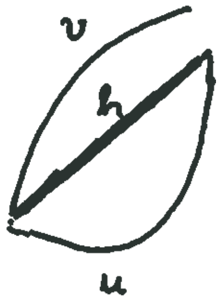
$$\boxed{\underbrace{\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right)}_{\Delta_p u} = 0}$$

p-Laplace eqn

$$\int_\Omega |\nabla u|^p dx = \min.$$



- Supersolutions: "$\Delta_p v \leq 0$"
- Solutions: $\Delta_p h = 0$
- Subsolutions: "$\Delta_p u \geq 0$"

$$\boxed{\frac{\partial v}{\partial t} = \nabla\cdot\left(|\nabla v|^{p-2}\nabla v\right)}$$

Evolutionary p-Laplace eqn
"p-parabolic eqn"
Non-Newtonian eqn of filtration

Supersolutions: "$\frac{\partial v}{\partial t} \geq \Delta_p v$"

$$\boxed{\frac{\partial v}{\partial t} = \Delta\left(|v|^{m-1}v\right)}$$

POROUS MEDIUM EQN.

$$\frac{\partial v}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left\{ \left| \sum_{k,m=1}^{n} a_{km}(x) \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_m} \right|^{\frac{p-2}{2}} a_{ij}(x) \frac{\partial u}{\partial x_j} \right\}$$

where $\sum a_{ij}(x)\zeta_i\zeta_j \geq \varepsilon|\zeta|^2$

$$\frac{\partial v}{\partial t} = \nabla\cdot\left(|\nabla v|^{p-2}\nabla v\right)$$

The BARENBLATT SOLUTION is

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-\frac{n}{\lambda}}\left[C - \frac{p-2}{p}\lambda^{\frac{1}{1-p}}\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}} \\ 0, \quad \text{when } t \leqq 0. \end{cases}$$

Here $p > 2$. It has a compact support in the x-variable at each fixed instant t. "Disturbances propagate with finite speed."

$$\int_0^1 \int_{|x|<1} |\nabla \mathcal{B}(x,t)|^p \, dx\, dt = \infty$$

J. KINNUNEN – P. LINDQVIST

– Annali della Scuola Normale Superiore di Pisa

– Annali di Matematica Pura ed Applicata

P. L – J. MANFREDI – Diff. Int. Equations

PROPOSITION Suppose that v is superharmonic in $\mathbb{R}^n$. Then the Sobolev derivative ∇v exists and

$$\int_{B_R} |\nabla v|^q dx < \infty$$

whenever $0 < q < \dfrac{n}{n-1}$.

RIESZ'S THEOREM $\Rightarrow$ PROPOSITION

$$\Delta v = 0$$

$$\nabla \cdot (|\nabla v|^{p-2} \nabla v) = 0$$

$$0 < q < \frac{n(p-1)}{n-1}$$

THE STATIONARY EQN

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

DEF Let $u \in W^{1,p}_{loc}(\Omega)$, $1 < p < \infty$. We say that u is a <u>weak</u> <u>solution</u> in Ω, if

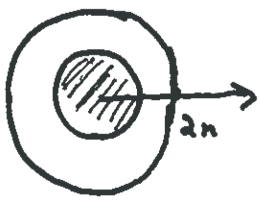
$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla \eta\rangle dx = 0$$

for all $\eta \in C_0^\infty(\Omega)$. If, in addition, u is continuous, it is called a p-<u>harmonic</u> <u>function</u>.

In fact, $u \in C^{1,\alpha}_{loc}(\Omega)$, $\alpha = \alpha(n,p)$
Ural'tseva

<u>Harnack's ineq.</u> If u is p-harmonic and $u \geq 0$ in the ball B_{2r}, then

$$\max_{\overline{B}_r}(u) \leq C_{n,p} \min_{\overline{B}_r}(u)$$



DIRICHLET'S PROBLEM

$\Omega \subset \mathbb{R}^n$ a bounded domain;

$f \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ represents the boundary values;

$$\begin{cases} \Delta_p u = 0 \text{ in } \Omega, \\ u - f \in W_0^{1,p}(\Omega). \end{cases}$$

The problem has a unique solution $u \in W^{1,p}(\Omega) \cap C(\Omega)$. The p-harmonic function u is obtained as the <u>minimizer</u> of the "p-energy":

$$\int_\Omega |\nabla u|^p dx \leq \int_\Omega |\nabla(u+\eta)|^p dx$$

whenever $\eta \in C_0^\infty(\Omega)$.

If the boundary $\partial\Omega$ is regular enough (a Lipschitz boundary will do), then the boundary values are attained in the classical sense:

$$\lim_{x \to \xi} u(x) = f(\xi), \quad \xi \in \partial\Omega.$$

OBSTACLE PROBLEM

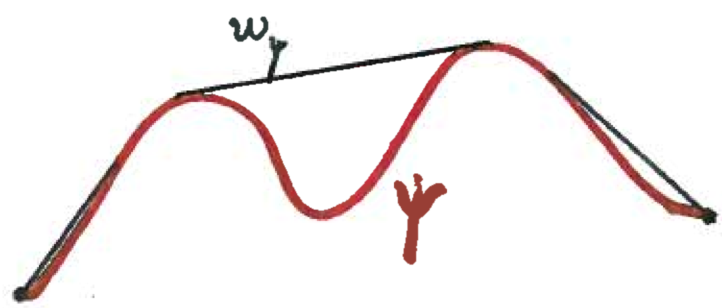
$\psi \in C(\bar{D}) \cap W^{1,p}(D)$ is the obstacle

$$\mathcal{F}_\psi = \{ w \in C(\bar{D}) \cap W^{1,p}(D) \mid w \geq \psi, \; w|_{\partial D} = \psi|_{\partial D} \}$$

THM If D is regular enough, there is a unique minimizer w_ψ such that

$$\int_D |\nabla w_\psi|^p dx = \min_{w \in \mathcal{F}_\psi} \int_D |\nabla w|^p dx.$$

In the open set $\{x \mid w_\psi(x) > \psi(x)\}$, where the obstacle does not hinder, w_ψ is a p-harmonic function



- weak supersolutions (test functions under the integral sign)
- p-superharmonic functions (defined via a comparison principle)
- viscosity supersolutions (test functions evaluated at points of contact)

DEF. We say that a function $v : \Omega \longrightarrow (-\infty, +\infty]$ is p-superharmonic in Ω if

