C.I.M.E. Lecture Series in Cetraro

Discrete-time nonautonomous dynamical systems

or equivalently

Nonautonomous Difference Equations

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$6 + \varepsilon$ Lectures

on Nonautonomous Difference Equations

Lecture ε: Autonomous Difference Equations

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous mapping. A difference equation of the form

$$\boxed{x_{n+1} = f(x_n) \qquad (1)}$$

is called a first-order autonomous difference equation on $\mathbb{R}^d$.

There is no loss of generality in the restriction to first-order difference equations (1), since higher-order difference equations can be reformulated as (1) by the use of an appropriate higher dimensional state space.

Successive iteration of the difference equation (1) generates the solution mapping $\pi : \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\boxed{x_n = \pi(n, x_0) = f^n(x_0) := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x_0),}$$

which satisfies the initial condition and semigroup property

$$\boxed{\begin{array}{c} \pi(0, x_0) = x_0, \qquad \pi(n, \pi(m, x_0)) = \pi(n+m, x_0) \\ \text{for all } n,\, m \in \mathbb{Z}^+, x_0 \in \mathbb{R}^d. \end{array}}$$

The solution mapping π usually forms only a semigroup under composition rather than a group since the mapping f need not be invertible.

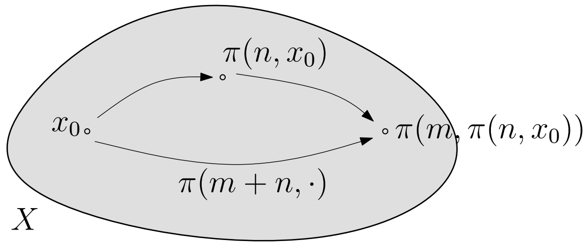

Figure 1: Semigroup property ii) of a discrete-time semidynamical system $\pi : \mathbb{Z}^+ \times X \to X$

The solution mapping π generates a discrete-time semidynamical system on $\mathbb{R}^d$.

In general, the state space could be a metric space (X, d).

Definition 1. *A mapping $\pi : \mathbb{Z}^+ \times X \to X$ satisfying*

i) $\pi(0, x_0) = x_0$ *for all* $x_0 \in X$,

ii) $\pi(m + n, x_0) = \pi(m, \pi(n, x_0))$ *for all* m, $n \in \mathbb{Z}^+$ *and* $x_0 \in X$,

iii) the mapping $x_0 \mapsto \pi(n, x_0)$ *is continuous for each* $n \in \mathbb{Z}^+$,

is called a (discrete-time) autonomous semidynamical system or a semigroup on the state space X.

If $\mathbb{Z}^+$ is replaced by $\mathbb{Z}$, then π is called a (discrete-time) autonomous dynamical system on the state space X.

An autonomous semidynamical system π on X is equivalent to a first order autonomous difference equation on X with the right-hand side f defined by

$$f(x) := \pi(1, x) \quad \text{for all } x \in X.$$

Autonomous dynamical systems need not be generated by autonomous difference equation.

The space $X = \{1, \cdots, r\}^{\mathbb{Z}}$ of bi-infinite sequences $x = \{k_n\}_{n\in\mathbb{Z}}$ with $k_n \in \{1, \cdots, r\}$ is a compact metric space with the metric

$$d(x, x') = \sum_{n\in\mathbb{Z}} (r+1)^{-|n|} |k_n - k'_n| .$$

The group of left shift operators $\theta_n := \theta^n$ for $n \in \mathbb{Z}$ on X, where the mapping $\theta : X \to X$ is defined by

$$\theta(\{k_n\}_{n\in\mathbb{Z}}) = \{k_{n+1}\}_{n\in\mathbb{Z}},$$

forms an autonomous dynamical system on X.

Invariant sets and attractors

The Hausdorff separation $\mathrm{dist}_X(A, B)$ of nonempty compact subsets $A, B \subseteq X$ is defined as

$$\mathrm{dist}(A, B) := \max_{a \in A} \mathrm{dist}(a, B) = \max_{a \in A} \min_{b \in B} d(a, b)$$

and the Hausdorff metric on the space $\mathcal{H}(X)$ of nonempty compact subsets of X is defined by

$$H(A, B) = \max \{\mathrm{dist}(A, B), \mathrm{dist}(B, A)\}.$$

The dynamical behaviour of a semidynamical system is characterised by its invariant sets and what happens in neighbourhoods of such sets.

A nonempty subset A of X is called π-invariant if

$$\pi(n, A) = A \quad \text{for all } n \in \mathbb{Z}^+ \tag{2}$$

or, equivalently, if $f(A) = \pi(1, A) = A$.

Simple examples are

- equilibria (steady state solutions) and

- periodic solutions.

In the first case A consists of a single point, which must thus be a fixed point of the mapping f.

A solution with period r consists of a finite set of r distinct points $\{p_1, \ldots, p_r\}$ which are fixed points of the composite mapping f^r (but not for an f^j with j smaller than r).

Invariant sets can also be much more complicated, e.g., fractal sets. Many are the ω-limit sets of some trajectory, i.e., defined by

$$\omega^+(x_0) = \{y \in X \ : \ \exists n_j \to \infty, \quad \pi(n_j, x_0) \to y\},$$

which is nonempty, compact and π-invariant, provided the forward trajectory $\{\pi(n, x_0);\ n \in \mathbb{Z}^+\}$ is a precompact subset of X and the metric space (X, d) is complete.

The asymptotic behaviour of a semidynamical system is characterised by its ω-limit sets, in general, and by its attractors and their associated absorbing sets, in particular.

An attractor is a nonempty π-invariant compact set A^* that attracts all trajectories starting in some neighbourhood $\mathcal{U}$ of A^*, i.e., with $\omega^+(x_0) \subset A^*$ for all $x_0 \in \mathcal{U}$ or, equivalently, with

$$\lim_{n\to\infty} \operatorname{dist}(\pi(n, x_0), A^*) = 0 \quad \text{for all } x_0 \in \mathcal{U}.$$

A^* is called a maximal or global attractor when $\mathcal{U}$ is the entire state space X. If a global attractor exists, then it must be unique.

Definition 2. *A nonempty compact subset A^* of X is a global attractor of the semidynamical system π on X if it is π-invariant and attracts bounded sets, i.e.,*

$$\lim_{n\to\infty} \operatorname{dist}(\pi(n, D), A^*) = 0 \quad \textit{for any bounded subset } D \subset X. \tag{3}$$

A simple example is the difference equation (1) on $X = \mathbb{R}$ with

$$f(x) := \max\{0, 4x(1-x)\}$$

for $x \in \mathbb{R}$. Then $A^* = [0,1]$ is invariant and $f(x_0) \in A^*$ for all $x_0 \in \mathbb{R}$, so A^* is the maximal attractor.

- The dynamics is very simple outside of the attractor, but chaotic inside it.

The existence and approximate location of a global attractor follows from that of more easily found absorbing sets, which typically have a convenient simpler shape such as a ball or ellipsoid.

Definition 3. *A nonempty compact subset B of X is called an absorbing set of a semidynamical system π on X if for every bounded subset D of X there exists a $N_D \in \mathbb{Z}^+$ such that*

$$\pi(n, D) \subset B \quad \textit{for all } n \geq N_D.$$

Theorem 1. *Suppose that a semidynamical system π on X has an absorbing set B. Then π has a unique global attractor $A^* \subset B$ given by*

$$A^* = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} \pi(n, B)}, \tag{4}$$

Lyapunov functions for autonomous attractors

A global attractor is, in fact, uniformly Lyapunov asymptotically stable.

The asymptotic stability of attractors and that of attracting sets in general can be characterised by a Lyapunov function.

Consider an autonomous semidynamical system π on a compact metric space (X, d), which is generated by an autonomous difference equation where $f : X \to X$ is globally Lipschitz continuous with Lipschitz constant $L > 0$.

Definition 4. *A nonempty compact subset $A \subsetneq X$ is called globally uniformly asymptotically stable if it is both*

i) Lyapunov stable, i.e., for all $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ with

$$\mathrm{dist}(x, A) < \delta \Rightarrow \mathrm{dist}(f^n(x), A) < \epsilon \quad \textit{for all } n \in \mathbb{Z}^+\,, \tag{5}$$

ii) globally uniformly attracting, i.e., for all $\epsilon > 0$, there exists an integer $N = N(\epsilon) > 1$ such that

$$\mathrm{dist}(f^n(x), A) < \epsilon \quad \textit{for all } x \in X,\ n \geq N\,. \tag{6}$$

Theorem 2. *(Diamond & Kloeden) Let $f : X \to X$ be globally Lipschitz continuous and let A be a nonempty compact subset of X. Then A is globally uniformly asymptotically stable for the dynamical system generated by f if and only if there exist*

i) *a Lyapunov function $V : X \to \mathbb{R}^+$,*

ii) *monotone increasing continuous functions $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = \beta(0) = 0$ and $0 < \alpha(r) < \beta(r)$ for all $r > 0$, and*

iii) *constants $K > 0$, $0 \leq q < 1$*

such that for all $x, y \in X$,

1) $|V(x) - V(y)| \leq Kd(x, y)$,

2) $\alpha(\mathrm{dist}(x, A)) \leq V(x) \leq \beta(\mathrm{dist}(x, A))$,

3) $V(f(x)) \leq qV(x)$.

Lecture 1: Nonautonomous Difference Equations

Difference equations on $\mathbb{R}^d$ of the form

$$\boxed{x_{n+1} = f_n\left(x_n\right), \qquad (1)}$$

in which the continuous mappings $f_n : \mathbb{R}^d \to \mathbb{R}^d$ on the right-hand side are allowed to vary with time n, are called nonautonomous difference equations.

Such nonautonomous difference equations arise quite naturally in many different ways. The mappings f_n in (1) may of course vary completely arbitrarily, but often there is some relationship between them or some regularity in the way in which they are given.

• For example, the mappings may all be the same as in the very special autonomous subcase or they may vary periodically within or be chosen irregularly from a finite family $\{g_1, \cdots, g_r\}$, in which case (1) can be rewritten as

$$x_{n+1} = g_{k_n}\left(x_n\right) \tag{2}$$

with the $k_n \in \{1, \ldots, r\}$ and $f_n = g_{k_n}$.

• As another example, a difference equation may involve a parameter $\lambda \in \Lambda$, which varies in time by choice or randomly giving rise to the nonautonomous difference equation

$$x_{n+1} = g(x_n, \lambda_n), \tag{3}$$

so $f_n(x) = g(x, \lambda_n)$ here for the prescribed choice of $\lambda_n \in \Lambda$.

Let

$$\mathbb{Z}^2_{\geq} := \{(n, n_0) \in \mathbb{Z}^2 \ : \ n \geq n_0\},$$

The nonautonomous difference equation (1) generates a solution mapping

$$\phi : \mathbb{Z}^2_{\geq} \times \mathbb{R}^d \to \mathbb{R}^d$$

through iteration, i.e.,

$$\phi(n_0, n_0, x_0) := x_0, \qquad \phi(n, n_0, x_0) := f_{n-1} \circ \cdots \circ f_{n_0}(x_0)$$

for all $n > n_0$ with $n_0 \in \mathbb{Z}$, and each $x_0 \in \mathbb{R}^d$.

This solution mapping satisfies the 2-parameter semigroup property

$$\phi(m, n_0, x_0) = \phi(m, n, \phi(n, n_0, x_0))$$

for all $(n, n_0) \in \mathbb{Z}^2_{\geq}$, $(m, n) \in \mathbb{Z}^2_{\geq}$ and $x_0 \in \mathbb{R}^d$.

In this sense, ϕ is called general solution of (1). Moreover, as the composition of continuous functions, the mapping $x_0 \mapsto \phi(n, n_0, x_0)$ is continuous for all $(n, n_0) \in \mathbb{Z}^2_{\geq}$.

Processes

Solution mappings of nonautonomous difference equations (1) are one of the main motivations for the process formulation of an abstract nonautonomous dynamical system on a metric state space (X, d) and time set $\mathbb{Z}$.

Definition 1. *(Dafermos, Hale) A (discrete-time) process on a state space X is a mapping $\phi : \mathbb{Z}^2_{\geq} \times X \to X$, which satisfies the initial value, 2-parameter evolution and continuity properties:*

i) $\phi(n_0, n_0, x_0) = x_0$ *for all* $n_0 \in \mathbb{Z}$ *and* $x_0 \in X$,

ii) $\phi(n_2, n_0, x_0) = \phi(n_2, n_1, \phi(n_1, n_0, x_0))$ *for all* $n_0 \leq n_1 \leq n_2$ *in* $\mathbb{Z}$ *and* $x_0 \in X$,

iii) the mapping $x_0 \mapsto \phi(n, n_0, x_0)$ *of* X *into itself is continuous for all* $n_0 \leq n$ *in* $\mathbb{Z}$.

The general nonautonomous case differs crucially from the autonomous in that the starting time n_0 is just as important as the time that has elapsed since starting, i.e., $n - n_0$.

Hence many of the concepts that have been developed and extensively investigated for autonomous dynamical systems in general and autonomous difference equations in particular are either too restrictive or no longer valid or meaningful.

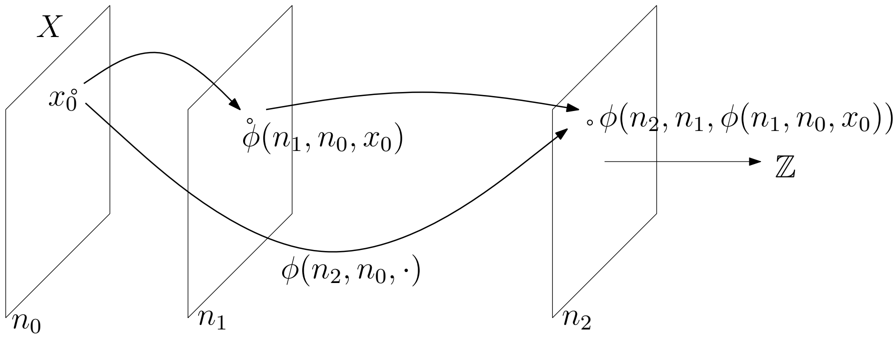

Figure 1: Property ii) of a discrete-time process $\phi : \mathbb{Z}^2_{\geq} \times X \to X$

Given a process ϕ on X there is an associated nonautonomous difference equation like (1) on X with mappings defined by

$$f_n(x) := \phi(n+1, n, x)$$

for all $x \in X$ and $n \in \mathbb{Z}$.

A process is often called a 2-parameter semigroup on X in contrast with the 1-parameter semigroup of an autonomous semidynamical system since it depends on both the initial time n_0 and the actual time n rather than just the elapsed time $n - n_0$.

> This abstract formalism of a nonautonomous dynamical system is a natural and intuitive generalisation of autonomous systems to nonautonomous systems.

Skew-product systems

The skew-product formalism of a nonautonomous dynamical system is somewhat less intuitive than the process formalism.

It represents the nonautonomous system as an autonomous system on the cartesian product of the original state space and some other space such as a function or sequence space on which an autonomous dynamical systems called the driving system acts.

- This driving system is the source of nonautonomity in the dynamics on the original state space.

Let (P, d_P) be a metric space with metric d_P and let $\theta = \{\theta_n : n \in \mathbb{Z}\}$ be a group of continuous mappings from P onto itself.

Thus θ is an autonomous dynamical system on P modeling the driving mechanism for the change in the mappings f_n on the right-hand side of a nonautonomous difference equation like (1), which will now be written as

$$x_{n+1} = f\left(\theta_n(p), x_n\right) \tag{4}$$

for $n \in \mathbb{Z}^+$, where $f : P \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous.

The corresponding solution mapping $\varphi : \mathbb{Z}^+ \times P \times \mathbb{R}^d \to \mathbb{R}^d$ is now defined by

$$\varphi(0, p, x) := x, \qquad \varphi(n, p, x) := f\left(\theta_{n-1}(p), \cdot\right) \circ \cdots \circ f\left(p, x\right)$$

for all $n \in \mathbb{N}$ and $p \in P$, $x \in \mathbb{R}^d$.

The mapping φ satisfies the cocycle property w.r.t. the driving system θ on P, i.e.,

$$\varphi(0,p,x) := x, \qquad \varphi(m+n,p,x) := \varphi\left(m,\theta_n(p),\varphi\left(n,p,x\right)\right) \tag{5}$$

for all m, $n \in \mathbb{Z}^+$, $p \in P$ and $x \in \mathbb{R}^d$.

Consider now a state space X instead of $\mathbb{R}^d$, where (X,d) is a metric space with metric d.
The above considerations lead to the following definition of a skew-product system as an alternative abstract formulation of a discrete nonautonomous dynamical system on the state space X.

Definition 2. *A (discrete-time) skew-product system (θ,ϕ) is defined in terms of a cocycle mapping φ on a state space X, driven by an autonomous dynamical system θ acting on a base space P.*

Specifically, the driving system θ on P is a group of homeomorphisms $\{\theta_n : n \in \mathbb{Z}\}$ under composition on P with the properties

i) $\theta_0(p) = p$ for all $p \in P$,

ii) $\theta_{m+n}(p) = \theta_m(\theta_n(p))$ for all m, $n \in \mathbb{Z}$ and $p \in P$,

iii) the mapping $p \mapsto \theta_n(p)$ is continuous for each $n \in \mathbb{Z}$,

and the cocycle mapping $\phi : \mathbb{Z}^+ \times P \times X \to X$ satisfies

I) $\varphi(0,p,x) = x$ for all $p \in P$ and $x \in X$,

II) $\varphi(m+n,p,x) = \varphi(m,\theta_n(p),\varphi(n,p,x))$ for all $m,n \in \mathbb{Z}^+$, $p \in P$, $x \in X$,

III) the mapping $(p,x) \mapsto \phi(n,p,x)$ is continuous for each $n \in \mathbb{Z}$.

