooler V_f;
- extrusion temperature T_E;
- extrusion density $\rho_E = \rho(T_E)$;
- specific extrusion heat $c_{pE} = c_p(T_E)$;
- extrusion surface tension $\sigma_E = \sigma(T_E)$;
- extrusion viscosity $\mu_E = \mu(T_E)$;
- thermal conductivity of extrusion $\lambda_E = \lambda(T_E)$;
- ambient temperature T_∞;
- axial extrusion velocity v_E;
- extrusion heat transfer coefficient h_E;

Molten glass IV

- characteristic fiber length L;
- characteristic radius $R_E = R(\cdot, 0)$

All quantities evaluated at the extrusion temperature T_E are denoted with a suffix E.
We suppose that the radiation effects/heat transfer are given by the formula of Kase-Matsuo for the heat transfert coefficient

$$h = \frac{\lambda_\infty}{R(z,t)} C \Big(\frac{2\rho_\infty v_z(z,t) R(z,t)}{\mu_\infty} \Big)^m, \tag{3}$$

Industrial data: For the surrounding air, we take as the air viscosity $\mu_\infty = 53.8E - 6$ Pa s; for the air density we take $\rho_\infty = 0.232$ kg/m^3 and the thermal conductivity of the air is set to be $\lambda_\infty = 0.084$ W/mK.

Some industrial data

	Value 1	Value 2		Value 1	Value 2
h_E	$10\frac{Wm^2}{K}$	$2.36\frac{Wm^2}{K}$	σ_E	0.1 N/m	0.37 N/m
T_E	$1227^o\ C$	$1145.15^o\ C$	L	$0.06m$	$0.06m$
ρ_E	$2.4 \cdot 10^3 \frac{kg}{m^3}$	$2.735 \cdot 10^3 \frac{kg}{m^3}$	T_∞	$27^o\ C$	$612^o\ C$
c_{pE}	$1046.6\frac{J}{kgK}$	$1591.23\frac{J}{kgK}$	λ_E	$1,0\frac{W}{mK}$	$3\frac{W}{mK}$
C	$1,117$	0.42	m	0.137	0.334
μ_E	$83\frac{Nsec}{m^2}$	$179.17\frac{Nsec}{m^2}$	R_E	$0.000838m$	$0.000838m$
v_E	$0.0031\frac{m}{sec}$	$0.0007\frac{m}{sec}$	T_g	$627^o\ C$	$635^o\ C$