(i) v is finite in a dense subset

(ii) v is lower semicontinuous

(iii) in each subset $D \subset\subset \Omega$ v obeys the Comparison Principle: if $h \in C(\overline{D})$ is p-harmonic in D, then

$$v|_{\partial D} \geq h|_{\partial D} \Longrightarrow v \geq h \text{ in } D$$

F. RIESZ $(p=2)$; $v \not\equiv \infty$ can replace (i).

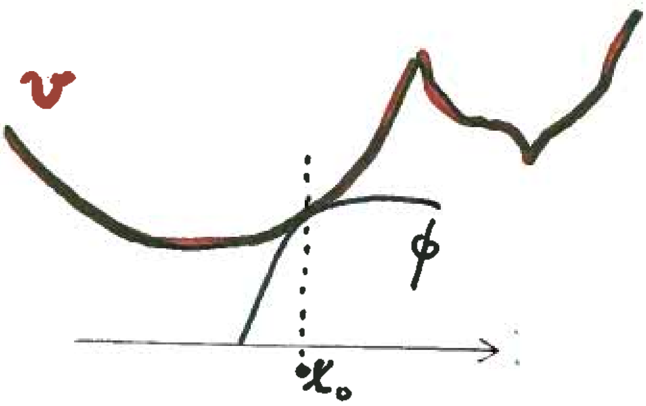
DEF. Let $p \geq 2$. We say that $v : \Omega \longrightarrow (-\infty, +\infty]$ is a <u>viscosity supersolution</u> in Ω if

(i) v is finite in a dense subset

(ii) v is lower semicontinuous

(iii) whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $v(x_0) = \phi(x_0)$ and $v(x) > \phi(x)$, when $x \neq x_0$, we have

$$\Delta_p \phi(x_0) \leq 0.$$



ϕ is touching from below at x_0

$$\Delta_p \phi \equiv |\nabla \phi|^{p-4} \left\{ |\nabla \phi|^2 \Delta \phi + (p-2) \sum_{i,j=1}^{n} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right\}$$

EXAMPLES

$$(n-p)\,|x|^{-\frac{n-p}{p-1}} \qquad (n \neq p)$$

$$\log\left(\frac{1}{|x|}\right) \qquad (n = p)$$

$$V(x) = \int \frac{\rho(y)\,dy}{|x-y|^{n-2}} \qquad (p = 2,\ n \geq 3)$$

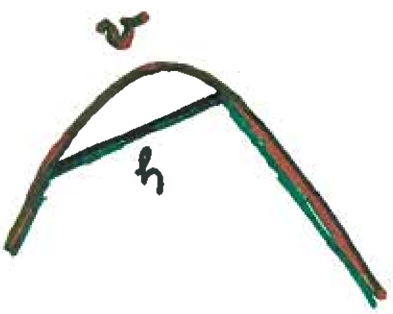
$$V(x) = \sum_{j=1}^{\infty} \frac{c_j}{|x-q_j|^{\frac{n-p}{p-1}}} \qquad (2 < p < n)$$

$q_1, q_2, \ldots$ rational points

$$V(x) = \int \frac{\rho(y)\,dy}{|x-y|^{\frac{n-p}{p-1}}} \qquad (2 < p < n)$$

$$v = \min\{v_1(x), \ldots, v_m(x)\}$$

$$v_k = \min\{v(x), k\}$$



PERRON MODIFICATION

$$v_{(\varepsilon)}(x) = \inf_y \left\{ v(y) + \frac{|x-y|^2}{2\varepsilon} \right\}$$

INFIMAL CONVOLUTION

LEMMA For $v \in C(\Omega) \cap W^{1,p}(\Omega)$ the following conditions are equivalent

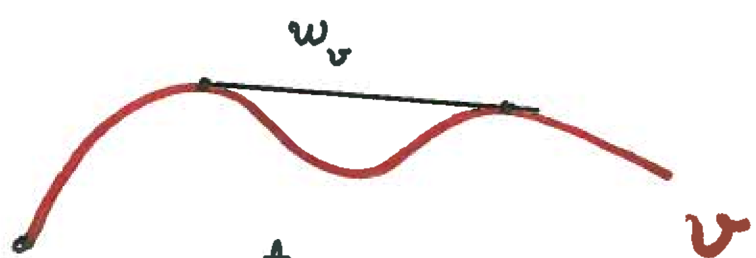
(i) $$\int_\Omega |\nabla v|^p dx \leq \int_\Omega |\nabla (v+\eta)|^p dx, \quad \eta \geq 0$$
$$\eta \in C_0^\infty(\Omega),$$

(ii) $$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle dx \geq 0, \quad \text{—''—},$$

(iii) v is p-superharmonic.

If, in addition, $\Delta_p v$ is continuous, then an equivalent condition is
$$\Delta_p v \leq 0.$$

About the <u>proof</u>: The crucial part is the sufficiency of (iii). Assume that v satisfies the Comparison Principle.



v acts as an obstacle

Choose a regular subdomain $D \subset\subset \Omega$ and let $w_v \geq v$ denote the solution in the class $\mathcal{F}_v(D)$ to the obstacle problem.

Then

$$\int_D \langle |\nabla w_v|^{p-2} \nabla w_v, \nabla \eta \rangle \geq 0$$

$$\eta \geq 0, \ \eta \in C_0^\infty(D)$$

CLAIM $\quad v = w_v$.

On the boundary of the set

$$\{w_v > v\}$$

$w_v = v$ and in the set w_v is p-harmonic. By the Comparison Principle $v \geq w_v$. $\lightning$

Consequences for $v \in C(\Omega) \cap W^{1,p}(\Omega)$

$$\boxed{\int_\Omega \zeta^p |\nabla v|^p dx \leq p^p \left(\operatorname*{osc}_{\zeta \neq 0}(v)\right)^p \int_\Omega |\nabla \zeta|^p dx}$$

CACCIOPPOLI

$v \geq 0$, $\alpha > 0$

$$\boxed{\int_\Omega \zeta^p v^{-1-\alpha} |\nabla v|^p dx \leq \left(\frac{p}{\alpha}\right)^p \int_\Omega v^{p-1-\alpha} |\nabla \zeta|^p dx}$$

$\alpha = p-1$ yields

$$\boxed{\int_\Omega |\zeta \nabla \log v|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla \zeta|^p dx}$$

Test functions

$\eta = (C - v(x))\zeta(x)^p$

$\eta = v^{-\alpha}\zeta^p$ ($v(x)+\varepsilon$ first)

$\eta = v^{1-p}\zeta^p$

APPROXIMATION

$0 \le v(x) \le L$ in Ω

v is lower semicontinuous

$$v_\varepsilon(x) = \inf_{y\in\Omega}\left\{v(y) + \frac{|x-y|^2}{2\varepsilon}\right\}$$

- $v_\varepsilon(x) \nearrow v(x)$ as $\varepsilon \longrightarrow 0+$
- $v_\varepsilon(x) - \dfrac{|x|^2}{2\varepsilon}$ loc. concave
- v_ε loc. Lipschitz continuous
- ∇v_ε exists in Sobolev's sense and $v_\varepsilon \in C(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$

If v, in addition, is p-superharmonic (a visc. supersolution), so is v_ε, though the domain shrinks to

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \sqrt{2L\varepsilon}\}.$$

PROPOSITION The approximant v_ε is a viscosity supersolution in Ω_ε.

Proof:
$$v_\varepsilon(x) = \inf_{y\in\Omega}\left\{v(y) + \frac{|x-y|^2}{2\varepsilon}\right\}$$
$$= v(x^*) + \frac{|x-x^*|^2}{2\varepsilon}$$

$$\frac{|x-x^*|^2}{2\varepsilon} \le v_\varepsilon(x) \le v(x) \le L \implies$$

$$|x-x^*| \le \sqrt{2L\varepsilon} < \operatorname{dist}(x, \partial\Omega)$$

Hence $x^* \in \Omega$, if $x \in \Omega_\varepsilon$.

