The Maslov index and global bifurcation for nonlinear boundary value problems *

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Preliminary version

1 Starting point and motivations

Consider

$$\begin{cases} -u'' + q(t)u = f(t, u, u', \lambda), \\ u(0) = 0 = u(\pi) \end{cases} \tag{1.1}$$

where $q \in C([0, \pi])$, $q(t) \geq 0$ for all t and $f \in C([0, \pi] \times \mathbb{R}^3)$. Assume

$$f(t, \xi, \eta, \lambda) = \lambda a(t)\xi + h(t, \xi, \eta, \lambda),$$

being $a(\cdot)$ continuous and positive and

$$h(t, \xi, \eta, \lambda) = o(\sqrt{\xi^2 + \eta^2}),\ (\xi, \eta) \to (0, 0),$$

uniformly in $t \in [0, \pi]$ and λ in bounded subsets of $\mathbb{R}$.

We are concerned with the problem of the existence and multiplicity of (classical) solutions to (1.1).

In what follows, Σ will denote the set of solutions to (1.1).

We shall go through three steps:

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(I) Embed the problem in an abstract bifurcation framework and use/develop a global bifurcation result.
(II) Study "the linear theory".
(III) Use a "topological invariant" and obtain a priori estimates.

As for step (I), recall

Theorem 1.1 (Rabinowitz [36]). *Let $L \in \mathcal{K}(X)$; let H be a completely continuous nonlinear operator s.t. $H(\lambda, u) = o(||u||)$, $u \to 0$ uniformly in bounded λ-intervals; consider*

$$u = \lambda L u + H(\lambda, u), \quad \lambda \in (a, b); \tag{1.2}$$

then, if $\bar{\mu}$ is an eigenvalue of L of odd multiplicity it follows that Σ contains a continuum C such that $(\bar{\mu}^{-1}, 0) \in C$ and either

$(A1)$ *there exists $(\lambda_n, u_n) \in C$ such that*

$$|\lambda_n| + ||u_n|| \to +\infty \quad \text{or } \lambda_n \to a \quad \text{or} \quad \lambda_n \to b$$

or

$(A2)$ *there exists $(\hat{\mu}, 0) \in C$ such that $\hat{\mu}^{-1}$ is an eigenvalue and $\hat{\mu}^{-1} \neq \bar{\mu}^{-1}$.*

As for step (II), recall that the linear eigenvalue problem

$$\begin{cases} -u'' + q(t)u = \lambda a(t)u, \ t \in [0, \pi], \\ u(0) = 0 = u(\pi) \end{cases}$$

has a sequence of simple eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \dots$$

such that $\lim_{k \to +\infty} \lambda_k = +\infty$. Moreover, if φ_k denotes the normalized eigenfunction associated to λ_k positive in a neighbourhood of 0, then φ_k has $(k-1)$ simple zeros in $(0, \pi)$.
Step (III) is the key point in the direction of eliminating the second alternative in Rabinowitz theorem. To this end, we set $\Sigma^* = \{u : u \,\text{satisfies}\,(1.1)\} \cup \{\lambda_k\}_{k=1}^{+\infty}$. We shall show the existence of a continuous map $\Phi : \Sigma^* \to \mathbb{Z}$ s.t. if we consider $C \subset \Sigma^*$ in Rabinowitz theorem then $\Phi(\lambda, u) = const$, for every $(\lambda, u) \in C$. This topological invariant is defined as follows.
Consider the nonlinear problem

$$\begin{cases} -u'' + q(t)u = f(t, u, u', \lambda), \ t \in [0, \pi], \\ u(0) = 0 = u(\pi). \end{cases}$$

Define $\Sigma^* \to \mathbb{Z}$ by setting

$$\Phi(\lambda, u) = \begin{cases} card(u^{-1}(0) \cap (0, \pi)) & u \neq 0 \\ (k-1) & (\lambda, u) = (\lambda_k, 0). \end{cases}$$

It is important to remark the relation between the definition of $\Phi(\lambda, u)$, being u a solution to the nonlinear problem, and the linear eigenvalue problem. Indeed, $(k-1)$ is exactly the number of (simple) zeros of the eigenfunction φ_k associated to the k-th eigenvalue λ_k.
We now give an elementary example which introduces the reader to the concepts we shall use and generalize in what follows.
Consider

$$\begin{cases} -u'' = \lambda u \\ \\ u(0) = 0 = u(\pi). \end{cases}$$

The number of zeros of $\varphi_2(t) = \sin 2t$ (the eigenfuction associated to $\lambda_2 = 4$) is equal to 1. Moreover, if γ denotes the orbit $(u(t), u'(t))$ in the phase plane s.t. $(u(0), u'(0)) = (0, 2)$, then

$$Ind(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} = \frac{1}{2\pi} \int_\gamma \frac{xdy - ydx}{x^2 + y^2} = -1$$

(we use the parametrization $x(t) = \rho(t) \sin\theta(t), y(t) = \rho(t) \cos\theta(t)$).
Note also that for the angular coordinate $\theta(\cdot)$ we have

$$\theta(t) = k(t)\pi + \alpha(t),$$

being $k(t) \in \mathbb{N}$ and $\alpha(t) \in (0, \pi]$. In particular, since $k(\pi) = 1$ and $\alpha(\pi) = \pi$, then the number of zeros of u in $(0, \pi)$ is equal to $k(\pi)$.
Let us now state a global bifurcation result for (1.1).

Theorem 1.2 (Rabinowitz [36]). *For every k, $(\lambda_k, 0)$ is a bifurcation point for the nonlinear BVP*

$$\begin{cases} -u'' + q(t)u = f(t, u, u', \lambda) \\ \\ u(0) = 0 = u(\pi). \end{cases}$$

The bifurcating branches $C_k \subset \mathbb{R} \times C^1([0, \pi], \mathbb{R})$ are unbounded in $\mathbb{R} \times C^1([0, \pi], \mathbb{R})$; moreover, if $(\lambda, u) \in C_k$ and $u \neq 0$, then u has $(k-1)$ simple zeros in $(0, \pi)$.

2 Linear symplectic preliminaries

Consider a linear space V of finite dimension and a bilinear, antisymmetric and nondegenerate form $\omega : V \times V \to \mathbb{R}$. The space (V, ω) is called a symplectic vector space. The classical example is $(\mathbb{R}^{2N}, \omega_0)$, where, setting $z = (x, y), z' = (x', y')$, it is $\omega_0(z, z') = xy' - x'y$. We shall denote the standard symplectic matrix by

$$\begin{pmatrix} 0 & -\mathrm{Id} \\ \\ \mathrm{Id} & 0 \end{pmatrix}.$$

Note that $\omega_0(z, z') = (Jz)z'$. Every symplectic space (V, ω) has even dimension; moreover, there exists a symplectic isomorphism $\phi : (V, \omega) \to (\mathbb{R}^{2N}, \omega_0)$, i.e. a map such that $\omega_0(\phi(z), \phi(z')) = \omega(z, z')$ for all $z, z' \in V$. In what follows we shall write ω instead of ω_0.

Definition 2.1 *A subspace L of $\mathbb{R}^{2N}$ is called lagrangian if $\dim L = N$ and $\omega(z, z') = 0$ for all $z, z' \in L$.*

For example, the subspaces $L_1 = \{0\} \times \mathbb{R}^N$ and $L_2 = \mathbb{R}^N \times \{0\}$ are lagrangian subspaces. If $N = 1$, every line through the origin is a lagrangian subspace.
It is useful to know various ways to describe a lagrangian subspace. First, we can use a linear injective map $Z : \mathbb{R}^N \to \mathbb{R}^{2N}$ such that $L = \mathrm{Im}Z$, together with an additional condition which can be expressed by writing

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix},$$

being X, Y two $N \times N$ matrices s.t. Z has rank N. Precisely,

Proposition 2.2 *The subspace L is lagrangian if and only if* $\; Y^t X = X^t Y$.

