# REGULARITY ESTIMATES for NONLINEAR ELLIPTIC and PARABOLIC PROBLEMS

## Course organizers


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## Course description

The issue of regularity has obviously played a central role in the theory of Partial Differential Equations, almost since its inception, and despite the tremendous development, it still remains a very fruitful research field.

Regularity estimates for degenerate and singular elliptic and parabolic equations have developed considerably in recent years, in many unexpected and challenging directions.

Let us mention only few of the main contributions.

- The celebrated Wiener criterion for quasi–linear elliptic equations has recently been given a new proof based on a sharp Harnack inequality for supersolutions, which allows for a straightforward generalizations to sub–elliptic equations (see [17]).
- An intrinsic version of the parabolic Harnack inequality for non–negative solutions of quasi–linear degenerate parabolic equations, a well–known and long–standing open problem, has recently been established, opening the way to a thorough understanding of the structure of solutions of quasi–linear degenerate parabolic equations (see [4, 5]).
- Uniqueness, symmetry and uniform rectifiability in some free boundary problems for elliptic equations of $p$–laplacian type have been recently considered by Lewis and Vogel, showing that they are essential tools in solving difficult regularity problems (see [10, 11]).
- Regularity for solutions of $\Delta_\infty$ is a very active research field. A full proof of the $C^1$ regularity has been given in the two–dimensional case (see [15]), and Evans and Savin have recently announced the extension to $C^{1,\alpha}$ (see [19]), but the issue remains open for higher dimension, together with closely connected research topics (see [6, 7]). Moreover it is an outstanding open problem to understand how much of this theory can be carried over to the sub–elliptic framework. If in the case of Carnot groups preliminary results are promising, the case of general Hörmander vector fields is quite more challenging and remains open.

- Boundary Harnack inequality are a fundamental tool in studying regularity; the situation for the $p \neq 2$ case is a lot less understood than for the $p = 2$ case, but recent progress has been made, that sheds new light on the phenomena involved (see, for example, [9]). The characterization of $p$–harmonic measure is closely connected; under this point of view, just to quote an example, an interesting step forward is given in [12], where a 25–year–old conjecture is solved.
- New game theoretic interpretations of the elliptic $\Delta_p$ operator with $1 \leq p \leq \infty$, have been given in the last years (see [8, 13, 14]); this point of view is at its very beginning, but promises to be really enlightening, with particular regards to the study of $p$–harmonic measure and $p$–capacity.
- When structural estimates for elliptic and parabolic Partial Differential Equations are at disposal, then they can be fruitfully exploited in exploring the local structure of free boundaries, as shown, for example, in [3, 16, 18].
- A free boundary problem that has recently seen a new interest is the boundary obstacle problem (the so–called *Signorini problem*), which is also linked to the obstacle problem for the fractional laplacian or similar integral–differential operators; in Mathematical Finance, it is a model for perpetual american options, which are governed by jump processes (see [1, 2].)

Because of all these recent developments, and many other we did not mention here, it seems to us timely to trace an overview that would highlight emerging trends and issues of this fascinating research topic in a proper and effective way.

We aim at a course, that would be a reference point in this field and serve at the same time as a source of inspirations for young researchers, very much in the spirit and tradition of CIME courses. Hence we expect in the audience Ph.D. students and researchers in PDE's, Applied Mathematics and related fields, and, more generally, everybody who would like to interact with some of the leading experts in this field and to learn about the latest developments.

## Course Lecturers

We propose that each of the following five senior experts gives a series of 5/6–hour–lectures:

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## Title of the Courses and Short Abstracts

Emmanuele DiBenedetto: **Introduction to Regularity Theory for Degenerate Parabolic Equations in Divergence Form**
Abstract - Some recent techniques are introduced to investigate the local and global behavior of solutions of degenerate parabolic equations when their principal part fails to be coercive. The equations have to be regarded in their own intrinsic geometry, and the solutions have a limited degree of regularity. Identifying regularity classes as functions of the degenerate and/or singular structure of the PDE is part of an emerging theory which promises to yield an understanding of degeneracy and/or singularity in PDE's.

John Lewis: **Applications of Boundary Harnack Inequalities for $p$–Harmonic Functions and Related Topics**
Abstract -

Peter Lindqvist: **Regularity of Viscosity Supersolutions**
Abstract -

Juan J. Manfredi: **Introduction to random Tug–of–War Games and Partial Differential Equations**
Abstract - We present an introduction to the results of Peres, Schramm, Sheffield and Wilson [2008 JAMS] and Peres and Sheffield [Duke Math. J., to appear] on random Tug–of–War games, the $\infty$–Laplace operator and the $p$-Laplace operator. A tentative program is as follows:

- Probabilistic background (Kolmogorov theorem and Martingales).
- Game Theory background (Two–person zero–sum games).
- PDE background (The $p$–Laplacian for $1 < p \leq \infty$).

- $\infty$–harmonious and $p$–harmonious functions.

Sandro Salsa: **The Obstacle Problem for the Fractional Laplacian**
Abstract - We consider lower dimensional obstacle problems and the obstacle problem for the fractional laplacian. We show optimal regularity of the solution and analyze the structure of their free boundaries.

### Location and dates

We would like the course to be held in Cetraro at the end of June 2009, from Sunday $21^{st}$ to Saturday $27^{th}$, but of course we are more than willing to consider a different venue and/or a different week, if our suggestion could not be taken into consideration for organizational reasons.

### Covering of the costs

If this proposal is approved, we are very confident that we will be able to find further contributions to cover all the foreseeable expenses.

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