VISCOSITY PROOF. Fix x_0 so that $x_0^* \in \Omega$.

ϕ touches v_ε from below at x_0

$$\phi(x_0) = v_\varepsilon(x_0) = \frac{|x_0 - x_0^*|^2}{2\varepsilon} + v(x_0^*)$$
$$\phi(x) \le v_\varepsilon(x) \le \frac{|x-y|^2}{2\varepsilon} + v(y)$$

Verify that
$$\psi(x) = \phi(x + x_0 - x_0^*) - \frac{|x-x_0^*|^2}{2\varepsilon}$$
touches v from below at x_0^*

$$\underbrace{\operatorname{div}\left(|\nabla\psi(x_0^*)|^{p-2}\nabla\psi(x_0^*)\right)}_{=\operatorname{div}\left(|\nabla\phi(x_0)|^{p-2}\nabla\phi(x_0)\right)} \leq 0$$

v visc. sup. solution.

because $\nabla\psi(x_0^*) = \nabla\phi(x_0)$

$D^2\psi(x_0^*) = D^2\phi(x_0)$ $\square$

PROOF BY COMPARISON $D \subset\subset \Omega_\varepsilon$,

$h \in C(\bar{D})$ p-harmonic,

$$v_\varepsilon(x) \geq h(x) \quad \text{on } \partial D.$$

$$\frac{|x-y|^2}{2\varepsilon} + v(y) \geq h(x), \quad x \in \partial D, \ y \in D$$

$y = x + z$, z small

$$w(x) \equiv \underbrace{v(x+z) + \frac{|z|^2}{2\varepsilon}}_{\substack{p\text{-superharmonic} \\ \text{in } \Omega_\varepsilon}} \geq h(x), \quad x \in \partial D$$

Comparison Principle $\Longrightarrow$ $w \geq h$ in D.

$w(x_0) \geq h(x_0)$, choose $z = x_0^* - x_0$

$$v_\varepsilon(x_0) \geq h(x_0). \qquad \square$$

COR The approximant v_ε is a weak supersolution in Ω_ε, i.e.,

$$\int_{\Omega_\varepsilon} \langle |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla \eta \rangle \, dx \geq 0$$

when $\eta \geq 0$, $\eta \in C_0^\infty(\Omega_\varepsilon)$.

$$\int_\Omega \zeta^p |\nabla v_\varepsilon|^p dx \leq (pL)^p \int_\Omega |\nabla \zeta|^p dx$$

ε = small enough

COMPACTNESS $\Rightarrow$

∇v exists and

$\nabla v_\varepsilon \longrightarrow \nabla v$ weakly in $L^p_{loc}(\Omega)$

$$\int_\Omega \zeta^p |\nabla v|^p dx \leq (pL)^p \int_\Omega |\nabla \zeta|^p dx$$

THEOREM Suppose that v is a <u>bounded</u> viscosity supersoln in Ω. Then

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}\right) \in L^p_{loc}(\Omega) \text{ exists}$$

in Sobolev's sense. Moreover v is a weak supersolution in the sense that

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx \geq 0$$

whenever $\eta \geq 0$, $\eta \in C_0^\infty(\Omega)$.

<u>Proof</u>: We may assume $0 \leq v \leq L$

$$\int \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx \overset{?}{=} \lim_{\varepsilon \to 0+} \int \langle |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla \eta \rangle \, dx$$

$$\geq 0$$

[LEMMA $\nabla v_\varepsilon \longrightarrow \nabla v$ strongly in $L^p_{loc}(\Omega)$.]

$$\langle |b|^{p-2} b - |a|^{p-2} a, b-a \rangle \geq 2^{2-p} |b-a|^p$$

$$(p \geq 2)$$

$$\theta \in C_0^\infty(\Omega), \qquad \eta = (v - v_\varepsilon)\theta \ \geq 0$$

$$0 \leq \theta \leq 1$$

$$I_\varepsilon = \int_\Omega \langle |\nabla v|^{p-2}\nabla v - |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla((v - v_\varepsilon)\theta)\rangle \, dx$$

$$\leq \int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla((v-v_\varepsilon)\theta)\rangle \, dx \longrightarrow 0$$

because of the weak conv. $\nabla v_\varepsilon \longrightarrow \nabla v$

$$I_\varepsilon = \int_\Omega \langle |\nabla v|^{p-2}\nabla v - |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla v - \nabla v_\varepsilon\rangle \theta \, dx$$

$$+ \underbrace{\int_\Omega (v-v_\varepsilon)\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla\theta\rangle \, dx}$$

Majorized by

$$\underbrace{\left\{\int (v-v_\varepsilon)^p dx\right\}^{1/p}}_{\longrightarrow 0} \|\nabla\theta\|_\infty \overbrace{\left\{\|\nabla v\|_p^{p-1} + \|\nabla v_\varepsilon\|_p^{p-1}\right\}}^{\text{STAYS BOUNDED}}$$

Integrals only over the support of η

It follows that

$$\int_\Omega \Theta \left\langle |\nabla v|^{p-2} \nabla v - |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla v - \nabla v_\varepsilon \right\rangle dx$$

$$\longrightarrow 0. \qquad \square$$

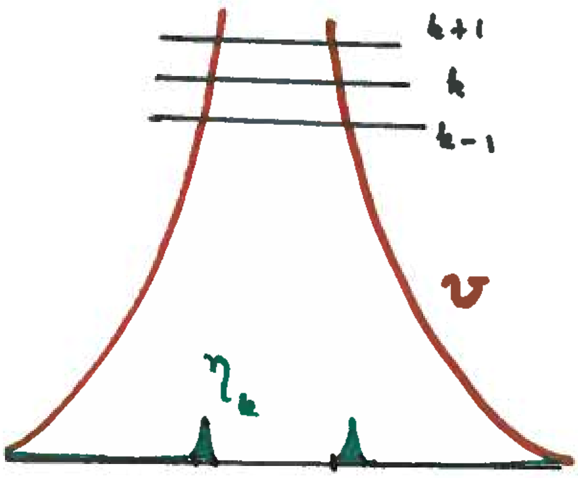
UNBOUNDED v $(1 < p \leq n)$

$$v_k = \min\{v(x), k\} \qquad k = 0, 1, 2, \ldots$$

$\Longrightarrow$ $v_k \in W^{1,p}_{loc}(\Omega)$ and v_k is a weak supersolution

The case $v \geq 0$, $v = 0$ on $\partial\Omega$ is easier

Assume $v \geq 0$, $v_k \in W^{1,p}_0(\Omega)$ to begin with; $k = 1, 2, 3, \ldots$.