PROOF. For $z \in L$ there exists a unique $u \in \mathbb{R}^N$ s.t. $z = (Xu, Yu)$. Then we have

$$\omega(z, z') = \omega((Xu, Yu), (Xu', Yu')) = Xu\; Yu' - Yu\; Xu' = (Y^t X - X^t Y)u\; u'.$$

■

The map Z is called a lagrangian frame.
Another useful expression for lagrangian subspaces is obtained as follows. Consider $G \in L(\mathbb{R}^N)$ and

$$L = \mathrm{Gr}\; G \;=\; \{(x, Gx), x \in \mathbb{R}^N\}.$$

Then we have

Proposition 2.3 *The subspace L is lagrangian if and only if the matrix G is symmetric.*

PROOF. Consider the frame

$$\begin{pmatrix} \mathrm{Id} \\ G \end{pmatrix}$$

and observe that $G^t\; \mathrm{Id} - \mathrm{Id}^t\; G = 0$. ■
Note that $\mathrm{Gr}\; G \cap (\{0\} \times \mathbb{R}^N) = \{0\}$. Moreover, it is useful to observe that if L is a lagrangian subspace s.t.

$$L \cap (\{0\} \times \mathbb{R}^N) = \{0\} \tag{2.1}$$

and if $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ is a frame for L then X is invertible and $L = \mathrm{Gr}\; Y\; X^{-1}$ (note that the lagrangian subspace $\{0\} \times \mathbb{R}^N$ does not satisfy (2.1)). Indeed, (2.1) implies that $\mathrm{Ker}\; X = \{0\}$ and hence X^{-1} is defined. Moreover,

$$L = \{(Xu, Yu), u \in \mathbb{R}^n\} = \{(XX^{-1}v, YX^{-1}v), v \in \mathbb{R}^n\} = \{(v, YX^{-1}v), v \in \mathbb{R}^n\} = \mathrm{Gr}\; Y\; X^{-1}$$

(it is easy to check that (2.1) implies that the matrix YX^{-1} is symmetric).

Theorem 2.4 *The space $\Lambda(2N)$ of lagrangian subspaces of $\mathbb{R}^{2N}$ is a C^∞-manifold of dimension $N(N+1)/2$.*

The space $\Lambda(2N)$ is called the lagrangian grassmannian.

3 The Maslov index for paths of lagrangian subspaces

We follow the paper [37] by Robbin-Salamon. Consider $L \in C^1([0,\pi], \Lambda(2N))$, a curve of lagrangian subspaces. The Maslov index will be defined by means of a quadratic form which we now describe.
Consider $\bar{t} \in [0,\pi]$ and $JL(\bar{t})$, a lagrangian complement of $L(\bar{t})$ (this means $L(\bar{t}) \cap JL(\bar{t}) = \{0\}$). Take $z \in L(\bar{t})$ and t in a neighbourhood of $\bar{t}$. Denote by $w(t) \in JL(\bar{t})$ the (unique) vector such that $z + w(t) \in L(t)$. For example, if $N = 1$ and $L(\bar{t}) = \mathbb{R} \times \{0\}$ and $z = (x, 0)$ then $w_2(t) = A(t)x$, for some real function A.
Then we can define

$$Q(L,\bar{t})(z) = \frac{d}{dt}\omega(z, w(t))_{|t=\bar{t}}.$$

This is a quadratic form in $L(\bar{t})$. In the particular case $N = 1$ and $L(\bar{t}) = \mathbb{R} \times \{0\}$ we have $Q(L,\bar{t})(z) = x\ A'(\bar{t})x$.

Proposition 3.1 *Let $Z(t)$ be a frame for $L(t)$. Then*

$$Q(L,\bar{t})(z) = X(\bar{t})u\ Y'(\bar{t})u - Y(\bar{t})u\ X'(\bar{t})u, \tag{3.1}$$

being u s.t. $z = Z(\bar{t})u \in L(\bar{t})$.

Proof. Without loss of generality, assume $L(\bar{t}) = \mathbb{R}^N \times \{0\}$. We know that, being $z = (X(\bar{t})u, Y(\bar{t})u)$ and $w(t) = (0, w_2(t))$, the condition $z + w(t) \in L(t)$ can be written as

$$Y(\bar{t})u + w_2(t) = Y(t)\ X(t)^{-1}\ X(\bar{t})u$$

(note that in a neighbourhood of $\bar{t}$ it is guaranteed that $L(t) \cap (\{0\} \times \mathbb{R}^N) = \{0\}$). Observe that

$$\omega(z, w(t)) = X(\bar{t})u\ w_2(t)$$

and

$$Q(L,\bar{t})(z) = X(\bar{t})u\ w_2'(t).$$

Moreover,

$$w_2'(t) = Y'(t)\ X(t)^{-1}\ X(\bar{t})u - Y(t)\ X(t)^{-1}\ X'(t)\ X(t)^{-1}\ X(\bar{t})u.$$

Then (3.1) follows using the fact that $Y^tX = X^tY$. ■

In general, the Maslov index is defined for a pair (L_1, L_2) of lagrangian planes. We shall consider the particular case $L_2 \equiv V = \{0\} \times \mathbb{R}^N$. To this end, we first give the following

Definition 3.2 *A point $\bar{t} \in [0,\pi]$ is a* crossing *for (L_1, V) if $L_1(\bar{t}) \cap V \neq \{0\}$.*

Then we set (in a nontrivial way)

$$\Gamma(L_1, V, \bar{t}) := Q(L_1, \bar{t})_{|L_1(\bar{t}) \cap V}.$$

If $\bar{t}$ is not a crossing for (L_1, V) then we mean that $\Gamma \equiv 0$.

Definition 3.3 *A crossing $\bar{t}$ is* regular *if the quadratic form Γ is nondegenerate.*

Recall that the *signature* of Γ is the difference between the number of positive eigenvalues and the number of negative eigenvalues.
Assume $N = 1$. Then if $\bar{t}$ is a crossing for (L_1, V) then $L_1(\bar{t}) = \{0\} \times \mathbb{R}$; in this case $L(t)$ is the graph of $x = B(t)y$, for some B, and the signature of Γ is determined by the sign of $B'(\bar{t})$ ($B'(\bar{t}) > 0$ implies that sign $\Gamma(L_1, V, \bar{t}) = -1$).

Remark 3.4 *Using the fact that the eigenvalues of a smooth family of symmetric matrices are continuous functions of t, we know that regular crossings are isolated. Thus, there are at most a finite number of regular crossings in* $[0, \pi]$.

Assume now that (L_1, V) has only regular crossings.

Definition 3.5 *The Maslov index of the pair (L_1, V) is defined as follows*

$$\mu(L_1, V) = \frac{1}{2}\mathrm{sign}(L_1, V, 0) + \sum_{0<t<\pi} \mathrm{sign}(L_1, V, t) + \frac{1}{2}\mathrm{sign}(L_1, V, \pi),$$

where the summation is taken over all crossings. Recall that we agree to think of $\Gamma \equiv 0$ in case 0 or π is not a crossing. Note also that $\mu(L_1, V) \in \mathbb{Z}/2$.
The restriction on the regularity of the crossings in the above definition can be eliminated (having in mind the definition of topological degree).

4 The Maslov index for paths of symplectic matrices

In this section we give the definition of a topological invariant which is used in a (generalized) phase plane analysis for differential equations. To this end, recall that the space of symplectic matrices is

$$Sp(2N) = \{A \in L(2N) : A^t J A = J\}.$$

Consider again $V = \{0\} \times \mathbb{R}^N$ and $\psi \in C^1([0, \pi], \mathrm{Sp}(2N))$.