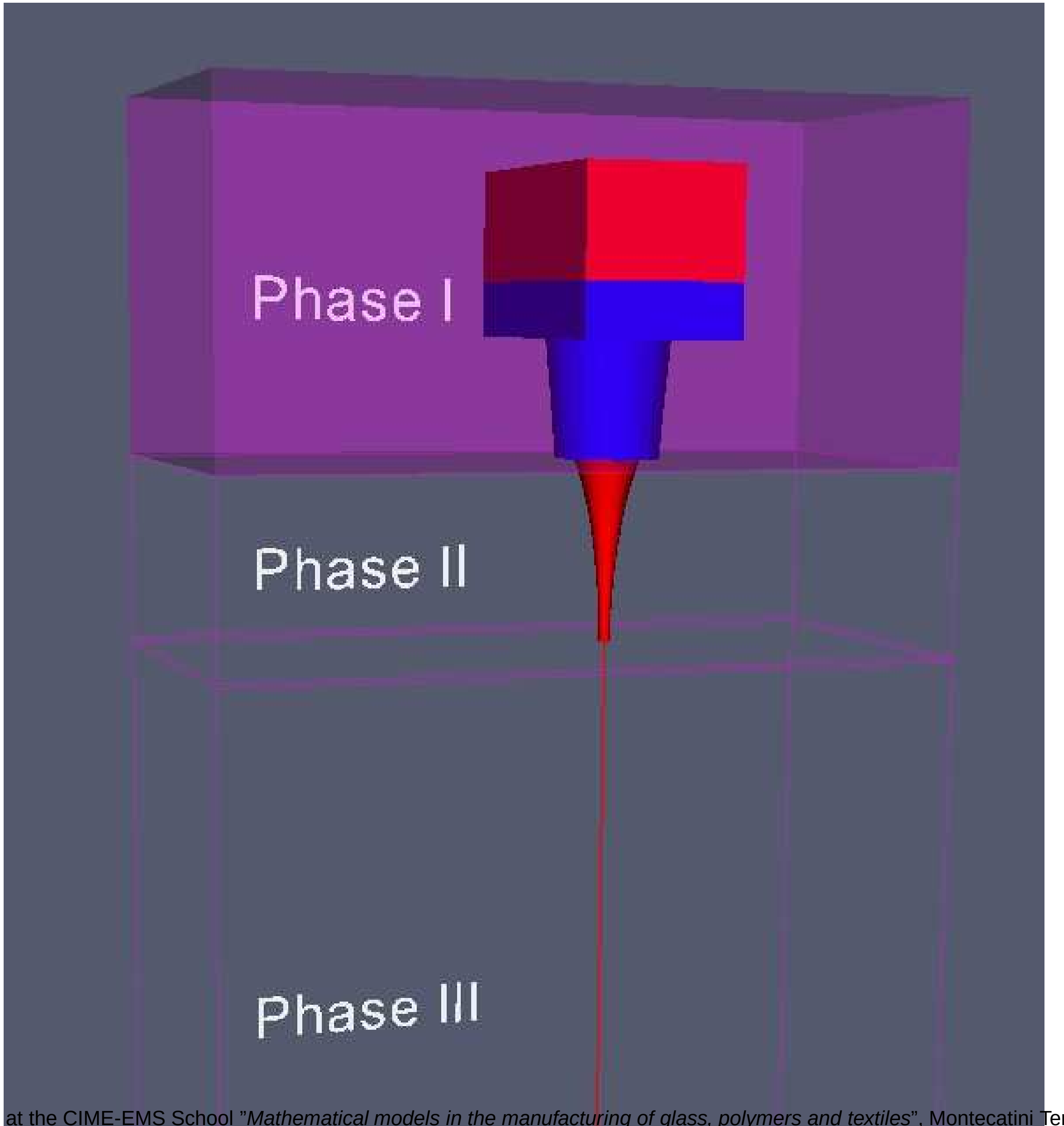
Data from the reference *G. Gupta, W.W. Schultz: Non-isothermal flows of Newtonian slender glass fibers. Int. J. Non-Linear Mech.,*

Some industrial data II

Vol. 33 (1998), p. 151-163. (Value 1), and from the reference *R. von der Ohe, Simulation of Glass Fiber Forming Processes, Thesis PhD (2005) - Aalborg University, 2005. pp. 192.* (Value 2)

	Value 1	Value 2
V_f	$14,46$ m/sec	35 m/sec
ε **Ca** $= \mu_E v_E R_E/(L\sigma_E)$	0.036	0.00475
Bo $= \rho_E g L^2/(\mu_E v_E)$	326	767.94
Re $= \rho_E v_E L/\mu_E$	$5.42 \cdot 10^{-3}$	$6.43 \cdot 10^{-4}$
Pe $= c_{pE}\rho_E v_E L/\lambda_E$	470.82	61.102
Bi $= h_E R_E/\lambda_E$	$8.38 \cdot 10^{-3}$	$1.98 \cdot 10^{-3}$
2 **Bi** / (**Pe** ε^2)	0.1845	0.332

I1a



Mathematical model I

Therefore, we need a mathematical model which describes the motion of a **mechanically incompressible, but thermally expansible viscous fluid.** Such model could be widely used in industrial simulations of flows of hot melted glasses, polymers etc.

The model for a mechanically incompressible, but thermally expansible viscous fluid could be thought as a particular case of the compressible heat-conducting Navier-Stokes system.

Nevertheless, the pressure isn't linked any more to the density and to the temperature and it plays the same role like in the incompressible case. Consequently, we should be careful with the thermodynamical modeling.