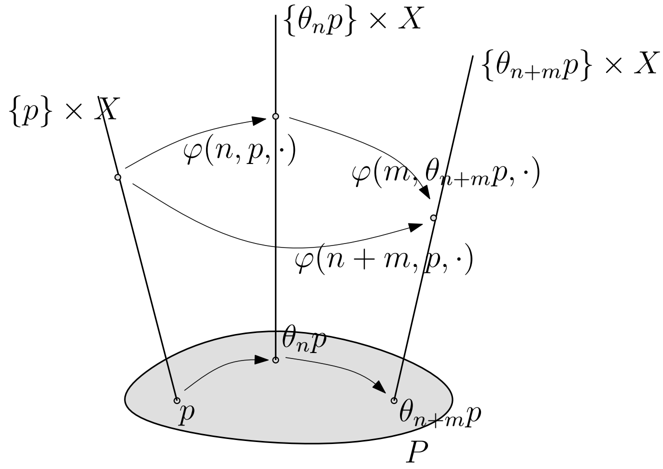

Figure 2: A discrete-time skew-product system (θ, φ) over the base space P

A difference equation of the form (4) can be obtained from a skew-product system by defining

$$f(p, x) := \varphi(1, p, x) \quad \text{for all } p \in P,\ x \in X.$$

A process ϕ admits a formulation as a skew-product system with $P = \mathbb{Z}$, the time shift $\theta_n(n_0) := n + n_0$ and the cocycle mapping

$$\phi(n, n_0, x) := \varphi(n + n_0, n_0, x) \quad \text{for all } n \in \mathbb{Z}^+,\ x \in X.$$

The real advantage of the somewhat more complicated skew-product system formulation of nonautonomous dynamical systems occurs when P is compact.

This never happens for a process reformulated as a skew-product system as above since the parameter space P then is $\mathbb{Z}$, which is only locally compact and not compact.

Examples

The examples above can be reformulated as skew-product systems with appropriate choices of parameter space P and the driving system θ.

Example 1. *A nonautonomous difference equation (1) with continuous right-hand sides* $f_n : \mathbb{R}^d \to \mathbb{R}^d$ *generates a cocycle mapping* φ *over the parameter set* $P = \mathbb{Z}$ *w.r.t. the group of left shift mappings* $\theta_j := \theta^j$ *for* $j \in \mathbb{Z}$*, where* $\theta(n) := n + 1$ *for* $n \in \mathbb{Z}$*. Here* φ *is defined by*

$$\varphi(0, n, x) := x \quad \textit{and} \quad \varphi(j, n, x) := f_{n+j-1} \circ \cdots \circ f_n(x) \quad \textit{for all } j \in \mathbb{N}$$

and $n \in \mathbb{Z}$*,* $x \in \mathbb{R}^d$*.*

The mappings $\varphi(j, n, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ *are all continuous.*

Example 2. *Let* $f : \mathbb{R}^d \to \mathbb{R}^d$ *be a continuous mapping used in an* <u>*autonomous*</u> *difference equation. The solution mapping* φ *defined by*

$$\varphi(0, x) := x \quad \textit{and} \quad \varphi(j, x) = f^j(x) := \underbrace{f \circ \cdots \circ f}_{j \text{ times}}(x) \quad \textit{for all } j \in \mathbb{N}$$

and $x \in \mathbb{R}^d$ *generates a semigroup on* $\mathbb{R}^d$*.*

It can be considered as a cocycle mapping w.r.t. a <u>*singleton*</u> *parameter set* $P = \{p_0\}$ *and the singleton group consisting only of identity mapping* $\theta := \mathrm{id}_P$ *on* P*.*

Since the driving system just sits at p_0*, the dependence on the parameter in* φ *can be suppressed.*

While the integers $\mathbb{Z}$ appears to be the natural choice for the parameter set in Example 1 and the choice is trivial in the autonomous case of Example 2, in the remaining examples the use of sequence spaces is more advantageous because such spaces are often compact.

Example 3. *The nonautonomous difference equation (2) with continuous mappings*

$$g_k : \mathbb{R}^d \to \mathbb{R}^d \quad \textit{for } k \in \{1, \cdots, r\}$$

generates a cocycle mapping over the parameter set $P = \{1, \cdots, r\}^{\mathbb{Z}}$ *of bi-infinite sequences* $p = \{k_n\}_{n\in\mathbb{Z}}$ *with* $k_n \in \{1, \cdots, r\}$ *w.r.t. the group of left shift operators* $\theta_n := \theta^n$ *for* $n \in \mathbb{Z}$*, where* $\theta(\{k_n\}_{n\in\mathbb{Z}}) = \{k_{n+1}\}_{n\in\mathbb{Z}}$.

The mapping φ *is defined by*

$$\varphi(0, p, x) := x \quad \textit{and} \quad \varphi(j, p, x) := g_{k_{j-1}} \circ \cdots \circ g_{k_0}(x) \quad \textit{for all } j \in \mathbb{N}$$

and $x \in \mathbb{R}^d$*, where* $p = \{k_n\}_{n\in\mathbb{Z}}$*, is a cocycle mapping.*

The parameter space $\{1, \cdots, r\}^{\mathbb{Z}}$ *here is a compact metric space with the metric*

$$d(p, p') = \sum_{n\in\mathbb{Z}} (r+1)^{-|n|} \, |k_n - k'_n| \,.$$

In addition, $\theta_n : P \to P$ *and* $\varphi(j, \cdot, \cdot) : P \times \mathbb{R}^d \to \mathbb{R}^d$ *are all continuous.*

Example 4. *As an example of a parametrically perturbed difference equation (3), consider the mapping* $g : \left[\frac{1}{2}, 1\right] \times \mathbb{R}^1 \mapsto \mathbb{R}^1$ *defined by*

$$g(\lambda, x) = \frac{|x| + \lambda^2}{1 + \lambda},$$

which is continuous in $x \in \mathbb{R}^1$ *and* $\lambda \in \left[\frac{1}{2}, 1\right]$.

Let $P = \left[\frac{1}{2}, 1\right]^{\mathbb{Z}}$ *be the space of bi-infinite sequences* $p = \{\lambda_n\}_{n \in \mathbb{Z}}$ *taking values in* $\left[\frac{1}{2}, 1\right]$, *which is a compact metric space with the metric*

$$d(p, p') = \sum_{n \in \mathbb{Z}} 2^{-|n|} |\lambda_n - \lambda'_n|,$$

and let $\{\theta_n, n \in \mathbb{Z}\}$ *be the group generated by the left shift operator* θ *on this sequence space (analogously to Example 3).*

The mapping φ *is defined by*

$$\varphi(0, p, x) := x \quad \text{and} \quad \varphi(j, p, x) := g(\lambda_{j-1}, \cdot) \circ \cdots \circ g(\lambda_0, x) \quad \text{for all } j \in \mathbb{N}$$

and $x \in \mathbb{R}^1$, *where* $p = \{\lambda_n\}_{n \in \mathbb{Z}}$, *is a cocycle mapping on* $\mathbb{R}^1$ *with parameter space* $\left[\frac{1}{2}, 1\right]^{\mathbb{Z}}$ *and the above shift operators* θ_n.

The mappings $\theta_n : P \to P$ *and* $\varphi(j, \cdot, \cdot) : P \times \mathbb{R}^d \to \mathbb{R}^d$ *are all continuous here.*

Skew-products as autonomous semidynamical systems

A skew-product system (θ, φ) can be reformulated as autonomous semidynamical system on the extended state space $\mathbb{X} := P \times X$. Define a mapping $\pi : \mathbb{Z}^+ \times \mathbb{X} \to \mathbb{X}$ by

$$\pi\left(n, (p, x_0)\right) := \big(n, (\theta_n(p), \phi(n, p, x_0))\big) \quad \text{for all } n \in \mathbb{Z}^+, (p, x_0) \in \mathbb{X}\,.$$

Note that the variable n in $\pi\left(n, (p, x_0)\right)$ is the time that has elapsed since starting at state (p, x_0).

Theorem 1. *π is an autonomous semidynamical system on $\mathbb{X}$.*

Proof. It is obvious that $\pi(n, \cdot)$ is continuous in its variables (p, x_0) for every $n \in \mathbb{Z}^+$ and satisfies the initial condition

$$\pi(0, (p, x_0)) = (p, \varphi(0, p, x_0)) = (p, x_0) \quad \text{for all } p \in P, x_0 \in X.$$

It also satisfies the 1-parameter semigroup property

$$\pi(m + n, (p, x_0)) = \pi\left(m, \pi(n, (p, x_0))\right) \quad \text{for all } m, n \in \mathbb{Z}^+,\ p \in P,\ x_0 \in X$$

since, by the group property of the driving system and the cocycle property of the skew-product,

$$\begin{aligned}\pi(m+n,(p,x_0)) &= (\theta_{m+n}(p),\varphi(m+n,p,x_0)) \\ &= \big(\theta_m\left(\theta_n(p)\right),\varphi(m,\theta_n(p),\varphi(n,p,x_0))\big) \\ &= \pi\big(m,(\theta_n(p),\varphi(n,p,x_0))\big) = \pi\big(m,\pi(n,(p,x_0))\big)\,.\end{aligned}$$

□

A process ϕ on X is also a skew-product on X with the shift operator θ on $P := \mathbb{Z}$ and thus generates an autonomous semidynamical system π on the extended state space $\mathbb{X} := \mathbb{Z} \times X$.

- This semidynamical system has some unusual properties.

In particular, π has no nonempty ω-limit sets and, indeed, no compact subset of $\mathbb{X}$ can be π-invariant.

Lecture 2: Nonautonomous invariant sets and attractors of processes

Invariant sets and attractors are important regions of state space that characterise the long term behaviour of a dynamical system.

A process $\phi : \mathbb{Z}^2_{\geq} \times X \to X$ on a metric state space (X, d) generates a solution $x_n = \phi(n, n_0, x_0)$ that depends on the starting time n_0 as well as the current time n and not just on the time $n - n_0$ that has elapsed since starting as in an autonomous system.

This has some profound consequences in terms of definitions and the interpretation of dynamical behaviour.

In particular, many concepts and results from the autonomous case are not valid or are too restrictive and exclude many interesting types of possible behaviour.

For example, it is too restrictive to consider a single subset A of X to be invariant under ϕ in the sense that

$$\phi(n, n_0, A) = A \quad \text{for all } n \geq n_0,\ n_0 \in \mathbb{Z},$$

which is equivalent to $f_n(A) = A$ for every $n \in \mathbb{Z}$, where the f_n are mappings in the corresponding nonautonomous difference equation

$$x_{n+1} = f_n(x_n).$$

Then, in general, neither the trajectory $\{\chi_n^* : n \in \mathbb{Z}\}$ of a solution χ^* that exists on all of $\mathbb{Z}$ nor a nonautonomous ω-limit set defined by

$$\omega^+(n_0, x_0) = \{y \in X \ : \ \exists n_j \to \infty, \ \phi(n_j, n_0, x_0) \to y\},$$

will be invariant in such a sense.

Moreover, such nonautonomous ω-limit sets exist in the infinite future in absolute time rather than in current time like autonomous ω-limit sets, so it is not so clear how useful or even meaningful dynamically they are.

Hence, the appropriate formulation of asymptotic behaviour of a nonautonomous dynamical system needs some careful consideration.

Lyapunov asymptotical stability of a solution of a nonautonomous system provides a clue. It needs the concept of an entire solution.

Definition 1. *An entire solution of a process ϕ on X is a sequence $\{\chi_k : k \in \mathbb{Z}\}$ in X such that*

$$\phi(n, n_0, \chi_{n_0}) = \chi_n \quad \textit{for all } n \geq n_0 \textit{ and all } n_0 \in \mathbb{Z},$$

or equivalently, $\chi_{n+1} = f_n(\chi_n)$ for all $n \in \mathbb{Z}$ in terms of the nonautonomous difference equation corresponding to the process ϕ.

Definition 2. *An entire solution χ^* of a process ϕ on X is said to be (globally) Lyapunov asymptotically stable if it is Lyapunov stable, i.e., for every $\epsilon > 0$ and $n_0 \in \mathbb{Z}$ there exists a $\delta = \delta(\epsilon, n_0) > 0$ such that*

$$d\left(\phi(n, n_0, x_0), \chi_n^*\right) < \epsilon \quad \text{for all } n \geq n_0 \text{ whenever } d\left(x_0, \chi_{n_0}^*\right) < \delta,$$

and attracting in the sense that

$$d\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \to 0 \quad \text{as } n \to \infty \tag{1}$$

for all $x_0 \in X$ and $n_0 \in \mathbb{Z}$.

Note: the limiting "target" χ_n^* exists for all time and is, in general, also changing in time as the limit is taken.

Nonautonomous invariant sets

Let χ^* be an entire solution of a process ϕ on a metric space X and consider the family $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ of singleton subsets $A_n := \{\chi_n^*\}$ of X.

By the definition of an entire solution it follows that

$$\phi\left(n, n_0, A_{n_0}\right) = A_n \quad \text{for all } n \geq n_0,\ n_0 \in \mathbb{Z}\,.$$

This suggests the following generalization of invariance for nonautonomous dynamical systems.

Definition 3. *A family $\mathcal{A} = \{A_n :\ n \in \mathbb{Z}\}$ of nonempty subsets of X is ϕ-invariant if*

$$\phi\left(n, n_0, A_{n_0}\right) = A_n \qquad \textit{for all } n \geq n_0 \textit{ and all } \ n_0 \in \mathbb{Z},$$

or, equivalently, if $f_n(A_n) = A_{n+1}$ for all $n \in \mathbb{Z}$ in terms of the corresponding nonautonomous difference equation.

A ϕ-invariant family consists of entire solutions.

Proposition 1. *A family $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ is ϕ-invariant if and only if for every pair $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$ there exists an entire solution χ such that $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for all $n \in \mathbb{Z}$.*

Moreover, the entire solution χ is uniquely determined provided the mapping $f_n(\cdot) := \phi(n+1, n, \cdot) : X \to X$ is one-to-one for every $n \in \mathbb{Z}$.

Proof

Sufficiency Let $\mathcal{A}$ be ϕ-invariant and pick an arbitrary $x_0 \in A_{n_0}$. For $n \geq n_0$ define the sequence $\chi_n := \phi(n, n_0, x_0)$. Then the ϕ-invariance of $\mathcal{A}$ yields $\chi_n \in A_n$.

On the other hand, $A_{n_0} = \phi(n_0, n, A_n)$ for $n \leq n_0$, so there exists a sequence $x_n \in A_n$ with $x_0 = \phi(n_0, n, x_n)$ and $x_n = \phi(n, n-1, x_{n-1})$ for all $n < n_0$.

Hence define $\chi_n := x_n$ for $n < n_0$ and χ becomes an entire solution with the desired properties.

If the mappings f_n are all one-to-one, then the sequence $\{x_n\}$ is uniquely determined.

Necessity Suppose for an arbitrary $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$ that there is an entire solution χ with $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for all $n \in \mathbb{Z}$.

Hence $\phi(n, n_0, x_0) = \phi(n, n_0, \chi_{n_0}) = \chi_n \in A_n$ for $n \geq n_0$.

From this it follows that $f_n(A_n) \subseteq A_{n+1}$.

The remaining inclusion $f_n(A_n) \supseteq A_{n+1}$ follows from the fact that $x_0 = \phi(n_0, n, \chi_n)$ $\in \phi(n_0, n, A_n)$ for $n \leq n_0$. $\square$

Forward and pullback convergence

The convergence

$$d\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \left(n_0 \text{ fixed}\right)$$

in the attraction property (1) in the definition of a Lyapunov asymptotically stable entire solution χ^* of a process ϕ will be called forward convergence (see Figure 1).

Forward convergence does not, however, provide convergence to a particular point $\chi_{n^*}^*$ for a fixed $n^* \in \mathbb{Z}$.

To obtain such convergence one has to start progressively earlier as given by pullback convergence (see Figure 2)

$$d\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \rightarrow 0 \quad \text{as } n_0 \rightarrow -\infty \quad (n \text{ fixed})$$

In terms of the elapsed time j, forward convergence can be rewritten as

$$d\left(\phi\left(n_0+j, n_0, x_0\right), \chi_{n_0+j}^*\right) \rightarrow 0 \quad \text{as } j \rightarrow \infty \tag{2}$$

for all $x_0 \in X$ and $n_0 \in \mathbb{Z}$, while pullback convergence becomes

$$d\left(\phi\left(n, n-j, x_0\right), \chi_n^*\right) \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

for all $x_0 \in X$ and $n \in \mathbb{Z}$.

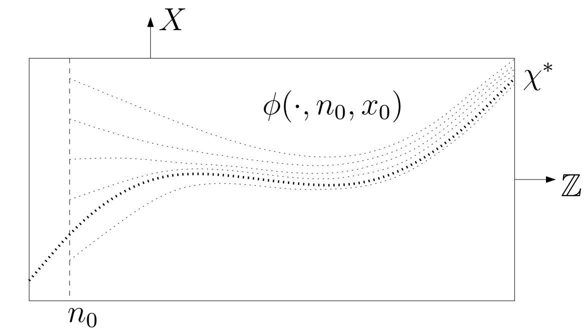

Figure 1: Forward convergence $n \to \infty$

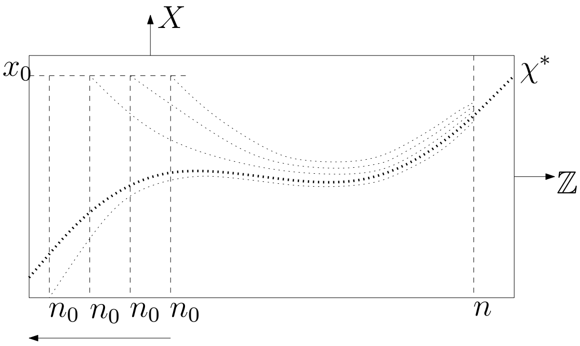

Figure 2: Pullback convergence $n_0 \to -\infty$

Pullback convergence makes use of information about the past of the nonautonomous dynamical system, while forward convergence uses information about the future.