Proposition 4.1 *The subspace $\psi(t)V$ is a lagrangian subspace.*

Proof. Write

$$\psi(t) = \begin{pmatrix} X_0(t) & X(t) \\ Y_0(t) & Y(t) \end{pmatrix},$$

where $X_0(t), Y_0(t), X(t), Y(t)$ are $N \times N$ block matrices. The subspace $\psi(t)V$ can be represented by means of $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$. Then it is easy to check that

$$\psi(t)^t \ J \ \psi(t) = J \implies Y(t)^t \ X(t) = X(t)^t \ Y(t).$$

■

We can now give the following

Definition 4.2 *The Maslov index of the symplectic path ψ is defined as follows*

$$\mu(\psi) = \mu(\psi \ V, V).$$

Note that
$$\dim(\psi(t)V \cap V) = \dim(\text{Ker}X(t)).$$
It is useful to observe that if $\bar{t}$ is a crossing and $z \in \psi(t)V \cap V$ then $z = (X(\bar{t})u, Y(\bar{t})u)$ with $X(\bar{t})u = 0$. Then, using (3.1),
$$\Gamma(\psi(\cdot)V, V, \bar{t})(z) = Q(\psi(\cdot)V, \bar{t})_{|\psi(\bar{t})V\cap V}(z) = -Y(\bar{t})u\ X'(\bar{t})u.$$
Let us now compute the Maslov index in an elementary situation. Consider the planar system
$$J\ z' = B(t)\ z, \tag{4.1}$$
where $B(t)$ is a 2×2 symmetric matrix. Let $(x(t), y(t))$ be the unique solution of (4.1) satisfying $(x(0), y(0)) = (0, 1)$. Denote by ψ the fundamental matrix solution (recall that $\psi(0) = \text{Id}$). We shall prove later, in a more general context, that $\psi(t)$ is a symplectic matrix, for all t. Let $V = \{0\} \times \mathbb{R}$. A frame for $\psi(t)V$ is given by $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Thus, $\bar{t}$ is a crossing for $(\psi(t)V, V)$ if and only if $x(\bar{t}) = 0$. Take
$$B(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$
Then (4.1) is equivalent to
$$\begin{cases} x' = t\ y \\ y' = -t\ x, \end{cases}$$
with $t \in [0, \pi]$. Note that $\psi(t)V = \begin{pmatrix} \sin(t^2/2) \\ \cos(t^2/2) \end{pmatrix}$. Observe that $\bar{t}$ is a crossing if and only if $\bar{t} = 0, \sqrt{2\pi}$. Being $\bar{t} = 0$ a non-regular crossing, we consider $\tilde{\psi}$ "near" ψ such that $\tilde{X}(0) = 0, \tilde{X}(\pi) = \sin(\pi^2/2), \tilde{X}'(0) > 0$. Then for $z = (0, y) \in \mathbb{R}^2$ we have
$$\Gamma(\tilde{\psi}(\cdot)V, V, 0)(z) = -\tilde{Y}(0)y\ \tilde{X}'(0)y < 0.$$
The crossing $\bar{t} = \sqrt{2\pi}$ is regular and
$$\Gamma(\tilde{\psi}(\cdot)V, V, \sqrt{2\pi})(z) = -\tilde{Y}(\sqrt{2\pi})y\ \tilde{X}'(\sqrt{2\pi})y < 0.$$
Hence,
$$\mu(\psi) = \mu(\tilde{\psi}) = \frac{1}{2}(-1) + (-1) + 0 = -\frac{3}{2}.$$
Note that the number of zeros of $x(t) = \sin(t^2/2)$ in $(0, \pi)$ is 1.
Let us now use elementary phase-plane analysis. If we write $x(t) = \sin\theta(t), y(t) = \cos\theta(t)$, then $\theta(t) = t^2/2$. The function $\theta(\cdot)$ is increasing in $[0, \pi]$; moreover, being $\theta(t) = k(t)\pi + \alpha(t)$, with $k(t) \in \mathbb{N}$ and $\alpha(t) \in (0, \pi]$, we have $k(\pi) = 1$.
Having in mind generalizations of this elementary computation, note that we can associate to $\theta(t)$ the angle $\Theta(t)$ s.t. $\Theta(t) = e^{2i\theta(t)}$, i.e. $\Theta(t) = \cos 2\theta(t) + i\ \sin 2\theta(t)$.
Finally, note that, for $\gamma = (x(t), y(t))$, the rotation number satisfies
$$\text{rot}(\gamma) = \frac{1}{2\pi}\int_\gamma \frac{xdy - ydx}{x^2 + y^2} = \frac{1}{2\pi}\int_0^\pi -\theta'(t)\ dt = -\frac{1}{2\pi}\int_0^\pi t\ dt = -\frac{\pi}{4}.$$

Exercise. Repeat the above argument in the case of the harmonic oscillator $u'' + \alpha u = 0, \alpha > 0$.

5 The number of moments of verticality

We follow the paper [2] by Arnold. Consider the system of $2N$ first order equations

$$J\ z' = B(t)\ z, \quad z \in \mathbb{R}^{2N}, \tag{5.1}$$

where $B(t)$ is a $2N \times 2N$ symmetric matrix. Assume that $B(t)$ is positive definite.

Definition 5.1 *We say that $\bar{t} \in [0, \pi]$ is a moment of verticality for (5.1) if the boundary value problem*

$$\begin{cases} J\ z' = B(t)\ z \\ \\ x(0) = 0 = x(\bar{t}) \end{cases} \tag{5.2}$$

($z = (x, y)$) has a nontrivial solution.

The number of moments of verticality is

$$j(B) = \sum_{0<t<\pi} m(t),$$

where the summation is taken over all moments of verticality. The fact that $j(B)$ is a finite sum will follow from Theorem 5.4 below.

Note that the mere definition of $j(B)$ does not require neither the symmetry of $B(t)$ nor its positive definiteness. The reason for the restrictions we make will be made clear in what follows.

In order to explain the relation between $j(B)$ and the Maslov index of a symplectic matrix, let us consider the matrix differential equation

$$\begin{cases} J\ \psi' = B(t)\ \psi \\ \\ \psi(0) = \mathrm{Id}. \end{cases} \tag{5.3}$$

Proposition 5.2 *The matrix $\psi(t)$ is symplectic.*

PROOF. Note that

$$\frac{d}{dt}\Big[\psi(t)^t\ J\ \psi(t)\Big] = \psi'(t)^t\ J\ \psi(t) + \psi(t)^t\ J\ \psi'(t) = (-J\ B(t)\ \psi(t))^t\ J\ \psi(t) + \psi(t)^t\ B(t)\ \psi(t) =$$

$$= \psi(t)^t\ B(t)\ J\ J\ \psi(t) + \psi(t)^t\ B(t)\ \psi(t) = -\psi(t)^t\ B(t)\ \psi(t) + \psi(t)^t\ B(t)\ \psi(t) = 0.$$

Hence,

$$\psi(0)^t\ J\ \psi(0) = \psi(t)^t\ J\ \psi(t)$$

for all t. Thus, being $\psi(0) = \mathrm{Id}$, it follows that $\psi(t)^t\ J\ \psi(t) = J$. ■

Note that in the above proof we only used the fact the $B(t)$ is symmetric.

Proposition 5.3 *A point $\bar{t}$ is crossing for $(\psi(\cdot)V, V)$ if and only if $\bar{t}$ is a moment of verticality for (5.1). Moreover, if we denote*

$$\psi(t) = \begin{pmatrix} X_0(t) & X(t) \\ \\ Y_0(t) & Y(t) \end{pmatrix},$$

then $m(\bar{t}) = \dim(\mathrm{Ker} X(\bar{t}))$.