Important function is the **thermal expansion coefficient** $\beta = \frac{d}{dT} \log \rho$ and its typical value β_R.

Mathematical model II

Following the lines from the related papers:
R.H. Hills, P.H. Roberts, *On the motion of a fluid that is incompressible in a generalized sense and its relationship to the Boussinesq Approximation*, Stability Appl. Anal. Continua, Vol. 3 (1991), pp. 205-212.
K.R. Rajagopal, M. Růžička, A.R. Srinivasa, *On the Oberbeck-Boussinesq approximation*, Math. Models Methods Appl. Sci, Vol. 6 (1996), p. 1157-1167.
R.Kh. Zeytounian, *The Bénard problem for deep convection: rigorous derivation of approximate equations*, Int. J. Engng. Sci., Vol. 27 (1989), pp. 1361-1366.
R.Kh. Zeytounian, *Joseph Boussinesq and his approximation: a contemporary view*, C.R. Mécanique, Vol. 331 (2003), p. 575-586. ,
we obtained in [1] a thermodynamically consistent derivation of the model.

Mathematical model III

This will be subject of the lecture by Dr. Farina. He will also discuss the linear stability of the model.

MODEL:

$$\text{div}\ \vec{v} = -\frac{K_\rho}{\rho(\vartheta)}(T_w - 1)\frac{D\vartheta}{Dt} \tag{4}$$

$$\rho(\vartheta)\frac{D\vec{v}}{Dt} = -\frac{K_\rho(T_w - 1)\vartheta}{\mathbf{Fr}^2}\rho(\vartheta)\vec{e}_3 - \nabla P + \frac{1}{\mathbf{Re}}\ \text{Div}\left\{2\mu(\vartheta)D(\vec{v}) - \frac{2\mu(\vartheta)}{3}\ \text{div}\ \vec{v}I\right\} \tag{5}$$

$$P = p + \rho_R g x_3,\ \rho(\vartheta) = \rho_R - \beta_R \rho_R T_R \left(\widetilde{T}_w - 1\right)\vartheta, \tag{6}$$

Mathematical model IV

$$\rho(\vartheta)c_{p_1}(\vartheta)\frac{D\vartheta}{Dt} = \left(\frac{|K_\rho|\,P_R}{\rho_R c_{pR} T_R\,(T_w-1)}\right)\frac{1+(T_w-1)\vartheta}{\rho(\vartheta)}\Big[\frac{DP}{Dt}$$
$$\frac{\rho_R g H}{P_R}v_3\Big] + \frac{1}{\mathbf{Pe}}\,\text{div}\,(\lambda\nabla\vartheta) + 2\frac{\mathbf{Ec}}{\textbf{Re}(T_w-1)}\mu(\vartheta)\Big(|D(\vec{v})|_2^2 -$$
$$\frac{1}{3}(\,\text{div}\,\vec{v})^2\Big), \qquad (7)$$

In the phenomena, we are considering, $T_w - 1$ is small but not negligible. Typically $T_w - 1$ is of order 10^{-1}

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$$\frac{1}{3}(\ \mathrm{div}\ \vec{v})^2\Big), \qquad (8)$$

In the phenomena, we are considering, $T_w - 1$ is small but not negligible. Typically $T_w - 1$ is of order 10^{-1} with

$$c_{p\,1}(\vartheta) = -\frac{1+(T_w-1)\vartheta}{(T_w-1)^2}\frac{d^2\psi}{d\vartheta^2},\ \ \psi = e - Ts,$$

(the specific Helmholtz free energy).

Mathematical model V

We list the non-dimensional characteristic numbers appearing in (4)–(7):

Re = $\frac{\rho_R V L}{\mu_R}$ is Reynolds' number. **Fr**= $\frac{V}{\sqrt{gL}}$ is Froude's number.

Pr = $\frac{\mu_R c_{pR}}{\lambda_R}$ is Prandtl's number. **Ec**= $\frac{V^2}{c_{pR} T_R}$ is Eckert's number.