Example 1. *The nonautonomous difference equation* $x_{n+1} = \frac{1}{2}x_n + g_n$ *on* $\mathbb{R}$ *has the solution mapping*

$$\phi(j + n_0, n_0, x_0) = 2^{-j}x_0 + \sum_{k=0}^{j} 2^{-j+k} g_{n_0+n},$$

for which <u>*pullback convergence*</u> *gives*

$$\phi(n_0, n_0 - j, x_0) = 2^{-j}x_0 + \sum_{k=0}^{j} 2^{-k} g_{n_0-k} \to \sum_{k=0}^{\infty} 2^{-k} g_{n_0-k} \quad \text{as } j \to \infty,$$

provided the infinite series here converges.

The limiting solution χ^* *is given by* $\chi^*_{n_0} := \sum_{k=0}^{\infty} 2^{-k} g_{n_0-k}$ *for each* $n_0 \in \mathbb{Z}$.

It is an <u>*entire solution*</u> *of the nonautonomous difference equation.*

In <u>autonomous</u> dynamical systems, forward and pullback convergence are <u>equivalent</u> since the elapsed time $n - n_0 \to \infty$ if either $n \to \infty$ with n_0 fixed or $n_0 \to -\infty$ with n fixed

In nonautonomous dynamical systems pullback convergence and forward convergence do not necessarily imply each other.

Example 2. *Consider the process ϕ on $\mathbb{R}$ generated $f_n = g_1$ for $n \leq 0$ and $f_n = g_2$ for $n \geq 1$, where the mappings g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ are given by*

$$g_1(x) := \frac{1}{2}x, \qquad g_2(x) := \max\{0, 4x(1-x)\}, \qquad \forall x \in \mathbb{R}.$$

Then ϕ is pullback convergent to the entire solution χ^* defined by $\chi^*_n \equiv 0$ for $n \in \mathbb{Z}$.

But χ^*_n is not forward convergent to χ^* or Lyapunov stable.

Forward and pullback attractors

Forward and pullback convergences can be used to define two distinct types of nonautonomous attractors for a process ϕ on a state space X.

Instead of a family $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ of singleton subsets $A_n := \{\chi_n^*\}$ for an entire solution χ^* of the process consider a ϕ-invariant family of $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty subsets A_n of X.

In this context forward convergence generalises to

$$\operatorname{dist}(\phi(n_0 + j, n_0, x_0), A_j) \to 0 \quad \text{as } j \to \infty \quad (n_0 \text{ fixed}) \tag{3}$$

and pullback convergence to

$$\operatorname{dist}(\phi(n, n - j, x_0), A_n) \to 0 \quad \text{as } j \to \infty \quad (n \text{ fixed}). \tag{4}$$

More generally, $\mathcal{A}$ is said to forward attracts (resp., pullback attracts) bounded subsets of X if x_0 is replaced by an arbitrary bounded subset D of X in (3) (resp. (4)).

Definition 4. *A ϕ–invariant family $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called a forward attractor if it forward attracts bounded subsets of X and a pullback attractor if it pullback attracts bounded subsets of X.*

- As a ϕ-invariant family $\mathcal{A}$ of nonempty compact subsets of X both pullback and forward attractors consist of entire solutions.

When the components subsets of a pullback attractor are uniformly bounded, i.e., if there exists a bounded subset B of X such that $A_n \subset B$ for all $n \in \mathbb{Z}$, then pullback attractors are characterised by the bounded entire solutions of the process.

Proposition 2. *A uniformly bounded pullback attractor $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ admits the dynamical characterisation: for each $n_0 \in \mathbb{Z}$*

$$x_0 \in A_{n_0} \Leftrightarrow \textit{ there exists a bounded entire solution } \chi \textit{ with } \chi_{n_0} = x_0.$$

Such a pullback attractor is therefore uniquely determined.

Proof

Sufficiency Pick $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$ arbitrarily. Then, due to the ϕ-invariance of the pullback attractor $\mathcal{A}$, by Proposition 1 there exists an entire solution χ with $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for each $n \in \mathbb{Z}$.

Moreover, χ is bounded since the component sets of the pullback attractor are uniformly bounded.

Necessity If there exists a bounded entire solution χ of the process ϕ, then the set of points $D_\chi := \{\chi_n : n \in \mathbb{Z}\}$ is bounded in X.

Since $\mathcal{A}$ pullback attracts bounded subsets of X, for each $n \in \mathbb{Z}$,

$$0 \leq \operatorname{dist}(\chi_n, \mathcal{A}_n) \leq \lim_{j \to \infty} \operatorname{dist}(\phi(n, n-j, D_\chi), A_n) = 0,$$

so $\chi_n \in A_n$. $\square$

Existence of pullback attractors

Absorbing sets can also be defined for pullback attraction. Wider applicability can be attained if they are also allowed to depend on time.

Definition 5. *A family* $\mathcal{B} = \{B_n : n \in \mathbb{Z}\}$ *of nonempty compact subsets of* X *is called a pullback absorbing family for a process* ϕ *on* X *if for each* $n \in \mathbb{Z}$ *and every bounded subset* D *of* X *there exists an* $N_{n,D} \in \mathbb{Z}^+$ *such that*

$$\phi(n, n-j, D) \subseteq B_n \quad \textit{for all } j \geq N_{n,D},\ n \in \mathbb{Z}.$$

The existence of a pullback attractor follows from that of a pullback absorbing family in the following generalisation of the theorem for autonomous global attractors.

The proof is simpler if the pullback absorbing family is assumed to be ϕ-positive invariant.

Definition 6. *A family* $\mathcal{B} = \{B_n : n \in \mathbb{Z}\}$ *of nonempty compact subsets of* X *is said to be* ϕ*-positive invariant if*

$$\phi(n, n_0, B_{n_0}) \subseteq B_n \quad \textit{for all } n \geq n_0.$$

Theorem 1. *Suppose that a process ϕ on a complete metric space (X, d) has a ϕ-positive invariant pullback absorbing family $\mathcal{B} = \{B_n : \ n \in \mathbb{Z}\}$. Then there exists a global pullback attractor $\mathcal{A} = \{A_n : \ n \in \mathbb{Z}\}$ with component sets determined by*

$$A_n = \bigcap_{j \geq 0} \phi\left(n, n-j, B_{n-j}\right) \quad \textit{for all } n \in \mathbb{Z}. \tag{5}$$

Moreover, if $\mathcal{A}$ is uniformly bounded, then it is unique.

Proof It is clear from the definition (5) of A_n that $A_n \subset B_n$ for each $n \in \mathbb{Z}$.

(i) First, it will be shown for any $n \in \mathbb{Z}$ that

$$\lim_{j \to \infty} \operatorname{dist}\left(\phi(n, n-j, B_{n-j}), A_n\right) = 0\,. \tag{6}$$

Assume to the contrary that there exist sequences $x_{j_k} \in \phi\left(n, n-j_k, B_{n-j_k}\right) \subset B_n$ and $j_k \to \infty$ such that $\operatorname{dist}(x_{j_k}, A_n) > \epsilon$ for all $k \in \mathbb{N}$.

The set $\{x_{j_k} : k \in \mathbb{N}\} \subset B_n$ is <u>relatively compact</u>, so there is a point $x_0 \in B_n$ and an index subsequence $k' \to \infty$ such that $x_{j_{k'}} \to x_0$. Now

$$x_{j_{k'}} \in \phi\left(n, n-j_{k'}, B_{n-j_{k'}}\right) \subset \phi\left(n, n-k, B_{n-k}\right)$$

for all $k_{j'} \geq k$ and each $k \geq 0$. This implies that

$$x_0 \in \phi\left(n, n-k, B_{n-k}\right) \quad \text{for all } k \geq 0\,.$$

Hence, $x_0 \in A_n$, which is a <u>contradiction</u>. This proves the assertion (6).

(ii) By (6), for every $\epsilon > 0$, $n \in \mathbb{Z}$, there exists a $N = N_{\epsilon,n} \geq 0$ such that

$$\mathrm{dist}\left(\phi(n, n-N, B_{n-N}), A_n\right) < \epsilon \,.$$

Let D be a bounded subset of X. The fact that $\mathcal{B}$ is a pullback absorbing family implies that $\phi\,(n, n-j, D) \subset B_n$ for all sufficiently large j.

Hence, by the cocycle property,

$$\phi\,(n, n-N-j, D) = \phi\,(n, n-N, \phi(n-N, n-N-j, D)) \subset \phi\,(n, n-N, B_{n-N})\ .$$

(iii) The ϕ-invariance of the family $\mathcal{A}$ will now be shown.

By (5), the set $F_m(n) := \phi\,(n, n-m, B_{n-m})$ is contained in B_n for every $m \geq 0$, and by definition, $A_{n-j} = \bigcap_{m\geq 0} F_m\,(n-j)$.

First, it will be shown that

$$\phi\left(n, n-j, \bigcap_{m\geq 0} F_m(n-j)\right) = \bigcap_{m\geq 0} \phi\,(n, n-j, F_m(n-j))\ . \tag{7}$$

One sees directly that "$\subset$" holds. To prove "$\supset$", let x be contained in the set on the right side.

Then for any $n \geq 0$, there exists an $x^m \in F_m(n-j) \subset B_{n-j}$ such that $x = \phi(n, n-j, x^m)$.

Since the sets $F_m(n-j)$ are compact and monotonically decreasing with increasing m, the set $\{x^m : m \geq 0\}$ has a limit point $\hat{x} \in \bigcap_{m \geq 0} F_m(n-j)$.

By the continuity of $\phi(n, n-j, \cdot)$, it follows that $x = \phi(n, n-j, \hat{x})$. Thus,

$$x \in \phi\left(n, n-j, \bigcap_{m \geq 0} F_m(n-j)\right) = \phi(n, n-j, A_{n-j}) .$$

Hence, equation (7), the compactness of $F_m(n-j)$ and the continuity of $\phi(n, n-j, \cdot)$ imply that

$$\begin{aligned}
\phi(n, n-j, A_{n-j}) &= \bigcap_{m \geq 0} \phi(n, n-j, F_m(n-j)) \\
&= \bigcap_{m \geq 0} \phi(n, n-j, \phi(n-j, n-j-m, B_{n-j-m})) \\
&= \bigcap_{m \geq 0} \phi(n, n-j-m, B_{n-j-m}) = \bigcap_{m \geq j} \phi(n, n-m, B_{n-m}) \supset A_n ,
\end{aligned}$$

which means that

$$A_n \subset \phi(n, n-j, A_{n-j}) , \quad j \in \mathbb{Z}^+ \quad \text{for all } n \in \mathbb{Z} . \tag{8}$$

Replacing n by $n-m$ in (8) and using the cocycle property gives

$$\begin{aligned} \phi\left(n, n-m, A_{n-m}\right) &\subset \phi\left(n, n-m, \phi\left(n-m, n-m-j, A_{n-m-j}\right)\right) \\ &= \phi\left(n, n-j, \phi\left(n-j, n-m-j, A_{n-m-j}\right)\right) \\ &\subset \phi\left(n, n-j, \phi(n-j, n-m-j, B_{n-m-j})\right) \\ &\subset \phi\left(n, n-j, B_{n-j}\right) \subset U_\epsilon(A_n) \end{aligned}$$

for all ϵ-neighborhoods $U_\epsilon(A_n)$ of A_n, where $\epsilon > 0$, provided that $j = J(\epsilon)$ is sufficiently large.

Hence, $\phi\left(n, n-m, A_{n-m}\right) \subset A_n$ for all $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}$.

With m replaced by j, this yields with (8) the ϕ-invariance of the family $\{A_n : n \in \mathbb{Z}\}$.

(iv) It remains to observe that if the sets in $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ are uniformly bounded, then the pullback attractor $\mathcal{A}$ is unique by Proposition 2. □

Remark 1. *There is no counterpart of Theorem 1 for nonautonomous forward attractors*

If the pullback absorbing family $\mathcal{B}$ is not ϕ-positive invariant, then the proof is somewhat more complicated and the component subsets of the pullback attractor of $\mathcal{A}$ are given by

$$A_n = \bigcap_{k\geq 0} \overline{\bigcup_{j\geq k} \phi\left(n, n-j, B_{n-j}\right)}.$$

However, the assumption in Theorem 1 that ϕ-positively invariant pullback absorbing systems is not a serious restriction.

Proposition 3. *Suppose $\mathcal{B} = \{B_n : n \in \mathbb{Z}\}$ is a pullback absorbing system for a process ϕ fulfilling $B_n \subset C$ for $n \in \mathbb{Z}$, where C is a bounded subset of X.*

Then there exists a ϕ-positively invariant pullback absorbing system $\widehat{\mathcal{B}} = \{\widehat{B}_n : n \in \mathbb{Z}\}$ containing $\mathcal{B} = \{B_n : n \in \mathbb{Z}\}$ component setwise.

Proof For each $n \in \mathbb{Z}$ define

$$\boxed{\widehat{B}_n := \overline{\bigcup_{j\geq 0} \phi(n, n-j, B_{n-j})}.}$$

Obviously $B_n \subset \widehat{B}_n$ for every $n \in \mathbb{Z}$.

To show positive invariance, the cocycle property is used in what follows.

$$\begin{aligned}
\phi(n+1, n, \widehat{B}_n) &= \overline{\bigcup_{j\geq 0} \phi(n+1, n, \phi(n, n-j, B_{n-j}))} \\
&= \overline{\bigcup_{j\geq 0} \phi(n+1, n-j, B_{n-j})} \\
&= \overline{\bigcup_{i\geq 1} \phi(n+1, n+1-i, B_{n+1-i})} \\
&\subseteq \overline{\bigcup_{i\geq 0} \phi(n+1, n+1-i, B_{n+1-i})} \;=\; \widehat{B}_{n+1},
\end{aligned}$$

so $\phi(n+1, n, \widehat{B}_n) \subseteq \widehat{B}_{n+1}$. By this and the cocycle property again

$$\phi(n+2, n, \widehat{B}_n) = \phi\left(n+2, n+1, \phi(n+1, n, \widehat{B}_n)\right) \subseteq \phi(n+2, n+1, \widehat{B}_{n+1}) \subseteq \widehat{B}_{n+2}.$$

The general positive invariance assertion then follows by induction.

The continuity of $\phi(n, n-j, \cdot)$ and the compactness of B_{n-j} imply that the set $\phi(n, n-j, B_{n-j})$ is compact for each $j \geq 0$ and $n \in \mathbb{Z}$.

Moreover, $B_{n-j} \subset C$ for each $j \geq 0$ and $n \in \mathbb{Z}$, so by the pullback absorbing property of $\mathcal{B}$ there exists an $N = N_{n,C} \in \mathbb{N}$ such that

$$\phi(n, n-j, B_{n-j}) \subset \phi(n, n-j, C) \subset B_n$$

for all $j \geq N$. Hence

$$\begin{aligned} \widehat{B}_n &= \overline{\bigcup_{j \geq 0} \phi(n, n-j, B_{n-j})} \\ &\subseteq B_n \bigcup \overline{\bigcup_{0 \leq j < N} \phi(n, n-j, B_{n-j})} = \overline{\bigcup_{0 \leq j < N} \phi(n, n-j, B_{n-j})}, \end{aligned}$$

which is compact as a finite union of compact sets, so $\widehat{B}_n$ is compact.

To see that $\mathcal{B}$ so constructed is pullback absorbing, let B be a bounded subset of X and fix $n \in \mathbb{Z}$. Since $\mathcal{B}$ is pullback absorbing, there exists an $N_{n,D} \in \mathbb{N}$ such that $\phi(n, n-j, D) \subset B_n$ for all $j \geq N_{n,D}$. But $B_n \subset \widehat{B}_n$, so

$$\phi(n, n-j, D) \subset \widehat{B}_n \quad \text{for all } j \geq N_{n,D}.$$

Hence $\widehat{B}$ is pullback absorbing as required. $\square$

Limitations of pullback attractors

Pullback attractors are based on the behaviour of a nonautonomous system in the past and may not capture the complete dynamics of a system when it is formulated in terms of a process. This was already indicated by Example 2 and will be illustrated here through some simpler examples.

First consider the autonomous scalar difference equation

$$x_{n+1} = \frac{\lambda x_n}{1 + |x_n|} \tag{9}$$

depending on a real parameter $\lambda > 0$. Its zero solution $x^* = 0$ exhibits a pitchfork bifurcation at $\lambda = 1$.

Its global dynamical behavior can be summarised as follows (see Figure 3):

• If $\lambda \leq 1$, then $x^* = 0$ is the only constant solution and is globally asymptotically stable.

Thus $\{0\}$ is the global attractor of the autonomous dynamical system generated by the difference equation (9).

• If $\lambda > 1$, then there exist two additional nontrivial constant solutions given by $x_\pm := \pm(\lambda - 1)$.

The zero solution $x^* = 0$ is an unstable steady state solution and the symmetric interval $A = [x_-, x_+]$ is the global attractor.

These constant solutions are the fixed points of the mapping $f(x) = \dfrac{\lambda x}{1 + |x|}$.