Sketch of the proof. Recall that $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ is a frame for $\psi(t)V$ and that $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ is a solution of the matrix differential equation (5.3). Then, for the conclusion it is sufficient to use the fact that $\dim(\psi(\bar{t})V \cap V) = \dim(\mathrm{Ker}X(\bar{t}))$ and they equal $m(\bar{t})$. ■

The result stated below establishes a relation between the Maslov index and the number of moments of verticality.

Theorem 5.4 (Boscaggin [7]).

$$\mu(\psi) = -\Big[\frac{N}{2} + j(B) + \frac{m(\pi)}{2}\Big]. \tag{5.4}$$

Proof. Remember that $z \in \psi(t)V \cap V$ can be represented by $z = (X(t)u, Y(t)u)$ with $X(t)u = 0$. In order to compute $\mu(\psi)$, let us note that

$$\Gamma(\psi(t)V, V, t)(z) = -Y(t)u\ X'(t)u.$$

Thus we need to compute $X'(t)$. To this end, we use (5.3). Indeed,

$$\psi'(t) = J^{-1}\ B(t)\ \psi(t) = \begin{pmatrix} 0 & \mathrm{Id} \\ & \\ -\mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ & \\ B_{12}(t) & B_{22} \end{pmatrix} \begin{pmatrix} X_0(t) & X(t) \\ & \\ Y_0(t) & Y(t) \end{pmatrix}$$

which implies

$$\begin{pmatrix} X_0'(t) & X'(t) \\ & \\ Y_0'(t) & Y'(t) \end{pmatrix} = \begin{pmatrix} \dots & B_{12}(t)X(t) + B_{22}(t)Y(t) \\ & \\ \dots & \dots \end{pmatrix}.$$

Thus,

$$\Gamma(\psi(t)V, V, t)(z) = -Y(t)u\ \Big(B_{12}(t)X(t) + B_{22}(t)Y(t)\Big)\ u = -B_{22}(t)^t\ Y(t)u\ Y(t)u = -B_{22}\ Y(t)u\ Y(t)u,$$

where we have used the fact that $B(t)$ is symmetric. Let us now focus on the r.h.s. of the above equality. Recall that if $z \in \psi(t)V \cap V$ then $z = (0, Y(t)u)$. Then we can write

$$B(t)z\ z = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ & \\ B_{12}(t) & B_{22}(t) \end{pmatrix} \begin{pmatrix} 0 \\ Y(t)u \end{pmatrix} \begin{pmatrix} 0 \\ Y(t)u \end{pmatrix} =$$

$$= \begin{pmatrix} B_{12}(t)\ Y(t)u \\ B_{22}(t)\ Y(t)u \end{pmatrix} \begin{pmatrix} 0 \\ Y(t)u \end{pmatrix} = B_{22}(t)\ Y(t)u\ \ Y(t)u.$$

Being $B(t)$ positive definite, it follows that $\Gamma(\psi(t)V, V, t)$ is negative definite in $\psi(t)V \cap V$. Hence, every crossing is regular and

$$\mathrm{sign}\Gamma(\psi(t)V, V, t) = -\dim(\psi(t)V \cap V) = -\dim\ (\mathrm{Ker}X(t)).$$

It is now sufficient to recall that for every $t \in (0, \pi]$ we have $\dim\ (\mathrm{Ker}X(t)) = m(t)$. For the completion of the proof, it is sufficient to consider $t = 0$. Recall that $\psi(0) = \mathrm{Id}$; thus $\psi(0)V \cap V \neq \{0\}$ and

$$\mathrm{sign}\Gamma(\psi(0)V, V, 0) = -\dim\ (\mathrm{Ker}X(0)) = -N.$$

■

As a consequence of the above theorem, $j(B)$ is a finite sum.
It is useful to observe that for the proof of the above theorem it is sufficient to assume that $B_{22}(t)$ is positive definite. This is the situation found in the case of second order systems of the form

$$X''(t) + A(t)\ X(t) = 0, \tag{5.5}$$

with $A(t)$ a $N \times N$ matrix. Indeed, system (5.5) can be written in the form (5.1) with

$$B(t) = \begin{pmatrix} A(t) & 0 \\ \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{5.6}$$

In the particular case when the matrix $A(t)$ is constant it is possible (cf. [33], in a more general setting) to compute the number of moments of verticality as a function of the eigenvalues of A. Indeed, up to a diagonalization, we can assume that $A = \mathrm{diag}\,(\lambda_1, \dots, \lambda_N)$; it is then possible to show that

$$j(B) = \sum_{i:\lambda_i>0} \left(\left\lceil \sqrt{\lambda_i} \right\rceil - 1 \right),$$

where, for every $r > 0$, we have set $\lceil r \rceil = n$ if $n - 1 < r \leq n$ for some $n \in \mathbb{N}$, $n \geq 1$.
If we now go back to

$$Jz' = \begin{pmatrix} t & 0 \\ \\ 0 & t \end{pmatrix},$$

formula (5.4) reads as: $-\dfrac{3}{2} = -(\dfrac{1}{2} + 1 + 0)$.

Remark 5.5 *It is worthmentioning the relation between the Maslov index and the Morse index. This question is deep and is part of the "Index theorem". Let us restrict ourselves to the elementary example* $u'' + 4u = 0, u(0) = 0 = u(\pi)$. *If we set*

$$B = \begin{pmatrix} 4 & 0 \\ \\ 0 & 1 \end{pmatrix},$$

we have $j(B) = 1$ *(in fact,* $u(t) = \sin 2t$ *is a solution). According to [38], the Morse index* $m(A)$*of the operator* $A : u \mapsto -u'' - 4u$ *defined in* $C^2_0([0, \pi])$ *is* 1*. Indeed,* 1 *is the number of negative eigenvalues of* A*. Generally speaking, for systems of second order equations* $X''(t)+A(t)\ X(t) = 0,\ t \in [0, \pi],\ X \in \mathbb{R}^N$*, the number of moments of verticality* $j(B)$*, being* B *as in* (5.6)*, equals the Morse index of*

$$\varphi_A(u) = \frac{1}{2} \int_0^\pi \left[-|u'(t)|^2 + (A(t)u(t), u(t)) \right] dt, \quad \forall\ u \in H^1_0([0, \pi])$$

(we refer to [1] and [31] for details).

6 Phase angles

We follow the results by Greenberg [22] and Atkinson [3]. We describe a so-called "generalized phase-plane analysis". As in the previous sections, we denote by

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

a frame for $\psi(t)V$. Then we can consider the matrix

$$\Theta(t) = (Y(t) + iX(t))\ (Y(t) - iX(t))^{-1}.$$

Observe that the fact that $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ has rank N implies that the matrix

$$\begin{pmatrix} Y(t) & X(t) \\ \\ -X(t) & Y(t) \end{pmatrix}$$

is invertible and thus the matrix $(Y(t)-iX(t))$ is invertible as well. We recall that a matrix M is called unitary if $\Theta^*(t)\ \Theta(t) = \mathrm{Id}$. The following fact will be useful in the sequel.

Lemma 6.1 *The matrix $\Theta(t)$ is unitary, for all t.*

The computation needed for the proof is based on the fact that $\psi(t)V$ is lagrangian, and this follows from the fact that $\psi(t)$ is symplectic.

Proposition 6.2 (Atkinson [3]). *There exist N continuous functions $\theta_j : [0,\pi] \to \mathbb{R},\ j = 1,\dots,N$, such that:*

(i) $e^{2i\theta_j(t)}$ *is an eigenvalue of $\Theta(t)$, for $j = 1,\dots,N$;*

(ii) $\theta_1(0) = 0 = \dots = \theta_N(0)$;

(iii) $\theta_1(t) \le \theta_2(t) \le \dots \le \theta_N(t) \le \theta_1(t) + 2\pi$.