$$\eta_k = (v_k - v_{k-1}) - (v_{k+1} - v_k)$$

$$\int_\Omega \langle |\nabla v_j|^{p-2} \nabla v_j , \nabla \eta_k \rangle \, dx \geq 0$$

$$k = 1, 2, \ldots, j-1$$

$$A_{k+1} = \int_\Omega \langle |\nabla v_j|^{p-2} \nabla v_j , \nabla v_{k+1} - \nabla v_k \rangle \, dx$$

$$\leq \int_\Omega \langle |\nabla v_j|^{p-2} \nabla v_j , \nabla v_k - \nabla v_{k-1} \rangle \, dx = A_k$$

$$A_{k+1} \leq A_k$$

$$A_k \leq A_1$$

ADD UP

$$\underbrace{\sum_{k=1}^{j} A_k}_{} \leq j A_1 = j \int_\Omega |\nabla v_1|^p \, dx$$

$$\int_\Omega |\nabla v_j|^p \, dx \leq \text{— " —}$$

$$(\dagger) \quad \boxed{\int_\Omega |\nabla v_j|^p \, dx \leq j \int_\Omega |\nabla v_1|^p \, dx}$$

$$1 < p < n$$

$$E_j = \{x \mid j < v(x) \le 2j\}$$

$$p^* = \frac{np}{n-p}$$ Estimate the mesures $|E_j|$.

$$j|E_j|^{\frac{1}{p^*}} \le \left(\int_{E_j} v_{2j}^{p^*}dx\right)^{\frac{1}{p^*}} \le \left(\int_\Omega v_{2j}^{p^*}dx\right)^{\frac{1}{p^*}}$$

$$\overset{\text{SOBOLEV}}{\le} S\left(\int_\Omega |\nabla v_{2j}|^p dx\right)^{\frac{1}{p}} \overset{(\dagger)}{\le} S(2jA_1)^{\frac{1}{p}}$$

$$\boxed{|E_j| \le CA_1^{\frac{n}{n-p}} j^{-\frac{n}{n-p}(p-1)}}$$

$$\int_\Omega v^\alpha dx \le |\Omega| + \sum_{j=1}^\infty \int_{E_{2^{j-1}}} v^\alpha dx$$

$$\le |\Omega| + \sum_{j=1}^\infty 2^{j\alpha}|E_{2^{j-1}}|$$

$$\lesssim \sum_{j=1}^\infty 2^{j\alpha} 2^{(j-1)\cdot\frac{-n}{n-p}(p-1)}$$

The geometric series converges when $\alpha < \frac{n}{n-p}(p-1)$.

We have proved

LEMMA: If $v \geq 0$ and $v_k \in W_0^{1,p}(\Omega)$, then

$$\int_\Omega |\nabla v_k|^p dx \leq k \int_\Omega |\nabla v_1|^p dx .$$

For $1 < p < n$,

$$\int_\Omega v^q dx \leq C_q \left(1 + \int_\Omega |\nabla v_1|^p dx\right)^{\frac{n}{n-p}}$$

whenever $q < \frac{n}{n-p}(p-1)$.

$$v_k = \min\{v, k\}$$

It remains to abandon the restriction with zero boundary values and to estimate

$$\int_\Omega |\nabla v_1|^p dx .$$

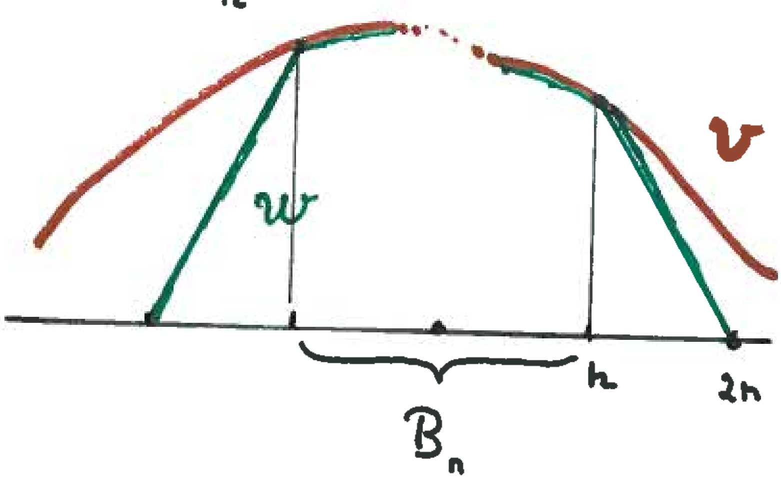
If $v \in C(\Omega) \cap W^{1,p}_{loc}(\Omega)$ is a weak supersolution and

$\overline{B_{2r}} \subset \Omega$, $\quad v \geq 0$ in B_{2r}

then
$$w = \begin{cases} v & \text{in } \overline{B_r} \\ h & \text{in } \overline{B_{2r}} \setminus \overline{B_r} \end{cases}$$

__is__ p-__superharmonic__ __in__ B_{2r}. Here h is the p-harmonic function in the annulus with boundary values
$$h\big|_{\partial B_{2r}} = 0, \quad h\big|_{\partial B_r} = v\big|_{\partial B_r}.$$

Hence $w \leq v$. IT IS ESSENTIAL THAT THE ORIGINAL v WAS DEFINED *EVEN* OUTSIDE $\overline{B_r}$



An easy estimate is

$$\int_{B_{2r}} |\nabla w|^p dx \leq c\|w\|_\infty^p r^{n-p}$$

Finally, if $v \in W^{1,p}_{loc}(\Omega)$ is merely semicontinuous and locally bounded, use v_ε (infimal convolution) and define

$$w_\varepsilon = \begin{cases} v_\varepsilon & \text{in } \overline{B_r} \\ h_\varepsilon & \text{in } \overline{B}_{2r} \setminus \overline{B}_r \end{cases}$$

as before. As $\varepsilon \longrightarrow 0$ we end up with a p-superharmonic function

$$w \in W^{1,p}_0(B_{2r}), \quad w = v \text{ in } B_r$$

$$\int_{B_{2r}} |\nabla w|^p dx \leq c\|w\|_\infty^p r^{n-p}$$

Now, do this with every $\min\{v(x), k\}$ in sight. In particular

$$\int_{B_{2r}} |\nabla w_1|^p \, dx \le c \cdot 1^p r^{n-p}$$

THEOREM Suppose that v is a visc. supersolution in Ω. Then

$$v \in L^q_{loc}(\Omega), \quad \text{whenever } q < \frac{n}{n-p}(p-1)$$

in the case $1 < p \le n$ and v is continuous if $p > n$. Moreover ∇v exists in Sobolev's sense and

$$\nabla v \in L^q_{loc}(\Omega), \quad \text{whenever } q < \frac{n}{n-1}(p-1)$$

in the case $1 < p \le n$. In the case $p > n$, we have $\nabla v \in L^p_{loc}(\Omega)$.

Finally

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx \ge 0$$

when $\eta \ge 0$, $\eta \in C_0^\infty(\Omega)$.