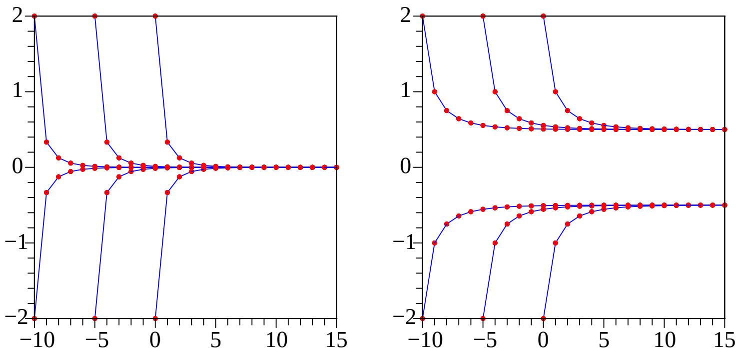

Figure 3: Trajectories of the autonomous difference equation (9) with $\lambda = 0.5$ (left) and $\lambda = 1.5$ (right)

Piecewise autonomous difference equation: Consider now the piecewise autonomous equation

$$x_{n+1} = \frac{\lambda_n x_n}{1 + |x_n|}, \qquad \lambda_n := \begin{cases} \lambda, & n \geq 0, \\ \lambda^{-1}, & n < 0 \end{cases} \tag{10}$$

for some $\lambda > 1$, which corresponds to a switch between the two autonomous problems (9) at $n = 0$.

The zero solution of the resulting nonautonomous system is the only bounded entire solution, so the pullback attractor $\mathcal{A}$ has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$.

Note that the zero solution seems to be "asymptotically stable" for $n < 0$ and then "unstable" for $n \geq 0$.

Moreover the interval $[x_-, x_+] = [-(\lambda - 1), (\lambda - 1)]$ looks like a global attractor for the whole equation on $\mathbb{Z}$, but it is not really one since it is not invariant or minimal for $n < 0$.

The nonautonomous difference equation (10) is asymptotically autonomous in both directions, but the pullback attractor does not reflect the full limiting dynamics, in particular in the forward time direction (see Figure 4 (left)).

Fully nonautonomous equation: If the parameters λ_n do not switch from one constant to another as above, but increase monotonically, e.g., such as

$$\lambda_n = 1 + \frac{0.9n}{1 + |n|},$$

then the dynamics is similar, although the limiting dynamics is not so obvious from the equation. See Fig. 4 (left).

Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a monotonically increasing sequence with $\lim_{k\to\pm\infty} \lambda_n = \bar{\lambda}^{\pm 1}$ for $\bar{\lambda} > 1$. The nonautonomous problem

$$x_{n+1} = f_n(x_n) := \frac{\lambda_n x_n}{1 + |x_n|}. \tag{11}$$

is asymptotically autonomous in both directions with the limiting autonomous systems given above.

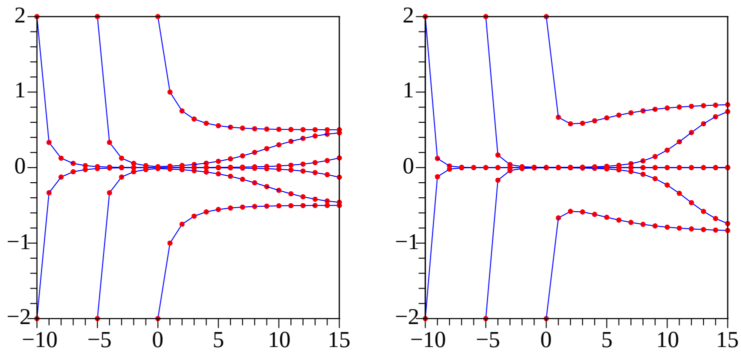

Figure 4: Trajectories of the piecewise autonomous equation (10) with $\lambda = 1.5$ (left) and the asymptotically autonomous equation (11) with $\lambda_k = 1 + \frac{0.9k}{1+|k|}$ (right)

Its pullback attractor $\mathcal{A}$ has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$ corresponding to the zero entire solution, which is the only bounded entire solution.

As above, the zero solution $x^* = 0$ seems to be "asymptotically stable" for $n < 0$ and then "unstable" for $n \geq 0$.

However, the forward limit points for nonzero solutions are $\pm(\bar{\lambda} - 1)$, neither of which is a solution at all. In particular, they are not entire solutions. See Figure 4 (right).

Remark 2. *Pullback attraction alone does not characterise fully the bounded limiting behaviour of a nonautonomous system formulated as a process.*

Something additional like nonautonomous limit sets, limiting equations or asymptotically invariant sets and eventual asymptotic stability or a mixture of these ideas is needed to complete the picture. However, this varies from example to example and is somewhat ad hoc.

Lecture 3: Nonautonomous invariant sets and attractors of skew-products

There are counterparts for skew-product systems of the concepts of invariance, forward and pullback convergence and forward and pullback attractors considered in the previous chapter for discrete-time processes.

Consider a discrete-time skew-product system (θ, φ) on $P \times X$, where (P, d_P) and (X, d) are metric spaces.

Definition 1. *A family $\mathcal{A} = \{A_p : p \in P\}$ of nonempty subsets of X is called φ-invariant for a skew-product system (θ, φ) on $P \times X$ if*

$$\varphi(n, p, A_p) = A_{\theta_n(p)} \quad \textit{for all } n \in \mathbb{Z}^+,\ p \in P.$$

It is called φ-positively invariant if

$$\varphi(n, p, A_p) \subseteq A_{\theta_n(p)} \quad \textit{for all } n \in \mathbb{Z}^+,\ p \in P.$$

Definition 2. *A family $\mathcal{A} = \{A_p : p \in P\}$ of nonempty compact subsets of X is called pullback attractor of a skew-product system (θ, φ) on $P \times X$ if it is φ-invariant and pullback attracts bounded sets, i.e.,*

$$\operatorname{dist}\left(\varphi(j, \theta_{-j}(p), D), A_p\right) = 0 \quad \textit{for } j \to \infty \tag{1}$$

for all $p \in P$ and all bounded subsets D of X.
It is called a forward attractor if it is φ-invariant and forward attracts bounded sets, i.e.,

$$\operatorname{dist}\left(\varphi(j, p, D), A_{\theta_j(p)}\right) = 0 \quad \textit{for } j \to \infty. \tag{2}$$

As with processes, the existence of a pullback for skew-product systems ensured by that of a pullback absorbing system.

Definition 3. *A family $\mathcal{B} = \{B_p : p \in P\}$ of nonempty compact subsets of X is called a pullback absorbing family for a skew-product system (θ, φ) on on $P \times X$ if for each $p \in P$ and every bounded subset D of X there exists an $N_{p,D} \in \mathbb{Z}^+$ such that*

$$\varphi\left(j, \theta_{-j}(p), D\right) \subseteq B_p \quad \text{for all } j \geq N_{p,D},\ p \in P.$$

The following result generalises the theorem for autonomous semidynamical systems and the first half is the counterpart of the corresponding Theorem for processes.

The proof is similar in the latter case, essentially with j and $\theta_{-j}(p)$ changed to n_0 and $n_0 - j$, respectively, but additional complications due to the fact that the pullback absorbing family is no longer assumed to be φ-positively invariant here.

Theorem 1. *Let (X, d) and (P, d_P) be complete metric spaces and suppose that a skew-product system (θ, φ) has a pullback absorbing set family $\mathcal{B} = \{B_p : p \in P\}$. Then there exists a pullback attractor $\mathcal{A} = \{A_p : p \in P\}$ with component sets determined by*

$$A_p = \bigcap_{n \geq 0} \overline{\bigcup_{j \geq n} \varphi\left(j, \theta_{-j}(p), B_{\theta_{-j}(p)}\right)}. \tag{3}$$

This pullback attractor is unique if its component sets are uniformly bounded.

The pullback attractor of a skew-product system (θ, φ) has some nice properties when its component subsets are contained in a common compact subset or if the state space P of the driving system is compact.

Proposition 1. *Suppose that* $A(P) := \overline{\bigcup_{p \in P} A_p}$ *is compact for a pullback attractor* $\mathcal{A} = \{A_p : p \in P\}$*. Then the set-valued mapping* $p \mapsto A_p$ *is upper semi-continuous in the sense that*

$$\mathrm{dist}\,(A_q, A_p) \to 0 \quad \textit{as } q \to p.$$

On the other hand, if P *is compact and the set-valued mapping* $p \mapsto A_p$ *is upper semi-continuous, then* $A(P)$ *is compact.*

Proof. First note that, since $A(P)$ is compact, the pullback attractor is uniformly bounded by a compact set and hence is uniquely determined.

Assume that the set-valued mapping $p \mapsto A_p$ is not upper semi-continuous.

Then there exists an $\epsilon_0 > 0$ and a sequence $p_n \to p_0$ in P such that $\mathrm{dist}\,(A_{p_n}, A_{p_0}) \geq 3\epsilon_0$ for all $n \in \mathbb{N}$.

Since the sets A_{p_n} are compact, there exists an $a_n \in A_{p_n}$ such that

$$\mathrm{dist}\,(a_n, A_{p_0}) = \mathrm{dist}\,(A_{p_n}, A_{p_0}) \geq 3\epsilon_0 \quad \text{for each } n \in \mathbb{N}\,. \tag{4}$$

By pullback attraction, $\operatorname{dist}\left(\varphi\left(m, \theta_{-m}(p_0), B\right), A_{p_0}\right) \leq \epsilon_0$ for $m \geq M_{B,\epsilon_0}$ for any bounded subset B of X.

By the φ-invariance of the pullback attractor, there exist $b_n \in A_{\theta_{-m}(p_n)} \subset A(P)$ for $n \in \mathbb{N}$ such that $\varphi\left(m, \theta_{-m}(p_n), b_n\right) = a_n$.

Since $A(P)$ is compact, there is a convergent subsequence $b_{n'} \to \bar{b} \in A(P)$.

Finally, by the continuity of $\theta_{-m}(\cdot)$ and of the cocycle mapping $\varphi(n, \cdot, \cdot)$,

$$d\left(\varphi(m, \theta_{-m}(p_{n'}), b_{n'}), \varphi(m, \theta_{-m}(p_0), \bar{b})\right) \leq \epsilon_0 \quad \text{for } n' \text{ large enough.}$$

Thus,

$$\begin{aligned}
\operatorname{dist}\left(a_{n'}, A_{p_0}\right) &= \operatorname{dist}\left(\varphi(m, \theta_{-m}(p_{n'}), b_{n'}), A_{p_0}\right) \\
&\leq d\left(\varphi(m, \theta_{-m}(p_{n'}), b_{n'}), \varphi(m, \theta_{-m}(p_0), \bar{b})\right) \\
&\quad + \operatorname{dist}\left(\varphi(m, \theta_{-m}(p_0), \bar{b}), A_{p_0}\right) \leq 2\epsilon_0\,,
\end{aligned}$$

which contradicts (4). Hence the set-valued mapping $p \mapsto A_p$ must be upper semi-continuous.

The remaining assertion follows since the image of a compact subset under an upper semi-continuous set-valued mapping is compact. $\square$

- Pullback attractors are <u>in general not</u> forward attractors.

<u>However</u>, when the state space P of the driving system is <u>compact</u>, then one has the following partial forward convergence result for the pullback attractor.

Theorem 2. *In addition to the assumptions of Theorem 1, suppose that P is a compact and suppose that the pullback absorbing family $\mathcal{B}$ is uniformly bounded by a compact subset C of X. Then*

$$\lim_{n\to\infty} \sup_{p\in P} \operatorname{dist}\left(\varphi(n, p, D), A(P)\right) = 0 \tag{5}$$

for every bounded subset D of X, where $A(P) := \overline{\bigcup_{p\in P} A_p}$.

Proof. First note that $A(P)$ is compact since the component subsets A_p are all contained in the common compact set C. This means also that the pullback attractor is unique.

Suppose to the contrary that the convergence (5) does not hold. Then there exist an $\epsilon_0 > 0$ and sequences $n_j \to \infty$, $\hat{p}_j \in P$ and $x_j \in C$ such that

$$\operatorname{dist}\left(\varphi(n_j, \hat{p}_j, x_j), A(P)\right) > \epsilon_0 \,. \tag{6}$$

Set $p_j = \theta_{n_j}(\hat{p}_j)$. By the compactness of P, there exists a convergent subsequence $p_{j'} \to p_0 \in P$.

From the pullback attraction, there exists an $n > 0$ such that

$$\operatorname{dist}\left(\varphi(n, \theta_{-n}(p_0), C), A_{p_0}\right) < \frac{\epsilon_0}{2} \,.$$

The cocycle property then gives

$$\varphi\left(n_j, \theta_{-n_j}(p_j), x_j\right) = \varphi\left(n, \theta_{-n}(p_j), \varphi\left(n_j - n, \theta_{-n_j}(p_j), x_j\right)\right)$$

for any $n_j > n$.

By the pullback absorption of $\mathcal{B}$, it follows that

$$\varphi\left(n_j - n, \theta_{-n_j}(p_j), x_j\right) \subset B_{\theta_{-n}(p_j)} \subset C\,,$$

and since C is compact, there is a further index subsequence j'' of j' (depending on n) such that

$$z_{n_{j''}} := \varphi\left(n_{j''} - n, \theta_{-n_{j''}}(p_{j''}), x_{j''}\right) \to z_0 \in C.$$

The continuity of the skew-product mappings in the p and x variables implies

$$\operatorname{dist}\left(\varphi(n, \theta_{-n}(p_{j''}), z_{n_{j''}}), \varphi(n, \theta_{-n}(p_0), z_0)\right) < \frac{\epsilon_0}{2}, \quad \text{when } n_{j''} > n(\epsilon_0)\,.$$

Therefore,

$$\begin{aligned} \epsilon_0 &> \operatorname{dist}\left(\varphi(n_{j''}, \theta_{-n_{j''}}(p_0), x_{j''}), A_{p_0}\right) \\ &= \operatorname{dist}\left(\varphi\left(n_{j''}, \hat{p}_{j''}, x_{j''}\right), A_{p_0}\right) \geq \operatorname{dist}\left(\varphi\left(n_{j''}, \hat{p}_{j''}, x_{j''}\right), A(P)\right)\,, \end{aligned}$$

which contradicts (6). Thus, the asserted convergence (5) must hold. $\square$

Comparison of nonautonomous attractors

Recall that the mapping $\pi : \mathbb{Z}^+ \times \mathbb{X} \to \mathbb{X}$ defined by

$$\pi(n,(p,x)) := (\theta_n(p), \varphi(n,p,x))$$

for all $j \in \mathbb{Z}^+$ and $(p,x) \in \mathbb{X} := P \times X$ forms an autonomous semidynamical system on the extended state space $\mathbb{X}$ with the metric

$$\mathrm{dist}_{\mathbb{X}}\left((p_1,x_1),(p_2,x_2)\right) = d_P(p_1,p_2) + d(x_1,x_2).$$

Proposition 2. *Suppose that $\mathcal{A}$ is a* <u>*uniform attractor*</u> *(i.e., uniformly attracting in both the forward and pullback senses) of a skew-product system (θ,φ) and that $\bigcup_{p\in P} A_p$ is precompact in X.*
Then the union $\mathbb{A} := \bigcup_{p\in P}\{p\} \times A_p$ is the <u>*global attractor*</u> *of the autonomous semidynamical system π.*

Proof. The π-invariance of $\mathbb{A}$ follows from the φ-invariance of $\mathcal{A}$, and the θ-invariance of P via

$$\pi(n,\mathbb{A}) = \bigcup_{p\in P}\{\theta_n(p)\} \times \varphi(n,p,A_p) = \bigcup_{p\in P}\{\theta_n(p)\} \times A_{\theta_n(p)} = \bigcup_{q\in P}\{q\} \times A_q = \mathbb{A}\,.$$

Since $\mathcal{A}$ is also a pullback attractor and $\bigcup_{p\in P} A_p$ is precompact in X (and P is compact too), the set-valued mapping $p \mapsto A_p$ is upper semi-continuous, which means that $p \mapsto F(p) := \{p\} \times A_p$ is also upper semi-continuous.

Hence, $F(P) = \mathbb{A}$ is a compact subset of $\mathbb{X}$.

Moreover, the definition of the metric $\mathrm{dist}_{\mathbb{X}}$ on $\mathbb{X}$ implies that

$$
\begin{aligned}
\mathrm{dist}_{\mathbb{X}}\left(\pi(n,(p,x)),\mathbb{A}\right) &= \mathrm{dist}_{\mathbb{X}}\left((\theta_n(p),\varphi(n,p,x)),\mathbb{A}\right) \\
&\leq \mathrm{dist}_{\mathbb{X}}\big((\theta_n(p),\varphi(n,p,x)),\{\theta_n(p)\}\times A_{\theta_n(p)}\big) \\
&= \mathrm{dist}_P\left(\theta_n(p),\theta_n(p)\right) + \mathrm{dist}\big(\varphi(n,p,x),A_{\theta_n(p)}\big) \\
&= \mathrm{dist}\big(\varphi(n,p,x),A_{\theta_n(p)}\big)\ ,
\end{aligned}
$$

where $\pi(n,(p,x)) = (\theta_n(p),\varphi(n,p,x))$.

The desired attraction to $\mathbb{A}$ w.r.t. π then follows from the forward attraction of $\mathcal{A}$ w.r.t. φ. $\square$

Without uniform attraction as in Proposition 2 a pullback attractor need not give a global attractor, but the following result does hold.

Proposition 3. *If $\mathcal{A}$ is a pullback attractor for a skew-product system (θ, φ) and $\bigcup_{p\in P} A_p$ is precompact in X, then $\mathbb{A} := \bigcup_{p\in P}\{p\} \times A_p$ is the maximal invariant compact set of the autonomous semidynamical system π.*

Proof. The compactness and π-invariance of $\mathbb{A}$ are proved in the same way as in first part of the proof of Proposition 2.