The functions θ_j are called "phase angles". The existence of θ_j satisfying Proposition 6.2 is proved in [3] using the fact that the eigenvalues of a symmetric matrix are continuous functions of t. It also follows from Kato Selection theorem (cf. Theorem 5.2 in [24]).
The relation between phase angles and the notion of moment of verticality is explained in the following

Theorem 6.3 *Let $\bar t \in [0,\pi]$. The following facts are equivalent:*

(a) *$\bar t$ is a moment of verticality for (5.1) of multiplicity m;*

(b) *1 is an eigenvalue of $\Theta(\bar t)$ of algebraic multiplicity m;*

(c) *there exist $j_1,\dots,j_m \in \{1,\dots N\}$ s.t. $\theta_{j_k}(\bar t) = 0 (\mathrm{mod}\pi),\ k = 1,\dots m$.*

Sketch of the proof. The equivalence between (b) and (c) is straightforward. Let 1 be an eigenvalue of $\Theta(\bar t)$ with algebraic multiplicity m. Being $\Theta(t)$ unitary, it follows that the geometric multiplicity coincides with the algebraic multiplicity; this means that $m = \dim(\mathrm{Ker}(\Theta(\bar t) - \mathrm{Id}))$. Note also that

$$\Big(\Theta(\bar t) - \mathrm{Id}\Big)\ \Big(Y(\bar t) - iX(\bar t)\Big) = \Big[\Big(Y(\bar t) + iX(\bar t)\Big)\ \Big(Y(\bar t) - iX(\bar t)\Big)^{-1} - \mathrm{Id}\Big]\ \Big(Y(\bar t) - iX(\bar t)\Big) =$$

$$= Y(\bar t) + iX(\bar t) - Y(\bar t) + iX(\bar t) = 2iX(\bar t).$$

Hence,

$$\Big(\Theta(\bar t) - \mathrm{Id}\Big) = 2iX(\bar t)\ \Big(Y(\bar t) - iX(\bar t)\Big)^{-1}$$

and

$$\mathrm{Ker}(\Theta(\bar t) - \mathrm{Id}) = \mathrm{Ker}(X(\bar t)) = m.$$

This is sufficient to conclude. ■

The following theorem is important for the development of a "generalized phase-plane analysis".

Theorem 6.4 (Atkinson [3]). *The functions θ_j are strictly increasing, for all $j = 1, \dots N$.*

After a sketch of the proof, we shall comment on the relation between the phase angles and (in case $N = 1$) the Prüfer coordinate $\theta(t) \in \mathbb{R}$ in the equation

$$\theta'(t) = B_{11}(t)\sin^2\theta(t) + 2B_{12}(t)\sin\theta(t)\cos\theta(t) + B_{22}(t)\cos^2\theta(t).$$

The crucial points for the proof of the above theorem are

Lemma 6.5 *Assume that $A(t)$ is a differentiable hermitian matrix s.t. $A'(t)$ is positive definite for all t. Then the eigenvalues of $A(t)$ are increasing functions of t.*

Lemma 6.6 *The matrix $\Theta(\cdot)$ satisfies the matrix differential equation*

$$\Theta'(t) = i\ \Omega(t)\ \theta(t),$$

where

$$\Omega(t) = 2\ \left[\Big(Y(t) - iX(t)\Big)^{-1}\right]^* \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} B(t) \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \Big(Y(t) - iX(t)\Big)^{-1}.$$

It is important to observe that the fact that $B(t)$ is positive definite implies that $\Omega(t)$ is positive definite.

Sketch of the proof of Theorem 6.4. Fix t_0; our aim is to prove that θ_j is increasing in a neighbourhood of t_0. To this end, consider α s.t. $e^{2i\alpha}$ is not an eigenvalue of $\Theta(t_0)$ and consider the matrix

$$M(t) = i\Big(e^{2i\alpha}\mathrm{Id} + \Theta(t)\Big)\ \Big(e^{2i\alpha}\mathrm{Id} - \Theta(t)\Big)^{-1}.$$

The matrix $M(t)$ is hermitian. If we denote by $\lambda_j(t)$ the eigenvalues of $M(t)$ then

$$\lambda_j(t) = i\ \frac{e^{2i\alpha} + e^{2i\theta_j(t)}}{e^{2i\alpha} - e^{2i\theta_j(t)}}.$$

Moreover, the function

$$\omega \mapsto i\ \frac{e^{2i\alpha} + \omega}{e^{2i\alpha} - \omega}$$

maps continuously the (clockwise oriented) unit circle to the (oriented) real line. Hence, for the proof it is sufficient to show that the matrix

$$M'(t) = 2\left[\Big(e^{2i\alpha}\mathrm{Id} + \Theta(t)\Big)^{-1}\right]^*\ \Omega(t)\ \Big(e^{2i\alpha}\mathrm{Id} - \Theta(t)\Big)^{-1}$$

is positive definite. This follows from the fact that $\Omega(t)$ is positive definite. ■

Let us consider the particular case $N = 1$. Then the system

$$Jz' = B(t)z \quad z = (x, y) \in \mathbb{R}^2$$

can be written as

$$\begin{cases} x' = B_{12}(t)x + B_{22}(t)y \\ \\ y' = -B_{11}(t)x + B_{12}(t)y \end{cases} \tag{6.1}$$

or (writing $x(t) = \rho(t)\sin\theta(t), y(t) = \rho(t)\cos\theta(t)$)

$$\theta'(t) = B_{11}(t)\sin^2\theta(t) + 2B_{12}(t)\sin\theta(t)\cos\theta(t) + B_{22}(t)\cos^2\theta(t). \tag{6.2}$$

It is easy to see that if $B(t)$ is positive definite then $\theta'(t) > 0$ for all t. Note also that if we deal with the scalar equation $x'' + \alpha(t)x = 0$ or, equivalently, with the planar system

$$\begin{cases} x' = y \\ \\ y' = -\alpha(t)x \end{cases} \quad \Longleftrightarrow \quad Jz' = \begin{pmatrix} \alpha(t) & 0 \\ \\ 0 & 1 \end{pmatrix} z, \tag{6.3}$$

the positive definiteness of $B(t)$ follows from the condition $\alpha(t) > 0$. It is easy to check, and very useful, that if $\bar{t}$ is such that $\theta(\bar{t}) = 0(\text{mod}\pi)$ then $\theta'(\bar{t}) = B_{22}(\bar{t})$; as a consequence, if we are interested in the monotonicity of θ in $\bar{t}$, then it is sufficient to require $B_{22}(\bar{t}) > 0$. Note that this condition is always satisfied when we deal with systems arising from second order equations; indeed, in this case we have $B_{22} \equiv 1$. Moreover, from (6.1) it follows that if $\theta(\bar{t}) = 0(\text{mod}\pi)$ then

$$B_{22}(\bar{t}) = \frac{x'(\bar{t})}{y(\bar{t})}.$$

Hence, if $x'(\bar{t}) > 0$ and $y(\bar{t}) > 0$ (or $x'(\bar{t}) < 0$ and $y(\bar{t}) < 0$) then $B_{22}(\bar{t}) > 0$ and $\theta'(\bar{t}) > 0$. Analogously, if $x'(\bar{t}) < 0$ and $y(\bar{t}) > 0$ (or $x'(\bar{t}) > 0$ and $y(\bar{t}) < 0$) then $B_{22}(\bar{t}) < 0$ and $\theta'(\bar{t}) < 0$. For this reason, if $\bar{t}$ is a zero of $x(\cdot)$ (or, equivalently, $\theta(\bar{t}) = 0(\text{mod}\pi)$) then we can speak about zeros "counted positively" and "counted negatively", respectively.

The possibility of weakening the assumption that $B(t)$ is positive definite to the positive definiteness of $B_{22}(t)$ also in the case $N \geq 2$ has been considered in [22] in the case of systems arising from systems of $(2N)$-th order equations.