ESTIMATION OF THE GRADIENT

$$\int_B |\nabla v_k|^q dx = \int_B v_k^{\frac{(1+\alpha)q}{p}} \left| \frac{\nabla v_k}{v_k^{\frac{1+\alpha}{p}}} \right|^q dx$$

$$\overset{\text{HÖLDER}}{\leq} \left\{ \int_B v_k^{\frac{(1+\alpha)q}{p-q}} dx \right\}^{1-\frac{q}{p}} \left\{ \underbrace{\int_B v_k^{-1-\alpha} |\nabla v_k|^p dx}_{\text{ESTIMATED PREVIOUSLY}} \right\}^{\frac{q}{p}}$$

Take $q < \frac{n}{n-1}(p-1)$ and fix α so that

$$\frac{(1+\alpha)q}{p-q} < \frac{n}{n-p}(p-1).$$

Continuing,

$$\leq \left\{ \int_B v^{\frac{(1+\alpha)q}{p-q}} dx \right\}^{1-q/p} C \left\{ \int_{2B} v_k^{p-1-\alpha} dx \right\}^{\frac{q}{p}}$$

We can take $v \geq 1$. Clearly the majorant is finite. Then, let $k \to \infty$. $\square$

DEF. Let Ω be a domain in $\mathbb{R}^{n+1}$.
Suppose that $h \in L^p(t_1, t_2, W^{1,p}(D))$ whenever $D_{t_1,t_2} = D \times (t_1, t_2) \subset\subset \Omega$ and that $h \in C(\Omega)$. We say that h is a <u>solution</u> in Ω, if

$$\int_{t_1}^{t_2}\int_D \left(|\nabla h|^{p-2}\nabla h \cdot \nabla\varphi - h\frac{\partial\varphi}{\partial t}\right)dx\,dt = 0$$

for all $\varphi \in C_0^\infty(D_{t_1,t_2})$, $D_{t_1,t_2} \subset\subset \Omega$.

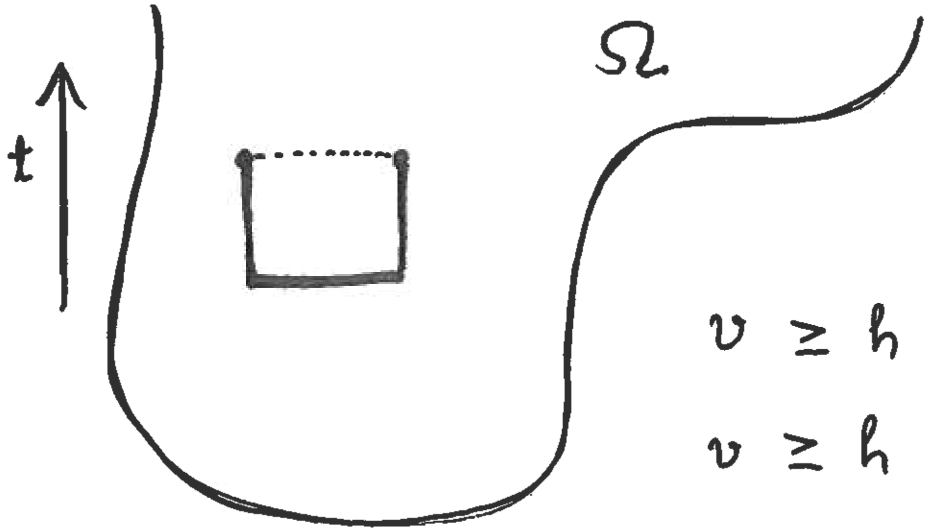
[DEF. A function $v : \Omega \longrightarrow (-\infty, \infty]$ is called p-SUPERPARABOLIC if

(i) v is <u>lower semicontinuous</u>

(ii) v is finite in a dense subset

(iii) v satisfies the (parabolic) <u>comparison principle</u> with respect to the solutions, on each cylindrical subdomain $D \times (t_1, t_2)$.]

VISCOSITY SUPERSOLUTIONS



$v \geq h$ on $\sqcup$ $\Longrightarrow$

$v \geq h$ in $\blacksquare$

$$v(x,t) = g(t) \quad \begin{cases} \text{MONOT. INCREASING,} \\ \text{LOWER SEMICONT.} \end{cases}$$

$$\mathcal{B}_\rho(x,t) \qquad v(x,t) + \frac{\varepsilon}{T-t} \qquad (0 < t < T)$$

$$v(x,t) = \min\{v_1(x,t), \ldots, v_k(x,t)\}$$

THEOREM At each point

$$v(x,t) = \operatorname*{ess\,liminf}_{\substack{(y,\tau)\to(x,t) \\ \tau < t}} v(y,\tau)$$

DEF. We say that $v : \Omega \longrightarrow (-\infty, \infty]$ is a <u>viscosity supersolution</u>, if

(i) v is lower semicontinuous

(ii) $v < \infty$ in a dense subset

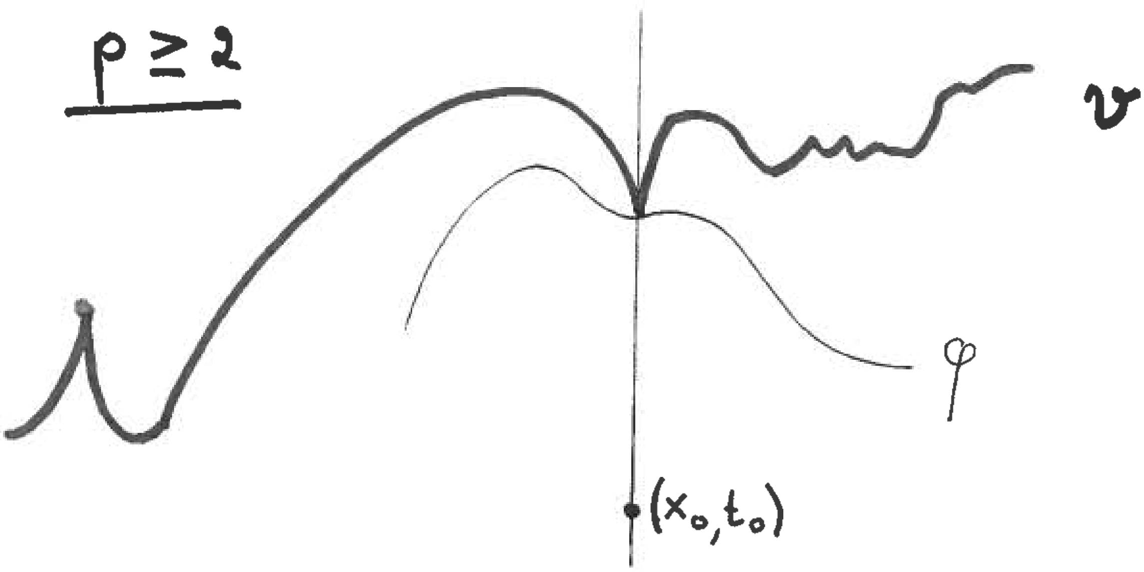
(iii) whenever (x_0, t_0) and $\varphi \in C^2(\Omega)$ are such that

$$v(x_0, t_0) = \varphi(x_0, t_0),$$
$$v(x, t) > \varphi(x, t), \text{ when } (x,t) \neq (x_0, t_0)$$

we have

$$\varphi_t(x_0, t_0) \geq \nabla \cdot \left(|\nabla \varphi(x_0, t_0)|^{p-2} \nabla \varphi(x_0, t_0) \right).$$

<u>$p \geq 2$</u>



THEOREM Let $p \geq 2$. Suppose that $v = v(x,t)$ is a p-superparabolic function in the domain Ω in $\mathbb{R}^n \times \mathbb{R}$. Then

$$v \in L^q_{loc}(\Omega) \text{ for all } q < p - 1 + \frac{p}{n}.$$

Moreover, the Sobolev derivative

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n} \right)$$

exists and

$$\nabla v \in L^q_{loc}(\Omega) \text{ whenever } q < p - 1 + \frac{1}{n+1}.$$

The summability exponents are sharp.