To prove that the compact invariant set $\mathbb{A}$ is maximal, let $\mathbb{C}$ be any other compact invariant set of the autonomous semidynamical system π.

Then $\mathbb{A}$ is a compact and φ-invariant family of compact sets, and by pullback attraction,

$$\operatorname{dist}\left(\varphi\left(n, \theta_{-n}(p), C_{\theta_{-n}(p)}\right), A_p\right) \leq \operatorname{dist}\left(\varphi\left(n, \theta_{-n}(p), K\right), A_p\right) \to 0$$

as $n \to \infty$, where $K := \overline{\bigcup_{p\in P} C_p}$ is compact.

Hence, $C_p \subseteq A_p$ for all $p \in P$, i.e., $\mathbb{C} := \bigcup_{p\in P}\{p\} \times C_p \subseteq \mathbb{A}$, which finally means that $\mathbb{A}$ is a maximal π-invariant set. $\square$

The set $\mathbb{A}$ here need not be the global attractor of π.

In the opposite direction, the global attractor of the associated autonomous semi-dynamical system always forms a pullback attractor of the skew-product system.

Proposition 4. *If the autonomous semidynamical system π has a global attractor*

$$\mathbb{A} = \bigcup_{p \in P} \{p\} \times A_p,$$

then $\mathcal{A} = \{A_p : p \in P\}$ is a pullback attractor for the skew-product system (θ, φ).

Proof. The sets P and $K := \bigcup_{p \in P} A_p$ are compact by the compactness of $\mathbb{A}$. Moreover, $\mathbb{A} \subset P \times K$, which is a compact set.

Now

$$\begin{aligned}
\operatorname{dist}(\varphi(n,p,x), K) &= \operatorname{dist}_P(\theta_n(p), P) + \operatorname{dist}(\varphi(n,p,x), K) \\
&= \operatorname{dist}_{\mathbb{X}}((\theta_n(p), \varphi(n,p,x)), P \times K) \\
&\leq \operatorname{dist}_{\mathbb{X}}(\pi(n,(p,x)), P \times K) \\
&\leq \operatorname{dist}_{\mathbb{X}}(\pi(n, P \times D), \mathbb{A}) \to 0 \text{ as } n \to \infty
\end{aligned}$$

for all $(p,x) \in P \times D$ and every arbitrary bounded subset D of X, since $\mathbb{A}$ is the global attractor of π.

Hence, replacing p by $\theta_{-n}(p)$ implies

$$\lim_{n\to\infty} \operatorname{dist}\left(\varphi(n, \theta_{-n}(p), D), K\right) = 0\,.$$

Then the system is pullback asymptotic compact.

By Theorem 12.12 in Kloeden & Rasmussen this is a sufficient condition for the existence of a pullback attractor $\mathcal{A}' = \{A'_p : p \in P\}$ with $\bigcup_{p\in P} A'_p \subset K$.

From Proposition 3, $\mathbb{A}' := \bigcup_{p\in P}\{p\} \times A'_p$ is the maximal π-invariant subset of $\mathbb{X}$, but so is the global attractor $\mathbb{A}$.

This means that $\mathbb{A}' = \mathbb{A}$. Thus, $\mathcal{A}$ is a pullback attractor of the skew-product system (θ, φ). $\square$

Limitations of pullback attractors revisited

The limitations of pullback attraction for processes were illustrated in previous lecture through the scalar nonautonomous difference equation

$$\boxed{\quad x_{n+1} = f_n(x_n) := \frac{\lambda_n x_n}{1 + |x_n|}, \quad (7)}$$

where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is an increasing sequence with $\lim_{n\to\pm\infty} \lambda_n = \bar{\lambda}^{\pm 1}$ for $\bar{\lambda} > 1$.

The pullback attractor $\mathcal{A}$ of the corresponding process has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$ corresponding to the zero entire solution, which is the only bounded entire solution.

The zero solution $x^* = 0$ seems to be "asymptotically stable" for $n < 0$ and then "unstable" for $n \geq 0$.

However the forward limit points for nonzero solutions are $\pm(\bar{\lambda} - 1)$, which both are not solutions at all. In particular, they are not entire solutions.

An elegant way to resolve the problem is to consider the skew-product system formulation of a nonautonomous dynamical system.

Its driving mechanism, which is responsible for the temporal change in the dynamics of the nonautonomous difference equation, includes the dynamics of the asymptotically autonomous difference equations above and their limiting autonomous systems.

The nonautonomous difference equation (7) can be formulated as a skew-product system with the driving system defined in terms of the shift operator θ on the space of bi-infinite sequences

$$\Lambda_L = \{\lambda = \{\lambda_n\}_{n\in\mathbb{Z}} : \lambda_n \in [0, L], \quad n \in \mathbb{Z}\}$$

for some $L > \bar{\lambda} > 1$. It yields a compact metric space with the metric

$$d_{\Lambda_L}(\lambda, \lambda') := \sum_{n\in\mathbb{Z}} 2^{-|n|} |\lambda_n - \lambda'_n| .$$

This is coupled with a cocycle mapping with values $x_n = \varphi(n, \lambda, x_0)$ on $\mathbb{R}$ generated by the difference equation (7) with a given coefficient sequence λ.

For the sequence λ in (7), the limit of the shifted sequences $\theta_n(\lambda)$ in the above metric as $n \to \infty$ is the constant sequence λ_+^* equal to $\bar{\lambda}$, while the limit as $n \to -\infty$ is the sequence λ_-^* with all components equal to $\bar{\lambda}^{-1}$.

The pullback attractor of the corresponding skew-product system (θ, φ) on $\Lambda \times \mathbb{R}$ consists of compact subsets A_λ of $\mathbb{R}$ for each $\lambda \in \Lambda_L$.

It is easy to see that $A_\lambda = \{0\}$ for any λ with components $\lambda_n < 1$ for $n \leq 0$, which includes the constant sequence λ_-^* as well as the switched sequence in (7). On the other hand, $A_{\lambda_+^*} = [-\bar{\lambda}, \bar{\lambda}]$.

- Here $\bigcup_{\lambda\in\Lambda_L} A_\lambda$ is precompact, so it contains all future limiting dynamics.

The pullback attractor of the skew-product system includes that of the process for a given bi-infinite coefficient sequence, but also includes its forward asymptotic limits and much more.

- The coefficient sequence set Λ_L includes all possibilities, in fact, far more than may be of interest in particular situation.

If one is interested in the dynamics of a process corresponding to a specific $\hat{\lambda} \in \Lambda_L$, then it would suffice to consider the skew-product system w.r.t. the driving system on the smaller space $\Lambda_{\hat{\lambda}}$ defined as the hull of this sequence, i.e., the set of accumulation points of the set $\{\theta_n(\hat{\lambda}) : n \in \mathbb{Z}\}$ in the metric space $(\Lambda_L, d_{\Lambda_L})$.

In particular, if $\hat{\lambda}$ is the specific sequence in (7), then the union

$$\bigcup_{\lambda \in \Lambda_{\hat{\lambda}}} A_\lambda = A_{\lambda_+^*} = [-\bar{\lambda}, \bar{\lambda}]$$

contains all future limiting dynamics, i.e.,

$$\lim_{n\to\infty} \mathrm{dist}\left(\varphi(n, \lambda, x), [-\bar{\lambda}, \bar{\lambda}]\right) = 0 \quad \text{for all } x \in \mathbb{R}.$$

The example described by nonautonomous difference equation (7) is asymptotically autonomous with

$$\Lambda_\lambda = \{\lambda_\pm^*\} \bigcup \{\theta_n(\lambda) : n \in \mathbb{Z}\}.$$

More generally, unlike the process formulation, the skew-product system and its pullback attractor includes all of the forward limiting dynamics.

Local pullback attractors

Less uniform behaviour such as parameter dependent domains of definition and local pullback attractors can be handled by introducing the concept of a basin of attraction system.

Let $Dom_p \subset X$ be the domain of definition of $f(p, \cdot)$ in the nonautonomous equation

$$x_{n+1} = f\left(\theta_n(p), x_n\right),$$

which requires $f(p, Dom_p) \subset Dom_{\theta(p)}$.

Then the corresponding cocycle mapping φ has the domain of definition $\mathbb{Z}^+ \times \bigcup_{p \in P} (\{p\} \times Dom_p)$.

Consequently one needs to restrict the admissible families of bounded sets in the pullback convergence to subsets of Dom_p for each $p \in P$.

Definition 4. *An ensemble $\mathfrak{D}_{ad}$ of families $\mathcal{D} = \{D_p : \ p \in P\}$ of nonempty subsets X is called admissible if*

i) D_p is bounded and $D_p \subset Dom_p$ for each $p \in P$ and every $\mathcal{D} = \{D_p : \ p \in P\}$ $\in \mathfrak{D}_{ad}$; and

ii) $\widehat{D}^{(1)} = \{D_p^{(1)} \ : \ p \in P\} \in \mathfrak{D}_{ad}$ whenever $\widehat{D}^{(2)} = \{D_p^{(2)} \ : \ p \in P\} \in \mathfrak{D}_{ad}$ and $D_p^{(1)} \subseteq D_p^{(2)}$ for all $p \in P$.

Further restrictions will allow one to consider local or otherwise restricted form of pullback attraction.

Definition 5. *A φ-invariant family $\mathcal{A} = \{A_p : \ p \in P\}$ of nonempty compact subsets of X with $A_p \subset Dom_p$ for each $p \in P$ is called a pullback attractor w.r.t. the basin of attraction system $\mathfrak{D}_{att}$ if $\mathfrak{D}_{att}$ is an admissible ensemble of families of subsets such that*

$$\lim_{j\to\infty} \operatorname{dist}\left(\varphi(j, \theta_{-j}(p), D_{\theta_{-j}(p)}), A_p\right) = 0 \tag{8}$$

for every $\mathcal{D} = \{D_p : \ p \in P\} \in \mathfrak{D}_{att}$.

In this case a pullback absorbing set system $\mathcal{B} = \{B_p : \ p \in P\}$ should also satisfy $\mathcal{B} \in \mathfrak{D}_{att}$ and the pullback absorbing property be modified to

$$\varphi\left(j, \theta_{-j}(p), D_{\theta_{-j}(p)})\right) \subseteq B_p$$

for all $j \geq N_{p,\mathcal{D}}$, $p \in P$ and $\mathcal{D} = \{D_p; \ p \in P\} \in \mathfrak{D}_{att}$.

A counterpart of Theorem 1 holds here: the pullback attractor is unique within the basin of attraction system.

But the skew-product system may have other pullback attractors within other basin of attraction systems, which may be either disjoint from or a proper sub-ensemble of the original basin of attraction system.

Example 1. *Consider the scalar nonautonomous difference equation*

$$x_{n+1} = f_n(x_n) := x_n + \gamma_n x_n \left(1 - x_n^2\right) \tag{9}$$

for given parameters $\gamma_n > 0$, $n \in \mathbb{Z}$.

First let $\gamma_n \equiv \bar{\gamma}$ *for all* $n \in \mathbb{Z}$, *so the system is* *autonomous*. *It has the attractor* $A^* = [-1, 1]$ *for the maximal basin of attraction* $(-1 - \bar{\gamma}^{-1}, 1 + \bar{\gamma}^{-1})$.

But if one *restricts attention* *further to the basin of attraction* $(0, 1 + \bar{\gamma}^{-1})$ *then the attractor is only* $A^{**} = \{1\}$.

- *Now let* γ_n *be variable with* $\gamma_n \in \left[\frac{1}{2}\bar{\gamma}, \bar{\gamma}\right]$ *for each* $n \in \mathbb{Z}$, *so the system is now* *nonautonomous* *and representable as a skew-product on the state space* $\mathbb{X} = \mathbb{Z} \times \mathbb{R}$ *with the parameter set* $P = \mathbb{Z}$.

Then $\mathcal{A}^* = \{A_n^* : n \in \mathbb{Z}\}$ *with* $A_n^* = [-1, 1]$ *for all* $n \in \mathbb{Z}$ *is the pullback attractor for the basin of attraction system* $\mathfrak{D}_{att}$ *consisting of all families* $\mathcal{D} = \{D_n : n \in \mathbb{Z}\}$ *satisfying* $D_n \subset (-1 - \bar{\gamma}^{-1}, 1 + \bar{\gamma}^{-1})$.

However, $\mathcal{A}^{**} = \{A_n^{**} : n \in \mathbb{Z}\}$ *with* $A_n^{**} = \{-1\}$ *for all* $n \in \mathbb{Z}$ *is the pullback attractor for the basin of attraction system* $\mathfrak{D}_{att}$ *consisting of all families* $\mathcal{D} = \{D_n : n \in \mathbb{Z}\}$ *with* $D_n \subset (0, 1 + \bar{\gamma}^{-1})$.

Lecture 4: Lyapunov functions for pullback attractors

A Lyapunov function characterising pullback attraction and pullback attractors for a discrete-time process in $\mathbb{R}^d$ will be constructed here.

Consider a nonautonomous difference equation

$$\boxed{x_{n+1} = f_n(x_n) \qquad (1)}$$

on $\mathbb{R}^d$, where the $f_n : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous mappings.

This generates a process $\phi : \mathbb{Z}^2_{\geq} \times \mathbb{R}^d \to \mathbb{R}^d$ through iteration by

$$\phi(n, n_0, x_0) = f_{n-1} \circ \cdots \circ f_{n_0}(x_0)$$

for all $n \geq n_0$ and each $x_0 \in \mathbb{R}^d$.

This satisfies the initial condition property

$$\phi(n_0, n_0, x_0) = x_0$$

for each $x_0 \in \mathbb{R}^d$ and all $n_0 \in \mathbb{Z}$; the 2-parameter semigroup property

$$\phi(n_2, n_0, x_0) = \phi(n_2, n_1, \phi(n_1, n_0, x_0))$$

for each $x_0 \in \mathbb{R}^d$ and $n_0 \leq n_1 \leq n_2$ in $\mathbb{Z}$; and the continuity property

$$x_0 \mapsto \phi(n, n_0, x_0) \quad \text{is -Lipschitz continuous for all } n \geq n_0.$$

The pullback attraction is taken with respect to a basin of attraction system, which is defined as follows for a process.

Definition 1. *A basin of attraction system* $\mathfrak{D}_{att}$ *consists of families* $\mathcal{D} = \{D_n : n \in \mathbb{Z}\}$ *of nonempty bounded subsets of* $\mathbb{R}^d$ *with the property that* $\mathcal{D}^{(1)} = \{D_n^{(1)} : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ *if* $\mathcal{D}^{(2)} = \{D_n^{(2)} : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ *and* $D_n^{(1)} \subseteq D_n^{(2)}$ *for all* $n \in \mathbb{Z}$.

Although somewhat complicated, the use of such a basin of attraction system allows both nonuniform and local attraction regions, which are typical in nonautonomous systems, to be handled.

Definition 2. *A* ϕ*-invariant family of nonempty compact subsets* $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ *is called a pullback attractor w.r.t. a basin of attraction system* $\mathfrak{D}_{att}$ *if it is pullback attracting*

$$\lim_{j \to \infty} (\phi(n, n-j, D_{n-j}), A_n) = 0 \tag{2}$$

for all $n \in \mathbb{Z}$ *and all* $\mathcal{D} = \{D_n : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$.

Obviously $\mathcal{A} \in \mathfrak{D}_{att}$.

A pullback absorbing neighbourhood system

The construction of the Lyapunov function requires the existence of a pullback absorbing neighbourhood family.

Lemma 1. *Let $\mathcal{A}$ be a pullback attractor with a basin of attraction system $\mathfrak{D}_{att}$ for a process ϕ.*
Then there exists a pullback absorbing neighbourhood system $\mathcal{B} \subset \mathfrak{D}_{att}$ of $\mathcal{A}$ w.r.t. ϕ. Moreover, $\mathcal{B}$ is ϕ-positive invariant.

Proof. For each $n_0 \in \mathbb{Z}$ pick $\delta_{n_0} > 0$ such that

$$B[A_{n_0}; \delta_{n_0}] := \{x \in \mathbb{R}^d : \ \mathrm{dist}(x, A_{n_0}) \leq \delta_{n_0}\}$$

so $\{B[A_{n_0}; \delta_{n_0}] : n_0 \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ and define

$$B_{n_0} := \overline{\bigcup_{j \geq 0} \phi(n_0, n_0 - j, B[A_{n_0 - j}; \delta_{n_0 - j}])}.$$

Obviously $A_{n_0} \subset \mathrm{int} B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$.

To show positive invariance the 2-parameter semigroup property will be used in what follows.

$$
\begin{aligned}
\phi(n_0+1, n_0, B_{n_0}) &= \overline{\bigcup_{j\geq 0} \phi(n_0+1, n_0, \phi(n_0, n_0-j, B[A_{n_0-j}; \delta_{n_0-j}]))} \\
&= \overline{\bigcup_{j\geq 0} \phi(n_0+1, n_0-j, B[A_{n_0-j}; \delta_{n_0-j}])} \\
&= \overline{\bigcup_{i\geq 1} \phi(n_0+1, n_0+1-i, B[A_{n_0+1-i}; \delta_{n_0+1-i}])} \\
&\subseteq \overline{\bigcup_{i\geq 0} \phi(n_0+1, n_0+1-i, B[A_{n_0+1-i}; \delta_{n_0+1-i}])} = B_{n_0+1},
\end{aligned}
$$

so $\phi(n_0+1, n_0, B_{n_0}) \subseteq B_{n_0+1}$.