We end this section with

Theorem 6.7

$$j(B) = k_1(\pi) + \cdots + k_N(\pi), \tag{6.4}$$

where $k_j(t)$ is such that $\theta_j(t) = k_j(t)\pi + \alpha_j(t)$, being $\alpha_j(t) \in (0, \pi]$ and $k_j(t) \in \mathbb{N}$.

Sketch of the proof. Statement (6.4) follows from the fact that the phase angles θ_j are increasing and by observing that $k_j(\bar{t})$ is the number of times $\theta_j(\bar{t})$ is a multiple of π, i.e. the number of moments of verticality. The relation between the multiplicity of a moment of verticality $\bar{t}$ and the number of phase angles θ_{j_k} which are multiples of π (cf. Theorem 6.3) is sufficient to conclude. ■

7 Dirac-type systems and hamiltonian-like systems

Consider a differential system of the form

$$\tau z = B(t)z, \quad z \in \mathbb{R}^{2N} \tag{7.1}$$

where

$$\tau z = 2q(t)Jz' + q'(t)Jz + P(t)z,$$

being $q \in C^1([0,\pi]), q(t) > 0$ for all t and $P(t)$ is a $2N \times 2N$ symmetric matrix. Systems of this form are called "Dirac-type" systems; we refer to [47] for details. See also [44], [45]. Note that an elementary change of variables transforms (7.1) into

$$Jw' = \frac{B(t) - P(t)}{2q(t)}\, w = \tilde{B}(t)w.$$

The matrix $\tilde{B}(t)$ is symmetric but if $B(t)$ is positive definite the matrix $\tilde{B}(t)$ does not have, in general, the same property. Thus the arguments developed in the previous sections do not follow immediately in case of Dirac-type systems. On the basis of phase-plane analysis and the classical notion of rotation number, we obtained some multiplicity results for nonlinear BVPs associated to *planar* systems of Dirac-type (cf. [8], [12] in case $t \in [0,\pi]$ and [13] for $t \in [1,+\infty)$). In the general case of $2N$-dimensional systems and $t \in [0,\pi]$, the possibility of developing a "complete linear theory" and an "Index theorem" has been developed by Boscaggin-Garrione [9] and is the object of a current research. More precisely, in [9] it is considered a system of the form

$$Jz' + A(t)Jz = B(t)z \tag{7.2}$$

where $t \in [0,\pi], z \in \mathbb{R}^{2N}$, the matrix $B(t)$ is symmetric and $A(t)$ is "hamiltonian-like", i.e. there exists $c \in \mathbb{R}$ such that $(A(t)J)^t = A(t)J + cJ$. The case $c = 0$ corresponds to the case when $A(t)$ is hamiltonian. Systems of the form (7.2) contain Dirac-type systems; indeed

$$\tau z = B(t)z \iff 2q(t)Jz' + q'(t)Jz + P(t)z = B(t)z \iff$$

$$\iff Jz' + \frac{q'(t)}{2q(t)}Jz + \frac{P(t)}{2q(t)}z = \frac{B(t)}{2q(t)}z$$

and $\frac{q'(t)}{2q(t)}\mathrm{Id}$ is hamiltonian-like (with $c = -2$). Boscaggin-Garrione show that in this general situation it is possible to define phase angles, to introduce an index using phase angles and to give a "spectral theorem" for (7.2). In [9] it is observed that the fundamental matrix ψ associated to (7.2) is symplectic-like, i.e. there exists $d \neq 0$ s.t. $\psi(t)^t\ J\ \psi(t) = dJ$; this is sufficient to guarantee two important facts. First, in this context the image through $\psi(t)$ of the vertical lagrangian $V = \{0\} \times \mathbb{R}^N$ is a lagrangian subspace; hence the Maslov index $\mu(\psi)$ is well defined. Moreover, the matrix $\Theta(t)$ (defined as in Section 6 using $\psi(t)$) is still unitary, and this is sufficient for the definition of phase angles. The index of (7.2) is then defined as follows:

$$i(A,B) = \sum_{j=1}^{N} \Big[\frac{\theta_j(t)}{2\pi}\Big].$$

For related questions, consult [14] and [27]. The main result in [9] is the following

Theorem 7.1 (Boscaggin-Garrione [9]). *Suppose that $f \in C([0,T],\mathbb{R})$, $B \in C([0,T],\mathcal{L}_s(\mathbb{R}^{2n}))$ with $B(t)$ positive definite for every $t \in [0,\pi]$. Then the linear boundary value problem*

$$\begin{cases} Jz' + A(t)Jz = \lambda B(t)z \\ x(0) = x(\pi) = 0 \end{cases} \qquad z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N$$

has a countable set of eigenvalues, with no accumulation points. Moreover, the eigenvalues can be ordered, according to their multiplicity, in a two-sided sequence $(\lambda_k)_{k\in\mathbb{Z}}$, such that

i) $\cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots$;

ii) $\lambda_k \to \pm\infty$ *for* $k \to \pm\infty$;

iii) $i(A, \lambda_k B) = \max_{\lambda_k=\lambda_j} j$, *for every* $k \in \mathbb{Z}$.

This theorem contains and extends to higher dimension the results for $N = 1$ in [8], [44], [45]. On the lines of [14] Boscaggin-Garrione also provide a formula which relates the index $i(A,B)$ to the Maslov index.

8 Another notion of Maslov index

Starting from the paper by Johnson-Moser [26], another notion of Maslov index has been developed. More recent improvements and developments can be found in the paper by Fabbri-Johnson-Nuñez [20] and references therein. In [26] it is considered

$$-u'' + q(t)u = \lambda u, \quad t \in \mathbb{R},$$

with q almost-periodic. Little is known about the related spectral theory [21]. Working in the framework of topological dynamics [40], a notion of rotation number can be introduced. The basic ingredient is the "hull of q":

$$H(q) = \overline{\{q_\tau(\cdot) = q(\cdot+\tau), \tau \in \mathbb{R}\}},$$

where the closure is meant in the sense of uniform convergence in compact subsets. A flow $\pi : H(q)\times\mathbb{R} \to H(q)$ can be defined in $H(q)$ as follows:

$$\pi(q,\tau) = q_\tau, \quad q_\tau(t) = q(t+\tau).$$

The function q can be embedded in a family of functions $Q : H(q) \to H(q)$ s.t. $Q(q_0) = q$. Then one can consider the equation

$$\theta'(t) = F(\tau(t),\theta(t)) = \cos^2\theta(t) - \Big[Q(q_{\tau(t)}) - \lambda\Big]\sin^2\theta(t) \tag{8.1}$$

and denote by $\theta(t,\tau_0,\theta_0)$ the (unique) solution to (8.1) satisfying $\theta(0) = \theta_0, \tau(0) = \tau_0$. Then one can prove

Theorem 8.1 (Johnson-Moser [26]). *There exists*

$$\alpha(\lambda) = \lim_{t\to+\infty} \frac{\theta(t,\tau_0,\theta_0) - \theta(0,\tau_0,\theta_0)}{t}$$

and this limit is independent of (τ_0,θ_0). It is a continuous increasing function of λ; moreover, $\lim_{\lambda\to+\infty}\alpha(\lambda) = +\infty$ and there exists λ^* such that $\alpha(\lambda) = 0$ for all $\lambda \le \lambda^*$.