Pe = **Re**·**Pr**$= \frac{V L \rho_R c_{pR}}{\lambda_R}$ is Peclet's number. **Ma**= $\frac{V}{c}$ is Mach's number.

K_ρ = $-\beta_R T_R$ is the thermal expansivity number.

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$K_\rho = -\beta_R T_R$ is the thermal expansivity number.

In particular,

$$\rho(\vartheta) = 1 + K_\rho (T_w - 1)\vartheta.$$

Mathematical model VI

We are interested in studying flows of very viscous heated fluids (liquid glasses, . . .). , i.e. the fluids in question are mechanically incompressible but thermally compressible.

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We are interested in studying flows of very viscous heated fluids (liquid glasses, $\ldots$). , i.e. the fluids in question are mechanically incompressible but thermally compressible. For the energy equation (7), this means that the Eckert number is very small (frequently of order 10^{-12}) and the Mach number is also quite small (of order 10^{-6}). Consequently, the terms containing the Eckert number are dropped in applications.

$$\rho(\vartheta) = 1 - \alpha\vartheta.$$

Next, we define the **Archimedes' number**

$$\mathbf{Ar} = \frac{|K_\rho|\,(T_w - 1)}{\mathbf{Fr}^2} = \frac{|K_\rho|\,(T_w - 1)}{V_R^2} gH, \;\; \Rightarrow \;\; \alpha = \mathbf{Ar}\,\mathbf{Fr}^2.$$

Mathematical model VII

We complete this modeling section by recalling formal derivations of the Oberbeck-Boussinesq system. In the applications we consider, the parameters are close to the conditions of the formal derivation of the Oberbeck-Boussinesq approximation from Schlichting's book, pages 86- 91:

$$T_w - 1 \approx 0, \quad V \approx 0, \quad \textbf{Ma} \approx 0 \quad \text{and} \quad \textbf{Ar} = \frac{-K_\rho}{\textbf{Fr}^2}(T_w - 1) = \mathcal{O}(1), \tag{9}$$

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where **Ar** is the Archimedes number. Having Archimedes' number of order 1 means that the buoyancy forces are important. Under these assumptions, the compressible Navier-Stokes system (4)-(7) could be approximated by the Oberbeck-Boussinesq system

Mathematical model VIII

$$\text{div } \vec{v}^{OB} = 0 \tag{11}$$

$$\frac{D\vec{v}^{OB}}{Dt} = -\textbf{Ar } \vartheta^{OB}\vec{e}_g - \nabla p_{mot}^{OB} + \frac{1}{\textbf{Re}} \text{ Div } \left\{2\mu(\vartheta^{OB})D(\vec{v}^{OB})\right\} \tag{12}$$

$$c_{p1}(\vartheta^{OB})\frac{D\vartheta^{OB}}{Dt} = \frac{1}{\textbf{Pe}} \text{ div } \left(\lambda(\vartheta^{OB})\nabla\vartheta^{OB}\right), \tag{13}$$

Mathematical model VIII

$$\text{div } \vec{v}^{OB} = 0 \tag{14}$$

$$\frac{D\vec{v}^{OB}}{Dt} = -\textbf{Ar } \vartheta^{OB}\vec{e}_g - \nabla p^{OB}_{mot} + \frac{1}{\textbf{Re}} \text{ Div } \{2\mu(\vartheta^{OB})D(\vec{v}^{OB})\} \tag{15}$$

$$c_{p1}(\vartheta^{OB})\frac{D\vartheta^{OB}}{Dt} = \frac{1}{\textbf{Pe}} \text{ div } \big(\lambda(\vartheta^{OB})\nabla\vartheta^{OB}\big), \tag{16}$$

In the situation which is of interest for us, the quantities from (9) are small but non-zero. Our goal is to study a model which generalizes the Oberbeck-Boussinesq approximation and reduces to it in the limit (9). Our equations read as follow:

Mathematical model IX

$$\text{div } \vec{v} = -\frac{K_\rho}{\rho(\vartheta)}(T_w - 1)\frac{D\vartheta}{Dt} \tag{17}$$

$$\rho(\vartheta)\frac{D\vec{v}}{Dt} = -\textbf{Ar } \vartheta\vec{e}_g - \nabla P + \frac{1}{\textbf{Re}} \text{ Div } \left\{2\mu(\vartheta)D(\vec{v})\right\} - \frac{2}{3\textbf{Re}}\nabla\Big(\mu(\vartheta) \text{ div } \vec{v}\Big) \tag{18}$$

$$\rho(\vartheta)c_{p1}(\vartheta)\frac{D\vartheta}{Dt} = \frac{1}{\textbf{Pe}} \text{ div } \Big(\lambda(\vartheta)\nabla\vartheta\Big), \tag{19}$$

Mathematical model IX

$$\text{div } \vec{v} = -\frac{K_\rho}{\rho(\vartheta)}(T_w - 1)\frac{D\vartheta}{Dt} \tag{20}$$

$$\rho(\vartheta)\frac{D\vec{v}}{Dt} = -\textbf{Ar } \vartheta\vec{e}_g - \nabla P + \frac{1}{\textbf{Re}} \text{ Div } \{2\mu(\vartheta)D(\vec{v})\} - \frac{2}{3\textbf{Re}}\nabla\big(\mu(\vartheta) \text{ div } \vec{v}\big) \tag{21}$$

$$\rho(\vartheta)c_{p1}(\vartheta)\frac{D\vartheta}{Dt} = \frac{1}{\textbf{Pe}} \text{ div } \big(\lambda(\vartheta)\nabla\vartheta\big), \tag{22}$$

We'll consider the system (17)-(19) in the realistic situation, when the parameter $\alpha = -K_\rho(T_w - 1)$ (the " **expansivity** " or the "**thermal expansion coefficient**") is small.

Boundary conditions for the energy equation

The glass thermal conductivity contains 2 terms:

$$\lambda = \lambda_T(T^0 K) + \lambda_{Ross}(T^0 K) \tag{23}$$

The 2nd term corresponds to the thermal conductivity due to the radiation effects. Expression for it comes from Rosseland's approximation and reads:

$$\lambda_{Ross}(T) = \frac{16\sigma_{SB} n^2 T^3}{3\beta_{Ross}} \; [W/(mK)], \tag{24}$$

with $\sigma_{SB} = 5,67038 \cdot 10^{-8}\ W/(m^2 K^4)$, $n \approx 1,5$ and $\beta_{Ross} = 132$. Consequently, the characteristic Rosseland's thermal conductivity is $\lambda_{RossR} = 12,68$.