This and the 2-parameter semigroup property again gives

$$
\begin{aligned}
\phi(n_0+2, n_0, B_{n_0}) &= \phi(n_0+2, n_0+1, \phi(n_0+1, n_0, B_{n_0}) \\
&\subseteq \phi(n_0+2, n_0+1, B_{n_0+1}) \subseteq B_{n_0+2}.
\end{aligned}
$$

The general positive invariance assertion then follows by induction.

The set $\phi(n_0, n_0-j, B[A_{n_0-j}; \delta_{n_0-j}])$ is compact for each $j \geq 0$ and $n_0 \in \mathbb{Z}$ by the continuity of $\phi(n_0, n_0-j, \cdot)$ and the compactness of $B[A_{n_0-j}; \delta_{n_0-j}]$.

Moreover, by pullback convergence, there exists an $N = N(n_0, \delta_{n_0}) \in \mathbb{N}$ such that

$$\phi(n_0, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}]) \subseteq B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$$

for all $j \geq N$. Hence

$$\begin{aligned} B_{n_0} &= \overline{\bigcup_{j \geq 0} \phi(n_0, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])} \\ &\subseteq B[A_{n_0}; \delta_{n_0}] \bigcup \overline{\bigcup_{0 \leq j < N} \phi(n_0, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])} \\ &= \overline{\bigcup_{0 \leq j < N} \phi(n_0, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])}, \end{aligned}$$

which is compact, so B_{n_0} is compact.

To see that $\mathcal{B}$ so constructed is pullback absorbing w.r.t. $\mathfrak{D}_{att}$, let $\mathcal{D} \in \mathfrak{D}_{att}$ and fix $n_0 \in \mathbb{Z}$.

Since $\mathcal{A}$ is pullback attracting, there exists an $N(\mathcal{D}, \delta_{n_0}, n_0) \in \mathbb{N}$ such that

$$\text{dist}\left(\phi(n_0, n_0 - j, D_{n_0-j}), A_{n_0}\right) < \delta_{n_0}$$

for all $j \geq N(\mathcal{D}, \delta_{n_0}, n_0)$. But $\phi(n_0, n_0 - j, D_{n_0-j}) \subset \text{int} B[A_{n_0}; \delta_{n_0}]$ and $B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$, so

$$\phi(n_0, n_0 - j, D_{n_0-j}) \subset \text{int} B_{n_0}$$

for all $j \geq N(\mathcal{D}, \delta_{n_0}, n_0)$. Hence $\mathcal{B}$ is pullback absorbing as required. $\square$

Necessary and sufficient conditions

The main result is the construction of a Lyapunov function that characterizes this pullback attraction.

Theorem 1. *Let the f_n be uniformly Lipschitz continuous on $\mathbb{R}^d$ for each $n \in \mathbb{Z}$ and let ϕ be the process that they generate. In addition, let $\mathcal{A}$ be a ϕ-invariant family of nonempty compact sets that is pullback attracting with respect to ϕ with a basin of attraction system $\mathfrak{D}_{att}$.*

Then there exists a Lipschitz continuous function $V : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}$ such that

Property 1 (upper bound): *For all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$*

$$V(n_0, x_0) \leq \mathrm{dist}(x_0, A_{n_0}); \tag{3}$$

Property 2 (lower bound): *For each $n_0 \in \mathbb{Z}$ there exists a function $a(n_0, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(n_0, 0) = 0$ and $a(n_0, r) > 0$ for all $r > 0$ which is monotonic increasing in r such that*

$$a(n_0, \mathrm{dist}(x_0, A_{n_0})) \leq V(n_0, x_0) \quad \textit{for all } x_0 \in \mathbb{R}^d; \tag{4}$$

Property 3 (Lipschitz condition): *For all $n_0 \in \mathbb{Z}$ and x_0, $y_0 \in \mathbb{R}^d$*

$$|V(n_0, x_0) - V(n_0, y_0)| \leq \|x_0 - y_0\|; \tag{5}$$

Property 4 (pullback convergence): *For all $n_0 \in \mathbb{Z}$ and any $\mathcal{D} \in \mathcal{D}_{att}$*

$$\mathrm{limsup}_{n\to\infty} \sup_{z_{n_0-n} \in D_{n_0-n}} V(n_0, \phi(n_0, n_0 - n, z_{n_0-n})) = 0. \tag{6}$$

In addition,

Property 5 (forward convergence): *There exists $\mathcal{N} \in \mathfrak{D}_{att}$, which is positively invariant under ϕ and consists of nonempty compact sets N_{n_0} with $A_{n_0} \subset \mathrm{int} N_{n_0}$ for each $n_0 \in \mathbb{Z}$ such that*

$$V(n_0+1, \phi(n_0+1, n_0, x_0)) \leq e^{-1} V(n_0, x_0) \tag{7}$$

for all $x_0 \in N_{n_0}$, and hence

$$V(n_0+j, \phi(j, n_0, x_0)) \leq e^{-j} V(n_0, x_0) \quad \textit{for all } x_0 \in N_{n_0},\ j \in \mathbb{N}. \tag{8}$$

Proof. The aim is to construct a Lyapunov function $V(n_0, x_0)$ that characterises a pullback attractor $\mathcal{A}$ and satisfies properties 1–5 of Theorem 1.
Define

$$\boxed{V(n_0, x_0) := \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \operatorname{dist}\left(x_0, \phi(n_0, n_0-n, B_{n_0-n})\right)}$$

for all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$, where

$$T_{n_0,n} = n + \sum_{j=1}^{n} \alpha^+_{n_0-j}$$

with $T_{n_0,0} = 0$.

Here $\alpha_n = \log L_n$, where L_n is the uniform Lipschitz constant of f_n on $\mathbb{R}^d$, and $a^+ = (a+|a|)/2$, i.e., the positive part of a real number a.

Note: $T_{n_0,n} \geq n$ and $T_{n_0,n+m} = T_{n_0,n} + T_{n_0-n,m}$ for n, $m \in \mathbb{N}$, $n_0 \in \mathbb{Z}$.

Proof of property 1

Since $e^{-T_{n_0,n}} \leq 1$ for all $n \in \mathbb{N}$ and $\operatorname{dist}(x_0, \phi(n, n_0 - n, B_{n_0-n}))$ is monotonically increasing from $0 \leq \operatorname{dist}(x_0, \phi(0, n_0, B_{n_0}))$ at $n = 0$ to $\operatorname{dist}(x_0, A_{n_0})$ as $n \to \infty$,

$$V(n_0, x_0) = \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) \leq 1 \cdot \operatorname{dist}(x_0, A_{n_0}).$$

Proof of property 3

$$\begin{aligned}
&|V(n_0, x_0) - V(n_0, y_0)| \\
&\quad = \left| \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) \right. \\
&\qquad\qquad \left. - \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \operatorname{dist}(y_0, \phi(n_0, n_0 - n, B_{n_0-n})) \right| \\
&\quad \leq \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \left| \operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) - \operatorname{dist}(y_0, \phi(n_0, n_0 - n, B_{n_0-n})) \right| \\
&\quad \leq \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \|x_0 - y_0\| \leq \|x_0 - y_0\|
\end{aligned}$$

since

$$|\operatorname{dist}(x_0, C) - \operatorname{dist}(y_0, C)| \leq \|x_0 - y_0\|$$

for any x_0, $y_0 \in \mathbb{R}^d$ and nonempty compact subset C of $\mathbb{R}^d$.

Proof of property 2

If $x_0 \in A_{n_0}$, then $V(n_0, x_0) = 0$ by Property 1, so assume that $x_0 \in \mathbb{R}^d \setminus A_{n_0}$.

Now the supremum in

$$V(n_0, x_0) = \sup_{n \geq 0} e^{-T_{n_0,n}} \text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right)$$

involves the product of an exponential decreasing quantity bounded below by zero and a bounded increasing function, since the sets $\phi(n_0, n_0 - n, B_{n_0-n})$ are a <u>nested</u> family of compact sets decreasing to $A_{n_0}$with increasing n. In particular,

$$\text{dist}\left(x_0, A_{n_0}\right) \geq \text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right) \quad \text{for all } n \in \mathbb{N}.$$

Hence there exists an $N^* = N^*(n_0, x_0) \in \mathbb{N}$ such that

$$\frac{1}{2}\text{dist}(x_0, A_{n_0}) \leq \text{dist}\left(x_0, \phi(n_o, n_0 - n, B_{n_0-n})\right) \leq \text{dist}(x_0, A_{n_0})$$

for all $n \geq N^*$, but not for $n = N^* - 1$. Then, from above,

$$\begin{aligned} V(n_0, x_0) &\geq e^{-T_{n_0,N^*}} \text{dist}\left(x_0, \phi(n_0, n_0 - N^*, B_{n_0-N^*})\right) \\ &\geq \frac{1}{2} e^{-T_{n_0,N^*}} \text{dist}\left(x_0, A_{n_0}\right). \end{aligned}$$

Define

$$N^*(n_0, r) := \sup\{N^*(n_0, x_0) : \text{dist}\left(x_0, A_{n_0}\right) = r\}$$

Now $N^*(n_0, r) < \infty$ for $x_0 \notin A_{n_0}$ with $\operatorname{dist}(x_0, A_{n_0}) = r$ and $N^*(n_0, r)$ is <u>nondecreasing</u> with $r \to 0$.

To see this note that by the triangle rule

$$\operatorname{dist}(x_0, A_{n_0}) \leq \operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) + \operatorname{dist}(\phi(n_0, n_0 - n, B_{n_0-n}), A_{n_0}).$$

Also by pullback convergence there exists an $N(n_0, r/2)$ such that

$$\operatorname{dist}(\phi(n_0, n_0 - n, B_{n_0-n}), A_{n_0}) < \frac{1}{2}r$$

for all $n \geq N(n_0, r/2)$.

Hence for $\operatorname{dist}(x_0, A_{n_0}) = r$ and $n \geq N(n_0, r/2)$,

$$r \leq \operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) + \frac{1}{2}r,$$

that is

$$\frac{1}{2}r \leq \operatorname{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})).$$

Obviously $N^*(n_0, r) \leq N^*(n_0, r/2)$.

Finally, define

$$a(n_0, r) := \frac{1}{2}r\ e^{-T_{n_0, N^*(n_0,r)}}. \tag{9}$$

Note that there is no guarantee here (without further assumptions) that $a(n_0, r)$ does not converge to 0 for fixed $r \neq 0$ as $n_0 \to \infty$.

Proof of property 4

Assume the opposite. Then there exists an $\varepsilon_0 > 0$, a sequence $n_j \to \infty$ in $\mathbb{N}$ and points $x_j \in \phi(n_0, n_0 - n_j, D_{n_0-n_j})$ such that $V(n_0, x_j) \geq \varepsilon_0$ for all $j \in \mathbb{N}$.

Since $\mathcal{D} \in \mathfrak{D}_{att}$ and $\mathcal{B}$ is pullback absorbing, there exists an $N = N(\mathcal{D}, n_0) \in \mathbb{N}$ such that

$$\phi(n_0, n_0 - n_j, D_{n_0-n_j}) \subset B_{n_0} \quad \text{for all } n_j \geq N.$$

Hence, for all j such that $n_j \geq N$, it holds $x_j \in B_{n_0}$, which is a compact set, so there exists a convergent subsequence $x_{j'} \to x^* \in B_{n_0}$. But also

$$x_{j'} \in \overline{\bigcup_{n \geq n_{j'}} \phi(n_0, n_0 - n, D_{n_0-n})}$$

and

$$\bigcap_{n_{j'}} \overline{\bigcup_{n \geq n_{j'}} \phi(n_0, n_0 - n, D_{n_0-n})} \subseteq A_{n_0}$$

by the definition of a pullback attractor. Hence $x^* \in A_{n_0}$ and $V(n_0, x^*) = 0$. But V is Lipschitz continuous in its second variable by property 3, so

$$\varepsilon_0 \leq V(n_0, x_{j'}) = \|V(n_0, x_{j'}) - V(n_0, x^*)\| \leq \|x_{j'} - x^*\|,$$

which contradicts the convergence $x_{j'} \to x^*$.

Hence property 4 must hold.

Proof of property 5

Define

$$N_{n_0} := \{x_0 \in B[B_{n_0};1] \ : \ \phi(1, n_0, x_0) \in B_{n_0+1}\},$$

where $B[B_{n_0};1] = \{x_0 \ : \ \mathrm{dist}(x_0, B_{n_0}) \leq 1\}$ is bounded because B_{n_0} is compact and $\mathbb{R}^d$ is locally compact, so N_{n_0} is bounded. It is also closed, hence compact, since $\phi(n_0+1, n_0, \cdot)$ is continuous and B_{n_0+1} is compact.

Now $A_{n_0} \subset \mathrm{int}B_{n_0}$ and $B_{n_0} \subset N_{n_0}$, so $A_{n_0} \subset \mathrm{int}N_{n_0}$. In addition,

$$\phi(n_0+1, n_0, N_{n_0}) \subset B_{n_0+1} \subset N_{n_0+1},$$

so $\mathcal{N}$ is positive invariant.

It remains to establish the exponential decay inequality (7). This needs the following <u>Lipschitz condition</u> on $\phi(n_0+1, n_0, \cdot) \equiv f_{n_0}(\cdot)$:

$$\|\phi(n_0+1, n_0, x_0) - \phi(n_0+1, n_0, y_0)\| \leq e^{\alpha_{n_0}} \|x_0 - y_0\|$$

for all x_0, $y_0 \in D_{n_0}$. It follows from this that

$$\mathrm{dist}(\phi(1, n_0, x_0), \phi(1, n_0, C_{n_0})) \leq e^{\alpha_{n_0}} \mathrm{dist}(x_0, C_{n_0})$$

for any compact subset $C_{n_0} \subset \mathbb{R}^d$.

From the definition of V,

$$\begin{aligned} V(n_0+1, \phi(n_0+1, n_0, x_0)) &= \sup_{n \geq 0} e^{-T_{n_0+1,n}} \mathrm{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-n, B_{n_0-n})) \\ &= \sup_{n \geq 1} e^{-T_{n_0+1,n}} \mathrm{dist}(\phi(1, n_0, x_0), \phi(n_0, n_0-n, B_{n_0-n})) \end{aligned}$$

since $\phi(n_0+1, n_0, x_0) \in B_{n_0+1}$ when $x_0 \in N_{n_0}$.

Hence re-indexing and then using the 2-parameter semigroup property and the Lipschitz condition on $\phi(1, n_0, \cdot)$

$$V(n_0+1, \phi(n_0+1, n_0, x_0))$$

$$\begin{aligned} &= \sup_{j \geq 0} e^{-T_{n_0+1,j+1}} \mathrm{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-j-1, B_{n_0-j-1})) \\ &= \sup_{j \geq 0} e^{-T_{n_0+1,j+1}} \mathrm{dist}(\phi(n_0+1, n_0, x_0), \phi(1, n_0, \phi(n_0, n_0-j, B_{n_0-j}))) \\ &\leq \sup_{j \geq 0} e^{-T_{n_0+1,j+1}} e^{\alpha_{n_0}} \mathrm{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \end{aligned}$$

Now $T_{n_0+1,j+1} = T_{n_0,j} + 1 - \alpha_{n_0}^+$, so

$$V(n_0+1, \phi(n_0+1, n_0, x_0))$$

$$\begin{aligned} &\leq \sup_{j \geq 0} e^{-T_{n_0+1,j+1}+\alpha_{n_0}} \mathrm{dist}(x_0, \phi(j, n_0-j, B_{n_0-j})) \\ &= \sup_{j \geq 0} e^{-T_{n_0,j}-1-\alpha_{n_0}^+ + \alpha_{n_0}} \mathrm{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \\ &\leq e^{-1} \sup_{j \geq 0} e^{-T_{n_0,j}} \mathrm{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \leq e^{-1} V(n_0, x_0), \end{aligned}$$

which is the desired inequality.

Moreover, since $\phi(1, n_0, x_0) \in B_{n_0+1} \subset N_{n_0+1}$, the proof continues inductively to give

$$V(n_0+j, \phi(j, n_0, x_0)) \leq e^{-j} V(n_0, x_0) \quad \text{for all } j \in \mathbb{N}.$$

This completes the proof of property 5 and hence of Theorem 1. $\square$

Comments on Theorem 1

Comment 1:

The forward convergence inequality (8) does not imply forward Lyapunov stability or Lyapunov asymptotical stability. Although

$$a(n_0 + j, \mathrm{dist}(\phi(n_0 + j, n_0, x_0), A_{n_0+j})) \leq e^{-j} V(n_0, x_0)$$

there is no guarantee (without additional assumptions) that

$$\inf_{j \geq 0} a(n_0 + j, r) > 0$$

for $r > 0$, so $\mathrm{dist}(\phi(n_0 + j, n_0, x_0), A_{n_0+j})$ need not become small as $j \to \infty$.

As a counterexample consider the process ϕ on $\mathbb{R}$ generated by the nonautonomous difference equation with $f_n = g_1$ for $n \leq 0$ and $f_n = g_2$ for $n \geq 1$, where the mappings $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$g_1(x) := \frac{1}{2}x, \qquad g_2(x) := \max\{0, 4x(1-x)\}$$

for all $x \in \mathbb{R}$.

Then the family $\mathcal{A}$ of subsets $A_{n_0} = \{0\}$ for all $n_0 \in \mathbb{Z}$ is pullback attracting for ϕ, but is not forward Lyapunov asymptotically stable.

Comment 2:

The forward convergence inequality (8) can be rewritten as

$$V(n_0, \phi(n_0, n_0 - j, x_{n_0-j})) \leq e^{-j} V(n_0 - j, x_{n_0-j}) \leq e^{-j} \mathrm{dist}(x_{n_0-j}, A_{n_0-j})$$

for all $x_{n_0-j} \in N_{n_0-j}$ and $j \in \mathbb{N}$.