The limit $\alpha(\lambda)$ in Theorem 8.1 is called the Rotation Number. Finally, it is important to mention that if $N(a,b)$ denotes the number of zeros of the solution $u(\cdot,\lambda)$ in $[a,b]$, then

$$\lim_{t\to+\infty} \frac{\pi\ N(a,b)}{b-a} = \alpha(\lambda).$$

9 Nonlinear first order systems in $\mathbb{R}^{2N}$

We follow the paper [11]. Consider

$$\begin{cases} z' = \lambda JS(t)z + JF(t,z,\lambda)z = JS(t,z(t),\lambda)z, \ z=(x,y)\in\mathbb{R}^{2N}, \\ \\ x(0) = 0 = x(\pi) \end{cases}$$

where $F(t,z,\lambda)$ is a symmetric matrix and $F(t,0,\lambda)=0$ for every t,λ and the linear eigenvalue problem

$$\begin{cases} z' = \lambda JS(t)z, \\ \\ x(0) = 0 = x(\pi). \end{cases}$$

The abstract bifurcation theorem we use is the classical Rabinowitz Theorem 1.1. Let us now focus on the linear theory. To this end, some preliminaries are in order. In what follows we denote by $\lambda_1,\dots,\lambda_{2N}$ the (real) eigenvalues of a matrix $S\in M_S^{2N}$ and let $D=\mathrm{diag}(\lambda_1,\dots,\lambda_N)$; moreover, let P be the orthogonal matrix such that

$$P^T SP = D.$$

We denote by $\mathcal{S}$ the class of constant matrices $S\in M_S^{2N}$ such that:

1. For every $i\neq j,\ i,\ j=1,\dots,N$, we have

$$\frac{\sqrt{\lambda_i\lambda_{N+i}}}{\sqrt{\lambda_j\lambda_{N+j}}} \notin \mathbf{Q}.$$

2. The matrix P has the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ \\ -P_{12} & P_{11} \end{pmatrix} \tag{9.1}$$

   and P_{11} is invertible.

3. The matrix $Q=-P_{12}P_{11}^{-1}$ is diagonal.

We remark that it would be possible to define a different class $\mathcal{S}$, suitable for the proof of our results, by requiring that P_{12} is invertible and by replacing the matrix Q with the matrix $Q'=-P_{11}P_{12}^{-1}$.

We recall that every symplectic orthogonal matrix P can we written in the form (9.1); we also observe that it is possible to find matrices S belonging to the class $\mathcal{S}$. Indeed, let us take $P_{11}\in M_S^N$ such that $P_{11}^2=$ Id and set $P_{12}=P_{11}$. Then, for every diagonal matrix $D=\mathrm{diag}(\lambda_1,\dots,\lambda_N)$ whose eigenvalues satisfy condition 1, the matrix $S=P\ D\ P^T$ belongs to the class $\mathcal{S}$.
In case $N\geq 2$ we have

Proposition 9.1 *Assume $S \equiv S(t)$ for all t, and denote by $\lambda_1, \dots, \lambda_{2N}$ the (real) eigenvalues of S. Assume that $S \in \mathcal{S}$. Then the eigenvalues of the linear problem are*

$$\lambda_{j,k} = \frac{k}{\sqrt{\lambda_j \lambda_{N+j}}}, \quad \forall\ j = 1, \dots, N,\ k \in \mathbb{Z}.$$

Moreover, for every $k \neq 0$ and for every $j = 1, \dots, N$, $\lambda_{j,k}$ is simple.

In case $N = 1$ we can write

Proposition 9.2 *The eigenvalues (if they exist) are simple. Moreover,*

$$S \in \mathcal{S} \iff det\ S > 0.$$

The main result of the "linear theory" is

Theorem 9.3 *Assume that $S \in C([0,\pi], \mathcal{S})$. Then the linear problem has a double sequence of eigenvalues μ_k such that*

$$\mu_k \to \pm\infty, \quad as \quad k \to \pm\infty.$$

Moreover, for every $k \in \mathbb{Z}, k \neq 0$,

$$j(-\mu_k S) = |k| - 1.$$

Let us now focus on the notion of index of a solution to the nonlinear problem

$$\begin{cases} z' = \lambda JS(t)z + JF(t,z,\lambda)z = JS(t,z(t),\lambda)z,\ z = (x,y) \in \mathbb{R}^{2N}, \\ \\ x(0) = 0 = x(\pi), \end{cases} \tag{9.2}$$

where $F(t,z,\lambda)$ is a symmetric matrix and $F(t,0,\lambda) = 0$ for every t, λ. For $(\lambda, z) \in \Sigma$, consider

$$\phi(\lambda, z) = j(-S(\cdot, z(\cdot), \lambda)),$$

where, for every solution (λ, z) of the nonlinear system, $j(-S(\cdot, z(\cdot), \lambda))$ denotes the number of moments of verticality of the linear system

$$Jw' = -S(t, z(t), \lambda)w,\ w = (u,v) \in \mathbb{R}^{2N}.$$

Before stating a global bifurcation result for the nonlinear problem (9.2), we have to introduce the following condition:
(F) exists $\xi : (a,b) \longrightarrow (0, +\infty)$ such that

$$||F(t,z,\lambda)|| \leq \xi(\lambda), \quad \forall\ (t,z,\lambda) \in [0,\pi] \times \mathbb{R}^{2N} \times (a,b). \tag{9.3}$$

Then we have

Theorem 9.4 *Assume that $S \in \mathcal{S}$ and $F(t,0,\lambda) = 0$, for every $t \in [0,\pi]$, $\lambda > 0$. Moreover, suppose that F satisfies (F). Then, for every $k \in \mathbb{N}$, $k \neq 0$, Σ contains a continuum C_k such that $(\mu_k, 0) \in C_k$, there exists $(\lambda_n, u_n) \in C_k$ such that*

$$\phi(\lambda, u) = k - 1, \quad for\ every\ (\lambda, u) \in C_k$$

and

$$\lambda_n + ||u_n|| \to +\infty \quad or\ \ \lambda_n \to 0^+.$$

10 Nonlinear Dirac-type systems in the half-line

We follow the paper [13]. Consider

$$Jz' + P(t)z = \lambda z + Q(t,z)z, \quad t \in [1,+\infty), \quad \lambda \in \mathbb{R}, \quad z = (u,v) \in \mathbb{R}^2,$$

where $P(t), Q(t,z)$ are continuous symmetric matrices. We denote by $\mathcal{P}$ the class of continuous maps $P : [1,+\infty) \longrightarrow M_S^{2,2}$ such that

$$\lim_{t\to+\infty} P(t) = \begin{pmatrix} \mu^- & 0 \\ 0 & \mu^+ \end{pmatrix} := P_0,$$

for some $\mu^- < \mu^+$, and there exists $q \geq 1$ such that

$$\int_1^{+\infty} |R(t)|^q \, dx < +\infty,$$

where $R(t) = P(t) - P_0$, for every $t \geq 1$. Moreover, we denote by $\mathcal{Q}$ the set of continuous functions $Q : [1,+\infty) \times \mathbb{R}^2 \longrightarrow M_S^{2,2}$ satisfying the conditions

1. there exist $\alpha \in L^\infty(1,+\infty)$, $\eta_1, \eta_2, \eta_{12} : \mathbb{R}^2 \longrightarrow \mathbb{R}$, continuous and with $\eta_1(0) = \eta_2(0) = \eta_{12}(0) = 0$, and $p \geq 1$ for which

$$\begin{aligned} &|Q_i(x,z)| \leq \alpha(x)\eta_i(z), \quad \forall\, x \geq 1, \quad z \in \mathbb{R}^2,\ i = 1,2, \\ &|Q_{12}(x,z)| \leq \alpha(x)\eta_{12}(z), \quad \forall\, x \geq 1, \quad z \in \mathbb{R}^2; \end{aligned} \tag{10.1}$$

2. for every compact $K \subset \mathbb{R}^2$ there exists $A_K > 0$ such that

$$||Q(x,z) - Q(x,z')|| \leq A_K ||z - z'||, \quad \forall\, x \geq 1, \quad z, z' \in K. \tag{10.2}$$

We will be interested in solutions z satisfying a boundary condition of the form $v(1) = 0$ and belonging to the space $H^1(1,+\infty)$; in particular, the solutions are convergent to zero at infinity.