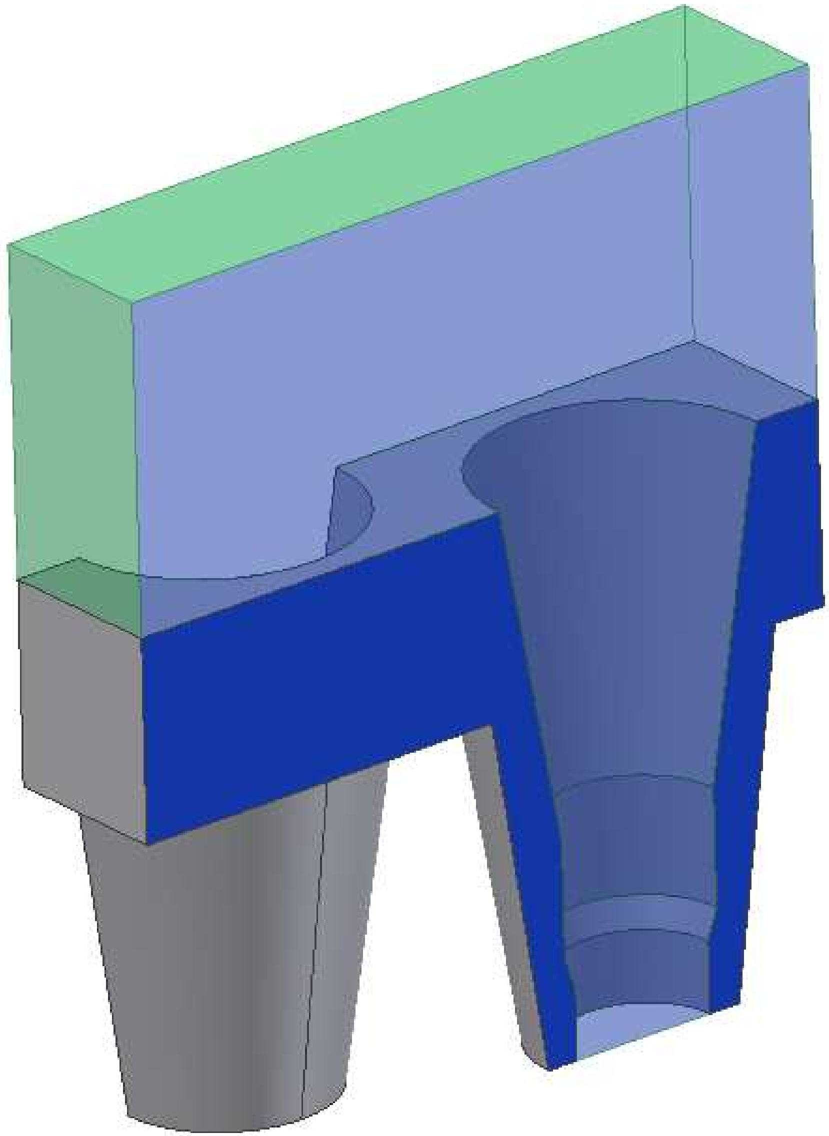
Boundary conditions for the energy equation I

Furthermore, for standard thermal conductivity we have $\lambda_{TR} = 2$, qui donne pour la conductivité thermique caractéristique $\lambda_R = 14,68$, leading to Peclet's number of order one in the interior of the die.

We note that The die is considered as a cylindrical domain $\Omega = \{r < R(x_3, \phi) \le 1\} \times [0, 2\pi] \times (0, 1)$ in $I\!R^3$. $R : [0, 1] \times [0, 2\pi] \to (0, 1]$ is a C^∞-map. The boundary of Ω contains 3 distinct parts: the lateral boundary $\Gamma_{lat} = \{r = R(x_3, \phi\}$, the inlet boundary $\Gamma_{in} = \{x_3 = 1$ and $r \le R(1, \phi),\ \phi \in [0, 2\pi]\}$ and the outlet boundary $\Gamma_{out} = \{x_3 = 0$ and $r \le R(0, \phi),\ \phi \in [0, 2\pi]\}$.

$\alpha \ge 0$

Boundary conditions for the energy equation I



Boundary conditions for the energy equation I

At the lateral boundary Γ_{lat} we impose the condition of the heat convection with radiation:

$$-\frac{1}{\mathbf{Pe}}\nabla T \cdot \vec{n} = \frac{Q_0}{c_{pR}\rho_R V}(T - \frac{T_{ext}}{T_R}) - \frac{\varphi_c}{c_{pR}\rho_R V T_R} + \frac{\mathcal{E} T_R^3}{c_{pR}\rho_R V}(T^4 - (\frac{T_{ext}}{T_R})^4) \quad (25)$$

Conclusion I

In the die, typical situation is the following:

- We deal with a flow with small expansivity and small Reynolds number. Archimedes number is of order 1. Therefore, the dominant term in the momentum equation is Stokes' operator with temperature dependent viscosity. Time changes are slow and they are observed at the time scale $t/$**Re**.
- Peclet's number is of order one and energy equation is a non-linear parabolic equation.
- Presence of geometrical singularities (angles etc) could lead to creation of vortices and interesting flow effects.
- Geometry influences strongly possible separation of the fiber in the interior of the die, leading to the well-known **fiber cold breakdown.**

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