Definition 3. *A family $\mathcal{D} \in \mathfrak{D}_{att}$ is called past–tempered w.r.t. $\mathcal{A}$ if*

$$\lim_{j\to\infty} \frac{1}{j} \log^+ \mathrm{dist}(D_{n_0-j}, A_{n_0-j}) = 0 \quad \textit{for all } n_0 \in \mathbb{Z}\,,$$

or equivalently if

$$\lim_{j\to\infty} e^{-\gamma j}\, \mathrm{dist}(D_{n_0-j}, A_{n_0-j}) = 0 \quad \textit{for all } n_0 \in \mathbb{Z},\ \gamma > 0.$$

This says that there is at most sub-exponential growth backwards in time of the starting sets.

For a past-tempered family $\mathcal{D} \subset \mathcal{N}$ it follows that

$$V(n_0, \phi(n_0, n_0 - j, x_{n_0-j})) \leq e^{-j}\, \mathrm{dist}(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$. Hence

$$a\,(n_0, \mathrm{dist}(\phi(n_0, n_0 - j, x_{n_0-j}), A_{n_0})) \leq e^{-j}\, \mathrm{dist}(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$.

Since n_0 is fixed in the lower expression, this implies the pullback convergence

$$\lim_{j\to\infty} \mathrm{dist}(\phi(n_0, n_0 - j, D_{n_0-j}), A_{n_0}) = 0.$$

Rate of pullback convergence

Since $\mathcal{B}$ is a pullback absorbing neighbourhood system, then for every $n_0 \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\mathcal{D} \in \mathfrak{D}_{att}$ there exists an $N(\mathcal{D}, n_0, n) \in \mathbb{N}$ such that

$$\phi(n_0 - n, n_0 - n - m, D_{n_0-n-m}) \subseteq B_{n_0-n} \quad \text{for all } m \geq N.$$

Thus

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \leq e^{-T_{n_0,n}} \text{dist}\,(B_{n_0}, A_{n_0})$$

for all $z_{n_0-m} \in D_{n_0-m}$, $m \geq n + N(\mathcal{D}, n_0, n)$ and $n \geq 0$.

It can be assumed that the mapping $n \mapsto n + N(\mathcal{D}, n_0, n)$ is monotonic increasing in n and is hence invertible.

Let the inverse of $m = n + N(\mathcal{D}, n_0, n)$ be $n = M(m) = M(\mathcal{D}, n_0, m)$. Then

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \leq e^{-T_{n_0,M(m)}} \text{dist}\,(B_{n_0}, A_{n_0})$$

for all $m \geq N(\mathcal{D}, n_0, 0) \geq 0$. Usually $N(\mathcal{D}, n_0, 0) > 0$.

This expression can be modified to hold for all $m \geq 0$ by replacing $M(m)$ by $M^*(m)$ defined for all $m \geq 0$ and introducing a constant $K_{\mathcal{D},n_0} \geq 1$ to account for the behaviour over the finite time set $0 \leq m < N(\mathcal{D}, n_0, 0)$. For all $m \geq 0$ this gives

$$\boxed{V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \leq K_{\mathcal{D},n_0} e^{-T_{n_0,M^*(m)}} \text{dist}\,(B_{n_0}, A_{n_0}).}$$

Lecture 5: Bifurcation in nonautonomous systems

The classical dynamical bifurcation theory focuses on autonomous difference equations

$$x_{n+1} = g(x_n, \lambda), \tag{1}$$

where right-hand side $g : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ depends on a parameter λ from some parameter space Λ, which is typically a subset of $\mathbb{R}^n$.

A central question is how stability and multiplicity of invariant sets of (1) change when the parameter λ is varied.

In the simplest case these invariant sets are fixed points or periodic solutions of (1).

Given some parameter value λ^*, a fixed point $x^* = g(x^*, \lambda^*)$ of (1) is called hyperbolic if the derivative $D_1 g(x^*, \lambda^*)$ has no eigenvalue on the complex unit circle $\mathbb{S}^1$.

- it is an easy consequence of the implicit function theorem that x^* has a unique continuation $x(\lambda) \equiv g(x(\lambda), \lambda)$ in a neighborhood of λ^*.

In particular, hyperbolicity rules out bifurcations understood as topological changes in the set $\{x \in \mathbb{R}^d : g(x, \lambda) = x\}$ near (x^*, λ^*) or a stability change of x^*.

On the other hand, eigenvalues on the complex unit circle give rise to various well understood autonomous bifurcation scenarios. e.g.,

- fold, transcritical or pitchfork bifurcations (eigenvalue 1)

- flip bifurcations (eigenvalue -1)

- the Sacker–Neimark bifurcation (a pair of complex conjugate eigenvalues for $d \geq 2$).

Hyperbolicity and simple examples

Even in the autonomous set up of (1), one easily encounters intrinsically nonautonomous problems, where the classical methods do not apply:

- *Investigate the behaviour of* (1) *along an entire reference solution* $(\chi_n)_{n\in\mathbb{Z}}$*, which is not constant or periodic.*

This is typically done using the (obviously nonautonomous) equation of perturbed motion

$$x_{n+1} = g(x_n + \chi_n, \lambda) - g(\chi_n, \lambda).$$

- *Replace the constant parameter* λ *in* (1) *by a sequence* $(\lambda_n)_{n\in\mathbb{Z}}$ *in* Λ*, which varies in time.*

The resulting parametrically perturbed equation

$$x_{n+1} = g(x_n, \lambda_n)$$

becomes nonautonomous.

- Both of the above problems fit into the framework of general nonautonomous difference equations

$$x_{n+1} = f_n(x_n, \lambda) \tag{Δ_λ}$$

with a sufficiently smooth right-hand side $f_n : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$, $n \in \mathbb{Z}$.

It will be supposed in addition that f_n and its derivatives map bounded subsets of $\mathbb{R}^d \times \Lambda$ into bounded sets uniformly in $n \in \mathbb{Z}$).

Generically, a nonautonomous equation (Δ_λ) does not have constant solutions and sequences of fixed points $x_n^* = f_n(x_n^*, \lambda^*)$ are usually not solutions of (Δ_λ).

This gives rise to the following question:

If there are no equilibria, what should bifurcate in a nonautonomous set up?

Example 1. *The autonomous difference equation* $x_{n+1} = \frac{1}{2}x_n + \lambda$ *has the unique fixed point* $x^*(\lambda) = 2\lambda$ *for all* $\lambda \in \mathbb{R}$.
Replace λ *by a bounded sequence* $(\lambda_n)_{n\in\mathbb{Z}}$ *and observe that the nonautonomous counterpart*

$$x_{n+1} = \tfrac{1}{2}x_n + \lambda_n$$

has a unique <u>*bounded entire solution*</u> $\chi_n^* := \sum_{k=-\infty}^{n-1} \left(\frac{1}{2}\right)^{n-k-1} \lambda_n$.

For the special case $\lambda_n \equiv \lambda$, *this solution reduces to the known fixed point* $\chi_n^* \equiv 2\lambda$.

This suggests that equilibria of autonomous equations persist as <u>bounded entire solutions</u> under parametric perturbations.

Theorem 1 below shows that this conjecture is generically true in the sense that the fixed point of (1) has to be hyperbolic in order to persist under parametric perturbations.

Example 2. *The linear difference equation* $x_{n+1} = x_n + \lambda_n$ *has the forward solution* $x_n = x_0 + \sum_{k=0}^{n-1} \lambda_n$, *whose boundedness requires the assumption that the real sequence* $(\lambda_n)_{n\geq 0}$ *is summable.*
Thus, the nonhyperbolic equilibria x^* *of* $x_{n+1} = x_n$ *do not necessarily persist as bounded entire solutions under arbitrary bounded parametric perturbations.*

Typical examples of nonautonomous equations having an equilibrium given by the trivial solution are equations of perturbed motion. Their variational equation along $(\chi_n)_{n\in\mathbb{Z}}$ is given by

$$x_{n+1} = D_1 g(\chi_n, \lambda) x_n.$$

Investigation of the behaviour of the trivial solution under variation of λ requires an appropriate nonautonomous concept of hyperbolicity.

Suppose that $A_n \in \mathbb{R}^{d\times d}$, $n \in \mathbb{Z}$, is a sequence of invertible matrices and consider a linear difference equation

$$x_{n+1} = A_n x_n \tag{2}$$

with the transition matrix

$$\Phi(n,l) := \begin{cases} A_{n-1}\cdots A_l, & l < n, \\ I, & n = 0, \\ A_n^{-1}\cdots A_{l-1}, & n < l. \end{cases}$$

Let $\mathbb{I}$ be a discrete interval and define $\mathbb{I}' := \{k \in \mathbb{I} :\ k+1 \in \mathbb{I}\}$.

An invariant projector for (2) is a sequence $P_n \in \mathbb{R}^{d\times d}$, $n \in \mathbb{I}$, of projections $P_n = P_n^2$ such that

$$A_{n+1}P_n = P_n A_n \quad \text{for all } n \in \mathbb{I}'.$$

Definition 1. *A linear difference equation* (2) *is said to admit an exponential dichotomy on* $\mathbb{I}$ *if there exists an invariant projector* P_n *and constants* $K \geq 0$, $\alpha \in (0,1)$ *such that for all* $n, l \in \mathbb{I}$

$$\|\Phi(n,l)P_l\| \leq K\alpha^{n-l} \quad \text{if } l \leq n,$$

$$\|\Phi(n,l)[\mathrm{id} - P_l]\| \leq K\alpha^{l-n} \quad \text{if } n \leq l.$$

Remark 1. *An autonomous difference equation* $x_{n+1} = Ax_n$ *has an exponential dichotomy if and only if the coefficient matrix* $A \in \mathbb{R}^{d\times d}$ *has no eigenvalues on the complex unit circle.*

An entire solution $(\chi_n)_{n\in\mathbb{Z}}$ of (Δ_λ) is called hyperbolic if the variational equation

$$x_{n+1} = D_1 f(\chi_n, \lambda) x_n \qquad (V_\lambda)$$

has an exponential dichotomy on $\mathbb{Z}$.

Let ℓ^∞ denote the space of bounded sequences in $\mathbb{R}^d$ with the supremum norm.

Theorem 1 (Continuation of entire solutions)**.** *If $\chi^* = (\chi_n^*)_{n\in\mathbb{Z}}$ is an entire bounded and hyperbolic solution of (Δ_{λ^*}), then there exists an open neighborhood $\Lambda_0 \subseteq \Lambda$ of λ^* and a unique function $\chi : \Lambda_0 \to \ell^\infty$ such that*

i) $\chi(\lambda^*) = \chi^*$,

ii) each $\chi(\lambda)$ is a bounded entire and hyperbolic solution of (Δ_λ),

iii) $\chi : \Lambda_0 \to \ell^\infty$ is as smooth as the functions f_n.

The proof formulates the difference equation (Δ_λ) as abstract equation $F(\chi, \lambda) = 0$ in the space ℓ^∞.

This is solved using the implicit mapping theorem, where the invertibility of the Fréchet derivative $D_1F(\chi^*, \lambda^*)$ is characterised by the hyperbolicity assumption on χ^*.

- Sufficient conditions for the occurence of a bifurcation require the hyperbolicity of χ^* to be violated.

The following characterisation of an exponential dichotomy is useful here.

Theorem 2. *A variational equation* (V_λ) *has an exponential dichotomy on* $\mathbb{Z}$ *if and only if the following conditions are fulfilled:*

i) (V_λ) *has an exponential dichotomy on* $\mathbb{Z}^+$ *with projector* P_n^+ *as well as an exponential dichotomy on* $\mathbb{Z}^-$ *with projector* P_n^-*,*

ii) $R(P_0^+) \oplus N(P_0^-) = \mathbb{R}^d$.

The following examples illustrate various scenarios how the conditions of Theorem 2 can be violated.

Example 3 (Pitchfork bifurcation). *The difference equation*

$$x_{n+1} = f_n(x_n, \lambda), \qquad f_n(x, \lambda) := \frac{\lambda x}{1 + |x|}.$$

is a prototype example for an <u>*autonomous pitchfork bifurcation*</u>*, where the unique asymptotically stable equilibrium* $x^* = 0$ *for* $\lambda \in (0, 1)$ *bifurcates into two asymptotically stable equilibria* $x_\pm := \pm(\lambda - 1)$ *for* $\lambda > 1$*.*

The variational equation $x_{n+1} = \lambda x_n$ *of the trivial solution becomes* <u>*nonhyperbolic*</u> *for* $\lambda = 1$*.*

Criterion (i) of Theorem 2 is <u>*violated*</u>*, since the variational equation does not admit a dichotomy on either* $\mathbb{Z}^+$ *nor* $\mathbb{Z}^-$*.*

This <u>*loss of hyperbolicity*</u> *causes an attractor bifurcation, since for*

- $\lambda \in (0, 1)$ *the set* $x^* = 0$ *is the* <u>*global attractor.*</u>

- $\lambda > 1$ *the trivial equilibrium* $x^* = 0$ <u>*becomes unstable*</u> *and the interval* $A = [x_-, x_+]$ *is the* <u>*global attractor.*</u>

Bifurcations of pullback attractors can occur as nonautonomous versions of pitchfork bifurcations.

Example 4 (Pullback attractor bifurcation)**.** *Consider the difference equation*

$$x_{n+1} = \lambda x_n - \begin{cases} \min\left\{a_n x_n^3, \frac{\lambda}{2} x_n\right\}, & x_n \geq 0, \\ \\ \max\left\{a_n x_n^3, \frac{\lambda}{2} x_n\right\}, & x_n < 0, \end{cases}$$

for parameter values $\lambda > 0$, where the sequence $(a_n)_{n\in\mathbb{Z}}$ is bounded and bounded away from zero.

There are neighbourhoods U and $V \supset U$ of 0 (independent of λ near $\lambda = 1$) such that in U the difference equation is given by $x_{n+1} = \lambda x_n - a_n x_n^3$, while outside of a set V it is given by $x_{n+1} = \frac{\lambda}{2} x_n$.

Moreover, the right-hand side lies between the functions $x \mapsto \frac{\lambda}{2} x$ and $x \mapsto \lambda x$ for fixed $n \in \mathbb{Z}$.

For $\lambda \in (0, 1)$ the global pullback attractor is given by the trivial solution, since points are contracted at each time step by the factor λ.

For $\lambda > 1$ the trivial solution is no longer attractive, but there exists a nontrivial pullback attractor for $\lambda \in (1, 2)$, since $\mathcal{B} = \{V : n \in \mathbb{Z}\}$ is pullback absorbing family.

The global pullback attractor changes its dimension at the parameter value $\lambda = 1$.

- *This is an example of a nonautonomous pitchfork bifurcation.*

In contrast, the following scenario is intrinsically nonautonomous.

Example 5 (shovel bifurcation)**.** *Consider the scalar difference equation*

$$x_{n+1} = a_n(\lambda)x_n, \qquad a_n(\lambda) := \begin{cases} \frac{1}{2} + \lambda, & n < 0, \\ \lambda, & n \geq 0, \end{cases} \tag{3}$$

with parameter $\lambda > 0$.

In order to understand the dynamics of (3)*, distinguish three cases:*

i) $\lambda \in (0, \frac{1}{2})$*: The equation* (3) *has an exponential dichotomy on* $\mathbb{Z}$ *with projector* $P_n \equiv 1$ *and the uniquely determined bounded entire solution is the trivial one;* (3) *is uniformly asymptotically stable.*

ii) $\lambda > 1$*: The equation* (3) *has an exponential dichotomy on* $\mathbb{Z}$ *with projector* $P_n \equiv 0$*. Again,* 0 *is the unique bounded entire solution;* (3) *is unstable.*

iii) $\lambda \in (\frac{1}{2}, 1)$*: In this situation,* (3) *has an exponential dichotomy on* $\mathbb{Z}^+$ *with projector* $P_n^+ \equiv 1$*, as well as an exponential dichotomy on* $\mathbb{Z}^-$ *with projector* $P_n^- = 0$.

Thus, condition (iii) in Theorem 2 is violated and 0 *is a nonhyperbolic solution. For this parameter regime, every solution of* (3) *is bounded. Moreover,* (3) *is asymptotically stable, but not uniformly asymptotically stable on the whole time axis* $\mathbb{Z}$.

- *The parameter values $\lambda \in \{\frac{1}{2}, 1\}$ are critical.*

In both situations, the number of bounded entire solutions to the linear difference equation (3) changes drastically.

Furthermore, there is a loss of stability in two steps: from uniformly asymptotically stable to asymptotically stable, and finally to unstable, as λ increases through the values $\frac{1}{2}$ and 1.

Hence, both values can be considered as bifurcation values, since the number of bounded entire solutions changes as well as their stability properties.

The next example requires the state space to be at least two-dimensional.