This choice is strictly related to the spectral properties of the linear operator $\tau z = Jz' + P(t)z$ and to the possibility to consider self-adjoint extensions of τ.

The abstract bifurcation result we shall apply is due to Stuart [41]; let us briefly describe this theorem. Consider a real Hilbert space B and let $A_0 : D(A_0) \longrightarrow B$ be an unbounded self-adjoint operator in B with

$$\sigma_{ess}(A_0) = (-\infty, \mu^-] \cup [\mu^+, +\infty),$$

for some $\mu^- < \mu^+$. Let H denote the real Hilbert space obtained from the domain of A_0 equipped with the graph topology and let us consider the nonlinear problem

$$A_0 u + M(u) = \lambda u, \ (\lambda, u) \in \mathbb{R} \times H,$$

where $M : H \longrightarrow B$ is a continuous and compact map such that

$$M(u) = o(||u||), \quad u \to 0.$$

Theorem 10.1 *Let $\mu \in (\mu^-, \mu^+)$ be an eigenvalue of A_0 of odd multiplicity and let C_μ denote the component of Σ containing $(\mu, 0)$. Then, C_μ has one of the following properties:*

(1) C_μ is unbounded in $\mathbb{R} \times H$.

(2) $\sup\{\lambda : \ (\lambda, u) \in C_\mu\} \geq \mu^+$ or $\inf\{\lambda : \ (\lambda, u) \in C_\mu\} \leq \mu^-$.

(3) C_μ contains an element $(\mu^*, 0) \in \Sigma$ with $\mu^* \neq \mu$.

This theorem is a straightforward variant of the result by Stuart, where the author considers the case of a linear operator A_0 satisfying $\sigma_{ess}(A_0) = [Q, +\infty)$, for some Q (as it happens for the one-dimensional Schrödinger operators).

As usual, we need some knowledge of the oscillatory properties of the solutions to the linear eigenvalue problem

$$Jz' + P(t)z = \lambda z. \tag{10.3}$$

Indeed, writing (10.3) as

$$\theta'(t) = (\lambda - p_1(t)) \cos^2 \theta(t) - 2p_{12}(t) \cos \theta(t) \sin \theta(t) + (\lambda - p_2(t)) \sin^2 \theta(t),$$

one can show that for every $\lambda \in (\mu^-, \mu^+)$ there exist $t_\lambda \geq 1$ and $k_\lambda \in \mathbb{Z}$ such that

$$k_\lambda \pi \leq \theta(t, \lambda) < (k_\lambda + 1)\pi, \quad \forall \ t \geq t_\lambda.$$

We are now in position to define the index associated to a nontrivial solution of the linear problem:

Definition 10.2 *Assume that $P \in \mathcal{P}$ and let $(\lambda, z) = (\lambda, u, v)$ be a nontrivial solution satisfying the $v(1) = 0$. We define*

$$i(\lambda, z) = \left[\frac{\theta(t_\lambda, \lambda)}{\pi}\right].$$

Then we can get

Proposition 10.3 *Let $\{\lambda_k\}_{k \in K}$ be the set of eigenvalues of A_0 in (μ^-, μ^+), for some $K \subset \mathbb{N}$. Then, there exist at most two indeces k_1 and $k_2 \in K$ such that*

$$i(\lambda_{k_1}) = i(\lambda_{k_2}) \ \text{ and } \ i(\lambda_j) \neq i(\lambda_m), \quad \forall \ j \neq m, \ j, m \in K \setminus \{k_1, k_2\},$$

where $i(\lambda_k)$ denotes the index of $(\lambda_k, z_{\lambda_k})$, being z_{λ_k} an eigenfunction associated to λ_k.

Let us now assume that

$$P(t) = \begin{pmatrix} -1 + V(t) & k/t \\ \\ k/t & 1 + V(t) \end{pmatrix}, \quad k \in \mathbb{N}.$$

Then we can guarantee the existence of eigenvalues; more precisely

Proposition 10.4 *Assume that $V \in C(1, +\infty)$ is a strictly increasing negative potential such that*

$$V(t) \sim \frac{c}{t^\alpha}, \quad t \to +\infty,$$

with $\alpha \in (0, 1]$. Then, the selfadjoint extension $A_0 : \{z \in H^1(1, +\infty) : \ v(1) = 0\} \longrightarrow L^2(1, +\infty)$ of the operator τ has a sequence of eigenvalues in $(-1, 1)$ converging to 1.

The hypotheses leading to the existence of eigenvalues are, in some sense, assumptions on the oscillatory behaviour of the linear problem when $\lambda = \mu^+$. In other words, the number of eigenvalues in (μ^-, μ^+) is related to the number of rotations of solutions of the linear problem for $\lambda = \mu^+$. When this number of rotation is infinite, an infinite sequence of eigenvalues accumulating to μ^+ does exist; when the solutions for $\lambda = \mu^+$ have a finite number of rotations in the phase-plane, then only finitely many eigenvalues fall in (μ^-, μ^+).
Let us now focus on the definition of the index of a solution to the nonlinear problem

$$Jz' + P(t)z = \lambda z + Q(t,z)z, \quad t \in [1, +\infty), \quad \lambda \in \mathbb{R}, \quad z = (u, v) \in \mathbb{R}^2. \tag{10.4}$$

We now give the following

Definition 10.5 *Assume that $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ and let (w, μ) be a solution of $Jz' + P(t)z = \lambda z + Q(t,z)z, \quad v(1) = 0, \quad t \in [1, +\infty)$.*
If $(\mu, w) \neq (\mu, 0)$, then the index of (μ, w) is defined as the index $i(\mu, w)$ of (μ, w) as a solution of the linear problem

$$\begin{cases} Jz' + P(t)z = \mu z + Q(t, w(t))z, \\ v(1) = 0. \end{cases}$$

If $(\mu, w) = (\mu, 0)$ and the linear problem

$$\begin{cases} Jz' + P(t)z = \mu z, \\ v(1) = 0 \end{cases}$$

has a nontrivial solution z_μ belonging to $H^1(1, +\infty)$, then the index of (μ, w) is defined as the index $i(\mu, z_\mu)$ of (μ, z_μ) as a solution of the above problem.

System (10.4) fits into the framework of the abstract bifurcation theorem with $H = D_0 = \{z = (u, v) \in H^1(1, +\infty) : \ v(1) = 0\}$, $B = L^2(1, +\infty)$ and M being the Nemitskii operator associated to Q, given by

$$M(u)(t) = Q(t, u(t))u(t), \quad \forall \ t \geq 1,$$

for every $u \in D_0$.
Moreover,

Proposition 10.6 *Assume that $Q \in \mathcal{Q}$ and that*

$$\lim_{x \to +\infty} \alpha(x) = 0, \tag{10.5}$$

where α is given in (10.1). Then $M : D_0 \longrightarrow L^2(1, +\infty)$ is a continuous compact map and satifies $M(u) = o(||u||), u \to 0$.

Finally, we state a global bifurcation result for the nonlinear problem (10.4).

Theorem 10.7 *Assume that $P \in \mathcal{P}$, $Q \in \mathcal{Q}$ and (10.5) hold true. Then for every $k \in K \setminus \{k_1, k_2\}$ there exists a continuum C_k of nontrivial solutions of (10.4) in $\mathbb{R} \times D_0$ bifurcating from $(\lambda_k, 0)$ and such that*

(1) C_k *is unbounded in* $\mathbb{R} \times D_0$*, or*

(2) $\sup\{\lambda : (\lambda, u) \in C_k\} \geq \mu^+$ *or* $\inf\{\lambda : (\lambda, u) \in C_k\} \leq \mu^-$.

Moreover, we have

$$i(\lambda, z) = i(\lambda_k), \quad \forall\ (\lambda, z) \in C_k.$$

11 Open problems

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