"APPLICATION":

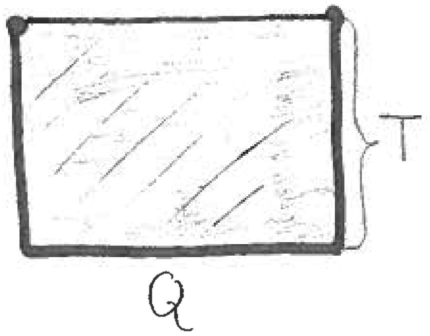
$$\frac{\partial v}{\partial t} - \nabla \cdot \left(|\nabla v|^{p-2} \nabla v \right) = \underset{\text{RADON MEASURE}}{\mu}$$

Boccardo, Dall'Aglio, Gallouët, Orsina 1997

When v is unbounded, consider

$$v_k(x,t) = \min\{v(x,t), k\}$$

for $k = 0, 1, 2, 3, \ldots$



$$\begin{cases} v(x,0) = 0, & x \in Q \\ v = 0 & \text{on } \partial Q \times [0,T] \end{cases}$$

$$\int_0^\tau \int_Q |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla \varphi_k \, dx\,dt + \int_0^\tau \int_Q \varphi_k \frac{\partial v_j}{\partial t} \, dx\,dt \geq 0$$

$$\varphi_k = (v_k - v_{k-1}) - (v_{k+1} - v_k)$$

$$k = 1, 2, 3, \ldots, j \qquad \varphi_k \geq 0$$

T. KILPELÄINEN – J. MALÝ

$$\int_0^\tau \int_Q \left(|\nabla v_j|^{p-2} \nabla v_j \cdot \nabla (v_{k+1} - v_k) + (v_{k+1} - v_k) \frac{\partial v_j}{\partial t} \right) dx\,dt$$

$$\leq \int_0^\tau \int_Q |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla (v_k - v_{k-1}) + (v_k - v_{k-1}) \frac{\partial v_j}{\partial t} dx\,dt$$

"
$$a_{k+1}(\tau) \leq a_k(\tau)$$
"

$$\sum_{k=1}^{j} a_k(\tau) \leq j\, a_1(\tau)$$

$$\int_0^\tau \int_Q |\nabla v_j|^p dx\,dt + \cancel{\frac{1}{2} \int_Q v_j^2(x,\tau)\, dx}$$

$$\leq \underbrace{j \int_0^T \int_Q |\nabla v_1|^p dx\,dt}_{\mathcal{O}(j)} + \underbrace{j \int_Q v_j(x,\tau)\, dx}_{\leq j^2 |Q|}$$

ITERATION FROM

$$(*) \quad \int_0^T \int_Q |\nabla v_j|^p \,dx\,dt \le K j^2 \quad (j = 1, 2, 3, \ldots)$$

$$K j$$

$$E_j = \{(x,t) \mid j \le v(x,t) \le 2j\}$$

$$\varkappa = 1 + \frac{2}{n}$$

$$j^{\varkappa p} |E_j| \le \iint_{E_j} v_{2j}^{\varkappa p} \,dx\,dt \le \int_0^T \int_Q v_{2j}^{\varkappa p} \,dx\,dt$$

SOBOLEV INEQ.

$$\le C \int_0^T \int_Q |\nabla v_{2j}|^p \,dx\,dt \cdot \left(\operatorname*{ess\,sup}_{0<t<T} \int_Q v_{2j}^2 \,dx \right)^{\frac{p}{n}}$$

$$\overset{(*)}{\le} C K j^2 \left(|Q| j^2 4 \right)^{\frac{p}{n}}$$

$$|E_j| \lesssim j^{2-p}$$

$p > 2$

$$\int_0^T \int_Q v^q \,dx\,dt \leq T|Q| + \sum_{j=1}^{\infty} \iint_{E_{2^{j-1}}} v^q \,dx\,dt$$

$$\leq T|Q| + \sum_{j=1}^{\infty} 2^{jq} \underbrace{|E_{2^{j-1}}|}_{\approx 2^{j(2-p)}} \overset{?}{<} \infty$$

CONVERGES IF $q < p-2$

$$\Longrightarrow \quad v \in L^q_{loc}, \quad q < p-2$$

$$p-1+\frac{p}{n}$$

$$\iint_{E_{2^{j-1}}} |\nabla v_\ell|^q \,dx\,dt \leq |E_{2^{j-1}}|^{1-\frac{q}{p}} \left(\int_0^T \int_Q |\nabla v_{2^{j-1}}|^p \,dx\,dt \right)^{\frac{q}{p}}$$

$$\Longrightarrow \quad \nabla v \in L^q_{loc}, \quad q < p-2$$

$$p-1+\frac{1}{n+1}$$

PASSAGE OVER $p-2$

POROUS MEDIUM EQN $(m \geq 1)$

$$\frac{\partial v}{\partial t} \geq \Delta\left(|v|^{m-1} v\right)$$

- $v \in L^q_{loc}$, $\quad 0 < q < m + \frac{2}{n}$

- $\nabla\left(|v|^{m-1} v\right)$ exists in Sobolev's sense and

- $\nabla\left(|v|^{m-1} v\right) \in L^q_{loc}$, whenever

$$0 < q < 1 + \frac{1}{1+mn}$$

NOTICE: NOT ∇v ITSELF !

APPROXIMATION

$$\frac{\partial v}{\partial t} = \Delta_p v$$

$$0 \le v(x,t) \le L, \qquad (x,t) \in \Omega$$

$$v_\varepsilon(x,t) = \inf_{(y,t)\in\Omega} \left\{ \frac{|x-y|^2 + (t-\tau)^2}{2\varepsilon} + v(y,\tau) \right\}$$

- $v_\varepsilon(x,t) \nearrow v(x,t)$
- $v_\varepsilon(x,t) - \dfrac{|x|^2 + t^2}{2\varepsilon}$ loc. concave
- $\dfrac{\partial v_\varepsilon}{\partial t}$ and ∇v_ε exist and belong to $L^\infty_{loc}(\Omega)$

PROPOSITION The approximant v_ε is a viscosity supersolution in

$$\Omega_\varepsilon = \{(x,t) \mid \operatorname{dist}(x,t) > \sqrt{2L\varepsilon}\}.$$

LEMMA The approximant v_ε is a weak supersolution in Ω_ε. That is

$$\iint \left(-v_\varepsilon \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla \varphi \rangle \right) dx\,dt$$

$$\geq 0, \quad \varphi \in C_0^\infty(\Omega_\varepsilon), \quad \varphi \geq 0.$$

(OBSTACLE PROBLEM NEEDED)

LEMMA (CACCIOPPOLI)

$$\iint_\Omega \zeta^p |\nabla v_\varepsilon|^p dx\,dt \leq C L^p \iint_\Omega |\nabla \zeta|^p dx\,dt + C L^2 \iint_\Omega \left| \frac{\partial \zeta^p}{\partial t} \right| dx\,dt$$

$\zeta \geq 0$, $\zeta \in C_0^\infty(\Omega_\varepsilon)$

<u>Proof:</u>

$$\varphi(x,t) = \left(L - v_\varepsilon(x,t)\right) \zeta^p(x,t) \qquad \square$$

CONCLUSION

$\nabla v \in L^p_{loc}(\Omega)$ exists $\quad (0 \le v \le L)$

$\nabla v_\varepsilon \longrightarrow \nabla v$ weakly in $L^p_{loc}(\Omega)$

No good bound on $\dfrac{\partial v_\varepsilon}{\partial t}$

<u>Ex:</u> $\quad v(x,t) = \begin{cases} 1, & \text{when } t > 0, \\ 0, & \text{when } t \le 0. \end{cases}$

$$\iint \left(-v_\varepsilon \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla \varphi \rangle \right) \ge 0$$

¿ $\varepsilon \to 0$? Passage to the limit ¿

$$\left| |b|^{p-2} b - |a|^{p-2} a \right|$$

$$\le (p-1) |b-a| \left(|b| + |a| \right)^{p-2}$$

+ Hölder

<u>ENOUGH</u> WITH

NOT p

$$\nabla v_\varepsilon \longrightarrow \nabla v \text{ in } L^{p-1}_{loc}(\Omega)$$

strongly.

THM Suppose that $v_1, v_2, v_3, \ldots$ is a sequence of Lipschitz cont. weak supersolutions such that

$$0 \le v_k \le L \quad \text{in } Q_T = Q \times (0,T),$$

$$v_k \longrightarrow v \quad \text{in } L^p(Q_T).$$

Then $\nabla v_1, \nabla v_2, \nabla v_3, \ldots$ is a Cauchy sequence in $L^{p-1}_{loc}(Q_T)$.

<u>Proof</u> $\delta > 0$

$$\operatorname{mes}\{|v_j - v_k| > \delta\} \overset{\text{TCHEBYCHEV}}{\le} \delta^{-p} \|v_j - v_k\|_p^p$$

$$\theta \in C_0^\infty(Q_T), \quad 0 \le \theta \le 1$$

$$\iint_{\theta = 1} |\nabla v_k|^p \,dx\,dt \underset{\text{Caccioppoli}}{\le} A^p \quad (k = 1, 2, \ldots)$$

$$v_k = v_{\varepsilon_k}$$

Fix k, j.

$$\iint \left(\langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \varphi \rangle - v_j \frac{\partial \varphi}{\partial t} \right) dx\,dt \geq 0$$

$$\varphi = (\delta - w_{jk})\,\theta$$

$$w_{jk} = \begin{cases} \delta, & v_j - v_k > \delta \\ v_j - v_k, & |v_j - v_k| \leq \delta \\ -\delta, & v_j - v_k < -\delta \end{cases}$$

$$|w_{jk}| \leq \delta, \quad \varphi \geq 0$$

In the eqn for v_k, use

$$(\delta + w_{jk})\,\theta$$

Add
~~Subtract~~ the equations and arrange the terms

$$\iint_{|v_j - v_k| \le \delta} \theta \langle |\nabla v_j|^{p-2} \nabla v_j - |\nabla v_k|^{p-2} \nabla v_k, \nabla v_j - \nabla v_k \rangle$$

$$\le \delta \int_0^T \int_Q \langle |\nabla v_j|^{p-2} \nabla v_j + |\nabla v_k|^{p-2} \nabla v_k, \nabla \theta \rangle$$

$$- \int_0^T \int_Q w_{jk} \langle |\nabla v_j|^{p-2} \nabla v_j - |\nabla v_k|^{p-2} \nabla v_k, \nabla \theta \rangle$$

$$+ \int_0^T \int_Q (v_j - v_k) \frac{\partial}{\partial t} (\theta w_{jk}) \longleftarrow \text{TERM } \underline{\mathrm{III}}$$

$$- \delta \int_0^T \int_Q (v_j + v_k) \frac{\partial \theta}{\partial t} \qquad = \underline{\mathrm{I}} + \underline{\mathrm{II}} + \underline{\mathrm{III}} + \underline{\mathrm{IV}}$$

$$\underline{\mathrm{III}} = \underbrace{\iint \theta \frac{\partial}{\partial t} \left(\frac{w_{jk}^2}{2} \right)}_{-\frac{1}{2} \iint w_{jk}^2 \frac{\partial \theta}{\partial t}} + \iint (v_j - v_k) w_{jk} \frac{\partial \theta}{\partial t}$$

$$\overline{\underline{III}} \le \frac{1}{2}\delta^2 \|\Theta_t\|_1 + 2L\delta\|\Theta_t\|_1$$

$$\le \delta C_3$$

$$\overline{\underline{IV}} \le 2\delta L \|\Theta_t\|_1 = \delta C_4$$

The terms $\underline{I}$, $\overline{\underline{II}}$ are easy

$$\underline{I} \le \delta C_1 \quad , \quad \overline{\underline{II}} \le \delta C_2 .$$

$$\underline{I} + \overline{\underline{II}} + \overline{\underline{III}} + \overline{\underline{IV}} \le C\delta$$

$$\iint\limits_{|v_j - v_k| \le \delta} \Theta |\nabla v_j - \nabla v_k|^p \le 2^{p-2}\delta C$$

$$\iint\limits_{|v_j - v_k| \le \delta} \Theta |\nabla v_j - \nabla v_k|^{p-1} = \mathcal{O}(\delta^{1-\frac{1}{p}})$$

$$\iint_{|v_j - v_k| > \delta} \Theta \, |\nabla v_j - \nabla v_k|^{p-1}$$

$$\leq \delta^{-1} \|v_j - v_k\|_p \left(\|\nabla v_j\|_p + \|\nabla v_k\|_p \right)^{p-1}$$

$$\leq (2A)^{p-1} \delta^{-1} \|v_j - v_k\|_p$$

FINALLY,

$$\int_0^T \int_Q \Theta \, |\nabla v_j - \nabla v_k|^{p-1} \longleftarrow \text{Independent of } \delta!$$

$$\leq O\left(\delta^{1-\frac{1}{r}}\right) + C_5 \, \delta^{-1} \underbrace{\|v_j - v_k\|_p}_{\to 0}$$

THM Let v be a bounded viscosity supersolution of $v_t = \Delta_p v$ in Ω, $p \geq 2$. Then

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right)$$

exists in Sobolev's sence, $\nabla v \in L^p_{loc}(\Omega)$, and

$$\iint \left(-v\varphi_t + \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle\right) dx\,dt \geq 0.$$

SOBOLEV'S INEQ

$u \in L^p(0,T; W_0^{1,p}(Q))$

$$\int_0^T \int_Q |u|^{p(1+\frac{2}{n})} dx\,dt$$

$$\leq C \int_0^T \int_Q |\nabla u|^p dx\,dt \left\{ \operatorname*{ess\,sup}_{0<t<T} \int_Q |u(x,t)|^2 dx \right\}^{\frac{p}{n}}$$

$Q \times (t_1, t_2)$

$\varphi = 0$ on $\partial Q \times [t_1, t_2]$

$$\int_{t_1}^{t_2} \int_Q \left(\langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle - v \frac{\partial \varphi}{\partial t} \right) dx\, dt$$

$$+ \int_Q v(x, t_2)\, \varphi(x, t_2)\, dx \geq \int_Q v(x, t_1)\, \varphi(x, t_1)\, dx$$

If v is zero on the lateral boundary, take $\varphi = v$ (!)

$$\boxed{\frac{1}{2} \underset{Q\ \text{PAST}}{\int} v(x, t_1)^2\, dx \leq \int_{t_1}^{t_2} \int_Q |\nabla v|^p\, dx\, dt + \frac{1}{2} \underset{Q\ \text{FUTURE}}{\int} v(x, t_2)^2\, dx}$$

START FROM

$$\int_0^{\cancel{T}\,t_1}\int_Q |\nabla v_j|^p\,dx\,dt \le j\int_0^T\int_Q |\nabla v_1|^p\,dx\,dt + j\int_Q v_j(x,\tau)\,dx$$

THIS WAS PROVED

$0 \le \underline{t_1} < \tau < T$ Integrate with respect to τ over $[t_1, T]$

$$(T-t_1)\int_0^{t_1}\int_Q |\nabla v_j|^p \le j(T-t_1)K + \underbrace{j\int_{t_1}^T\int_Q v_j(x,t)\,dx\,dt}_{\le j^{2-\varepsilon}\int_{t_1}^T\int_Q v^{\varepsilon}}$$

$$\int_0^{t_1}\int_Q |\nabla v_j|^p \le j^{2-\varepsilon}K \qquad (t_1 < T)$$

Estimating again the measures $|E_j|$, but starting with the bound

$$j^{\gamma} K \qquad (\gamma = 2 - \varepsilon)$$

in place of $j^2 K$, yields

$$|E_j| \lesssim j^{\gamma - p}$$
$$= j^{2 - p - \varepsilon} \qquad\qquad 0 < \varepsilon \leq 1$$

RESULT

$$\int_0^{t_2} \int_Q v^q \, dx\, dt < \infty, \quad \text{when}$$

$$0 < q < p - \gamma$$

$$q_1 = \varepsilon \qquad T$$

$$q_2 = p - 2 + \varepsilon \qquad t_1$$

$$q_3 = 2(p-2) + \varepsilon \qquad t_2$$

CONTINUE TILL

$$k(p-2)+\varepsilon > p-1$$

NOW

$$v \in L^{p-1}(Q_{T'}).$$

IN PARTICULAR
$v \in L^1$

$$\frac{1}{2}\int_Q v_j^2(x,t)\,dx \le \int_0^\tau \int_Q |\nabla v_j|^p\,dx\,dt + \frac{1}{2}\int_Q v_j^2(x,\tau)\,dx$$

PAST FUTURE

$$\operatorname*{ess\,sup}_{0<t<t_1} \int_Q v_j^2(x,t)\,dx \le 2\int_0^\tau \int_Q |\nabla v_j|^p\,dx\,dt + \int_Q v_j(x,\tau)\,dx$$

$$\le 2_j \int_0^\tau \int_Q |\nabla v_1|^p + 2_j \int_Q v_j(x,\tau)\,dx + \int_Q v_j(x,\tau)\,dx$$

$$\operatorname*{ess\,sup}_{0<t<t_1} \int_Q v_j^2(x,t)\,dx \qquad (t_1 < \tau < T)$$

$$\leq 2j \int_0^T \int |\nabla v_1|^p \,dx\,dt + \underbrace{3j \int_Q v_j(x,\tau)\,dx}_{*}$$

Integrate τ over $[t_1, T]$.

Then $(*)$ becomes : $\dfrac{3j}{T-t_1} \int_{t_1}^{T}\int_Q v_j \,dx\,dt$

We had

$$\int_0^{t_1}\int |\nabla v_j|^p \leq j \int\int |\nabla v_1|^p \qquad \underline{\varepsilon = 1}$$

$$+ \frac{j^{2-\varepsilon}}{T-t_1} \int_{t_1}^{T}\int_Q v^\varepsilon \,dx\,dt .$$

$$\boxed{\int_0^{t_1}\int_Q |\nabla v_j|^p + \operatorname*{ess\,sup}_{0<t<t_1} \int_Q v_j^2(x,t)\,dx \leq 3j \int_0^T \int |\nabla v_1|^p + \frac{4j}{T-t_1} \int_{t_1}^{T}\int_Q v} \approx K_j$$

LEMMA If

$$\int_0^T\int_Q |\nabla v_j|^p + \operatorname*{ess\,sup}_{0<t<T}\int v_j^2\,dx \leq j K$$

$$j = 1, 2, 3, \ldots,$$

then

$$v \in L^q(Q_T) \quad \text{whenever} \quad 0 < q < p - 1 + \frac{p}{n}$$

$$\nabla v \in L^q(Q_T) \quad \text{—''—} \quad 0 < q < p - 1 + \frac{1}{n+1}$$

<u>Proof</u>: $E_j = \{(x,t) \mid j \leq v(x,t) < 2j\}$

$$\varkappa = 1 + \frac{2}{n}$$

$$j^{\varkappa p}|E_j| \leq \iint_{E_j} v_{2j}^{\varkappa p} \leq \iint_{Q_T} v_{2j}^{\varkappa p}$$

$$\leq C \int_0^T\int_Q |\nabla v_{2j}|^p \left(\operatorname*{ess\,sup}_{0<t<T}\int_Q v_{2j}^2\right)^{p/n}$$

$$\leq C K^{1+\frac{p}{n}} j^{1+\frac{p}{n}}$$

$$|E_j| \lesssim j^{1-p-\frac{p}{n}}$$

$$\int_0^T\int_Q v^q\,dx\,dt \le T|Q| + \sum_{j=1}^{\infty}\iint_{E_{2^{j-1}}} v^q\,dx\,dt$$

$$\le T|Q| + \sum_{j=1}^{\infty} 2^{jq}|E_{2^{j-1}}|$$

$$\sim \sum_{j=1}^{\infty} 2^{j(q+1-p-\frac{p}{n})}$$

converges in the designed range

$$\iint |\nabla v_k|^q = \iint_{E_0} |\nabla v|^q + \sum_{j=1}^{\infty}\iint_{E_{2^{j-1}}} |\nabla v_k|^q$$

$$\lesssim \sum \left(\iint_{E_{2^{j-1}}} |\nabla v_k|^p\right)^{\frac{q}{p}} |E_{2^{j-1}}|^{1-q/p}$$

$$\le \sum 2^{(j-1)(1-\frac{q}{p})(1-p-\frac{p}{n})}\Big(\underbrace{\iint_{E_{2^{j-1}}} |\nabla v_{2^j}|^p}_{2^j K}\Big)^{\frac{q}{p}}$$