Example 6. *(Fold solution bifurcation) Consider the planar equation*

$$x_{n+1} = f_n(x_n, \lambda) := \begin{pmatrix} b_n & 0 \\ 0 & c_n \end{pmatrix} x_n + \begin{pmatrix} 0 \\ (x_n^1)^2 \end{pmatrix} - \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

with components $x_n = (x_n^1, x_n^2)$*, which depends on a parameter* $\lambda \in \mathbb{R}$ *and asymptotically constant sequences*

$$b_n := \begin{cases} 2, & n < 0, \\ \frac{1}{2}, & n \geq 0, \end{cases} \qquad c_n := \begin{cases} \frac{1}{2}, & n < 0, \\ 2, & n \geq 0. \end{cases} \tag{5}$$

The variational equation for (4) *corresponding to the trivial solution and the parameter* $\lambda^* = 0$ *reads*

$$x_{n+1} = D_1 f_n(0,0) x_n := \begin{pmatrix} b_n & 0 \\ 0 & c_n \end{pmatrix} x_n.$$

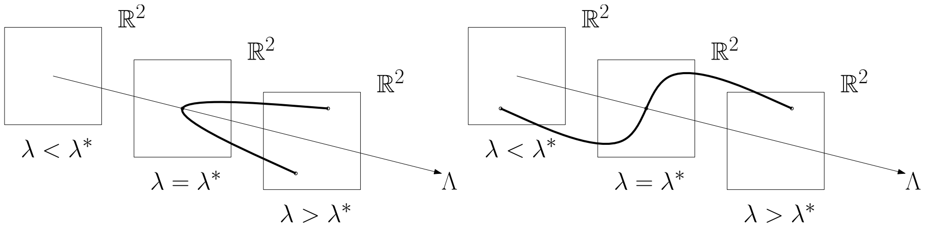

Figure 1: Left (supercritical fold): Initial values $\eta \in \mathbb{R}^2$ yielding a bounded solution $\phi_\lambda(\cdot; 0, \eta)$ of (4) for different parameter values λ.
Right (cusp): Initial values $\eta \in \mathbb{R}^2$ yielding a bounded solution $\phi_\lambda(\cdot; 0, \eta)$ of (7) for different parameter values λ

It admits an <u>*exponential dichotomy*</u> *on* $\mathbb{Z}^+$ *as well as on* $\mathbb{Z}^-$ *with corresponding invariant projectors*

$$P_n^+ \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_n^- \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This yields

$$R(P_0^+) \cap N(P_0^-) = \mathbb{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad R(P_0^+) + N(P_0^-) = \mathbb{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore condition (ii) of Theorem 2 is violated and the trivial solution to (4) *for* $\lambda = 0$ *is* <u>*not hyperbolic*</u>.

Let $\phi_\lambda(\cdot, 0, \eta)$ be the general solution to (4). Its first component ϕ_λ^1 is

$$\phi_\lambda^1(n, 0, \eta) = 2^{-|n|}\eta_1 \quad \textit{for all } n \in \mathbb{Z}, \tag{6}$$

while the variation of constants formula can be used to deduce the asymptotic representation

$$\phi_\lambda^2(n, 0, \eta) = \begin{cases} 2^n\left(\eta_2 + \frac{4}{7}\eta_1^2 - \lambda\right) + O(1), & n \to \infty, \\ \\ \frac{1}{2^n}\left(\eta_2 - \frac{1}{2}\eta_1^2 + 2\lambda\right) + O(1), & n \to -\infty. \end{cases}$$

Therefore, the sequence $\phi_\lambda(\cdot; 0, \eta)$ is bounded if and only if $\eta_2 = -\frac{4}{7}\eta_1^2 + \lambda$ and $\eta_2 = \frac{1}{2}\eta_1^2 - 2\lambda$ holds, i.e., $\eta_1^2 = \frac{7}{2}\lambda$, $\eta_2 = -\lambda$.

From the first relation, one sees that there exist two bounded solutions if $\lambda > 0$, the trivial solution is the unique bounded solution for $\lambda = 0$, and there are no bounded solutions for $\lambda < 0$.

See Figure 1 (left) for an illustration.

- *One can interpret $\lambda = 0$ as bifurcation point, since the number of bounded entire solutions increases from 0 to 2 as λ increases through 0.*

The method of explicit solutions can also be applied to the nonlinear equation

$$x_{n+1} = f_n(x_n, \lambda) := \begin{pmatrix} b_n & 0 \\ 0 & c_n \end{pmatrix} x_n + \begin{pmatrix} 0 \\ (x_n^1)^3 \end{pmatrix} - \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7}$$

However, using the variation of constants formula, one can show that the crucial second component of the general solution $\phi_\lambda(\cdot; 0, \eta)$ *for* (7) *satisfies*

$$\phi_\lambda^2(n, 0, \eta) = \begin{cases} 2^n \left(\eta_2 + \frac{8}{15}\eta_1^3 - \lambda\right) + O(1), & n \to \infty, \\ \frac{1}{2^n}\left(\eta_2 - \frac{2}{15}\eta_1^3 + 2\lambda\right) + O(1), & n \to -\infty. \end{cases}$$

Since the first component is given in (6)*,* $\phi_\lambda(\cdot; 0, \eta)$ *is bounded if and only if* $\eta_2 = -\frac{8}{15}\eta_1^3 + \lambda$ *and* $\eta_2 = \frac{2}{15}\eta_1^3 - 2\lambda$*, which in turn is equivalent to*

$$\eta_1 = \sqrt[3]{\frac{9}{2}\lambda}, \qquad \eta_2 = -\frac{7}{5}\lambda.$$

Hence, these particular initial values $\eta \in \mathbb{R}^2$ *given by the cusp shaped curve depicted in Figure 1 (right) lead to bounded entire solutions of* (7)*.*

Attractor bifurcation

A general bifurcation pattern is presented here, which ensures that, under certain conditions on the Taylor coefficients, a pullback attractor changes qualitatively when a parameter is varied.

This generalises the autonomous pitchfork bifurcation pattern, but usually only yields results for a local pullback attractor.

Definition 2 (Local pullback attractor)**.** *Consider a process ϕ on a metric state space (X, d). A ϕ-invariant family $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called a local pullback attractor if there exists an $\eta > 0$ such that*

$$\lim_{k\to\infty} \operatorname{dist}\big(\phi(n, n-k, B_\eta(A_{n-k})), A_n\big) = 0 \quad \textit{for all } n \in \mathbb{Z}\,.$$

The basin of attraction is chosen as a neighborhood of the local pullback attractor.

Suppose that (Δ_λ) is a scalar equation ($d = 1$) with the trivial solution for all parameters λ from an interval $\Lambda \subseteq \mathbb{R}$.

Let $\Phi_\lambda(n, l) \in \mathbb{R}$ denote the transition matrix of the corresponding variational equation

$$x_{n+1} = D_1 f_n(0, \lambda) x_n.$$

When dealing with attractor bifurcations, the hyperbolicity condition (i) in Theorem 2 will be violated.

- This provides a nonautonomous counterpart to the classical pitchfork bifurcation pattern.

Theorem 3 (Nonautonomous pitchfork bifurcation)**.** *Suppose that $f_n(\cdot,\lambda) : \mathbb{R} \to \mathbb{R}$ is invertible and of class C^4 with*

$$D_1^2 f_n(0,\lambda) = 0 \quad \textit{for all } n \in \mathbb{Z} \textit{ and } \lambda \in \Lambda.$$

Suppose there exists a $\lambda^* \in \mathbb{R}$ such that the following hypotheses hold.

- *Hypothesis on linear part: There exists a $K \geq 1$ and functions $\beta_1, \beta_2 : \Lambda \to (0,\infty)$ which are either both increasing or decreasing with $\lim_{\lambda\to\lambda^*} b_1(\lambda) = \lim_{\lambda\to\lambda^*} b_2(\lambda) = 1$ and*

$$\Phi_\lambda(n,l) \leq K\beta_1(\lambda)^{n-l} \quad \textit{for all } l \leq n,$$

$$\Phi_\lambda(n,l) \leq K\beta_2(\lambda)^{n-l} \quad \textit{for all } n \leq l$$

  and all $\lambda \in \Lambda$.

- *Hypothesis on nonlinearity: Assume that if the functions β_1 and β_2 are increasing, then*

$$-\infty < \liminf_{\lambda\to\lambda^*} \inf_{n\in\mathbb{Z}} D_1^2 f_n(0,\lambda) \leq \limsup_{\lambda\to\lambda^*} \sup_{n\in\mathbb{Z}} D_1^3 f_n(0,\lambda) < 0,$$

  and otherwise (i.e., if the functions β_1 and β_2 are decreasing) that

$$0 < \liminf_{\lambda\to\lambda^*} \inf_{n\in\mathbb{Z}} D_1^3 f_n(0,\lambda) \leq \limsup_{\lambda\to\lambda^*} \sup_{n\in\mathbb{Z}} D_1^2 f_n(0,\lambda) < \infty.$$

  Moreover, suppose that the remainder satisfies

$$\lim_{x\to 0} \sup_{\lambda\in(\lambda^*-x^2,\lambda^*+x^2)} \sup_{n\leq 0} x \int_0^1 (1-t)^3 D^4 f_n(tx,\lambda)\,dt = 0,$$

$$\limsup_{\lambda\to\lambda^*} \limsup_{x\to 0} \sup_{n\leq 0} \frac{Kx^3}{1-\min\{\beta_1(\lambda), \beta_2(\lambda)^{-1}\}} \int_0^1 (1-t)^3 D^4 f_n(tx,\lambda)\,dt < 3.$$

Then there exist $\lambda_- < \lambda^* < \lambda_+$ such that the following statements hold:

1) *If the functions β_1 and β_2 are increasing, the trivial solution is a local pullback attractor for $\lambda \in (\lambda_-, \lambda^*)$, which bifurcates to a nontrivial local pullback attractor $\{A_n^\lambda : n \in \mathbb{Z}\}$, $\lambda \in (\lambda^*, \lambda^+)$ with*

$$\lim_{\lambda \to \lambda^*} \sup_{n \leq 0} \mathrm{dist}(A_n^\lambda, \{0\}) = 0.$$

2) *If the functions β_1 and β_2 are decreasing, the trivial solution is a local pullback attractor for $\lambda \in (\lambda^*, \lambda^+)$, which bifurcates to a nontrivial local pullback attractor $\{A_n^\lambda : n \in \mathbb{Z}\}$, $\lambda \in (\lambda_-, \lambda^*)$ with*

$$\lim_{\lambda \to \lambda^*} \sup_{n \leq 0} \mathrm{dist}(A_n^\lambda, \{0\}) = 0.$$

Example 7. *Consider the nonautonomous difference equation*

$$x_{n+1} = \frac{\lambda x_n}{1 + \frac{b_n q}{\lambda} x_n^q}\,, \tag{8}$$

where $q \in \mathbb{N}$ and the sequence $(b_n)_{n\in\mathbb{N}}$ is positive and both bounded and bounded away from zero.

For $q = 1$, this difference equation can be transformed into the well-known Beverton–Holt equation, which describes the density of a population in a fluctuating environment. In this case, the system admits a nonautonomous transcritical bifurcation.

For $q = 2$, a nonautonomous pitchfork bifurcation occurs. The above Theorem can be applied, because the Taylor expansion of the right-hand side of (8) reads as

$$\lambda x_n + b_n x_n^{q+1} + O(x^{2q+1}),$$

and the remainder fulfills the conditions of the Theorem.

This means that for $\lambda \in (0,1)$, the trivial solution is a local pullback attractor, which undergoes a transition to a nontrivial local pullback attractor when $\lambda > 1$.

- *Note that the extreme solutions of the nontrivial local pullback attractor for $\lambda > 1$ are also local pullback attractors, which gives the interpretation of this bifurcation as a bifurcation of locally pullback attractive solutions.*

Solution bifurcation

In the previous section on attractor bifurcations the first hyperbolicity condition (i) in Theorem 2, given by exponential dichotomies on both semiaxes, was violated.

The present concept of solution bifurcation is based on the assumption that merely condition (ii) of Theorem 2 does not hold.

The variational difference equation (V_λ) is then intrinsically nonautonomous.

Indeed, if (V_λ) is almost periodic, then an exponential dichotomy on a semiaxis extends to the whole integer axis and the reference solution $\chi = (\chi_n)_{n\in\mathbb{Z}}$ becomes hyperbolic. For this reason the following bifurcation scenarios cannot occur for periodic or autonomous difference equations.

Hypothesis: *Suppose the variational equation (V_λ) admits an ED both on $\mathbb{Z}^+$ (with projector P_n^+) and on $\mathbb{Z}^-$ (with projector P_n^-) such that there exists nonzero vectors $\xi_1 \in \mathbb{R}^d$, $\xi_1' \in \mathbb{R}^d$ satisfying*

$$R(P_0^+) \cap N(P_0^-) = \mathbb{R}\xi_1, \qquad (R(P_0^+) + N(P_0^-))^\perp = \mathbb{R}\xi_1'. \tag{9}$$

Then solution bifurcation is understood as follows: Suppose that for a fixed parameter $\lambda^* \in \Lambda$, a difference equation (Δ_{λ^*}) has an entire bounded reference solution $\chi^* = \chi(\lambda^*)$.

One says that (Δ_λ) undergoes a bifurcation at $\lambda = \lambda^*$ along χ^*, or χ^* bifurcates at λ^*, if there exists a convergent parameter sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ with limit λ^* so that (Δ_{λ_n}) has two distinct entire solutions $\chi^1_{\lambda_n}, \chi^2_{\lambda_n} \in \ell^\infty$ satisfying

$$\lim_{n\to\infty} \chi^1_{\lambda_n} = \lim_{n\to\infty} \chi^2_{\lambda_n} = \chi^*.$$

The above Hypothesis allows a geometrical insight into the following abstract bifurcation results using invariant fiber bundles, i.e., nonautonomous counterparts to invariant manifolds:

Because (V_λ) *has an exponential dichotomy on* $\mathbb{Z}^+$*, there exists a stable fiber bundle* $\chi^* + \mathcal{W}^+_\lambda$ *consisting of all solutions to* (Δ_λ) *approaching* χ^* *in forward time.*

Here, $\mathcal{W}^+_\lambda$ is locally a graph over the stable vector bundle $\{R(P^+_n) : n \in \mathbb{Z}^+\}$.

Analogously, the dichotomy on $\mathbb{Z}^-$ guarantees an unstable fiber bundle $\chi^* + \mathcal{W}^-_\lambda$ consisting of solutions decaying to χ^* in backward time.

Then the <u>bounded entire solutions</u> to (Δ_λ) are contained in the intersection $(\chi^* + \mathcal{W}_\lambda^+) \cap (\chi^* + \mathcal{W}_\lambda^-)$. One concludes that the intersection of the fibers

$$S_\lambda := \chi_0^* + \mathcal{W}_{\lambda,0}^+ \cap \chi_0^* + \mathcal{W}_{\lambda,0}^- \subseteq \mathbb{R}^d$$

yields initial values for bounded entire solutions (see Figure 2).

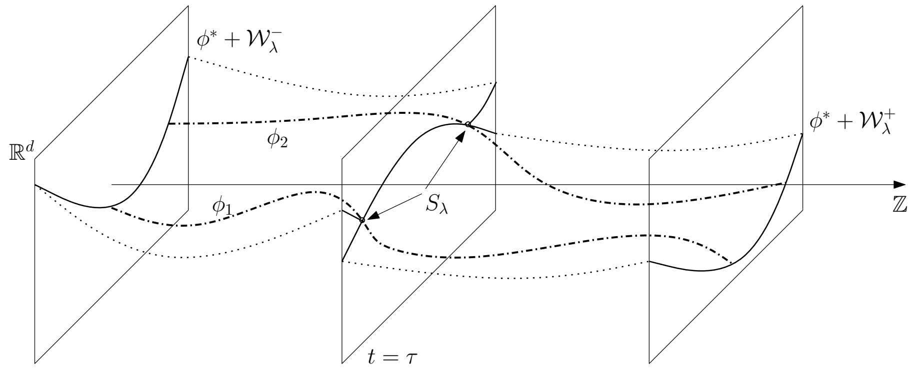

Figure 2: Intersection $S_\lambda \subseteq \mathbb{R}^d$ of the stable fiber bundle $\phi^* + \mathcal{W}_\lambda^+ \subseteq \mathbb{Z}^+ \times \mathbb{R}^d$ with the unstable fiber bundle $\phi^* + \mathcal{W}_\lambda^- \subseteq \mathbb{Z}^- \times \mathbb{R}^d$ at time $k = 0$ yields two bounded entire solutions ϕ_1, ϕ_2 to (Δ_λ) indicated as dotted dashed lines

It can be assumed using the equation of perturbed motion that $\chi^* = 0$. Suppose, in addition, that

$$f_n(0, \lambda) \equiv 0 \quad \text{on } \mathbb{Z},$$

which means that (Δ_λ) has the trivial solution for all $\lambda \in \Lambda$.

This yields the corresponding variational equation

$$x_{n+1} = D_1 f_n(0, \lambda) x_n$$

with transition matrix $\Phi_\lambda(n, l) \in \mathbb{R}^{d \times d}$.

Theorem 4 (Bifurcation from known solutions)**.** *Let $\Lambda \subseteq \mathbb{R}$ and suppose f_n is of class C^m, $m \geq 2$. If the transversality condition*

$$g_{11} := \sum_{n \in \mathbb{Z}} \langle \Phi_{\lambda^*}(0, n+1)' \xi_1', D_1 D_2 f_n(0, \lambda^*) \Phi_{\lambda^*}(n, 0) \xi_1 \rangle \neq 0 \tag{10}$$

is satisfied, then the trivial solution of a difference equation (Δ_λ) bifurcates at λ^*.

In particular, there exist a $\rho > 0$, open convex neighborhoods $U \subseteq \ell^\infty(\Omega)$ of 0, a subset $\Lambda_0 \subseteq \Lambda$ of λ^* and a C^{m-1}-function $\psi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ with

1) *$\psi(0) = 0$, $\lambda(0) = \lambda^*$ and $\dot{\psi}(0) = \Phi_{\lambda^*}(\cdot, 0)\xi_1$,*

2) *each $\psi(s)$ is a nontrivial solution of $(\Delta)_{\lambda(s)}$ homoclinic to 0, i.e.,*

$$\lim_{n \to \pm\infty} \psi(s)_n = 0.$$

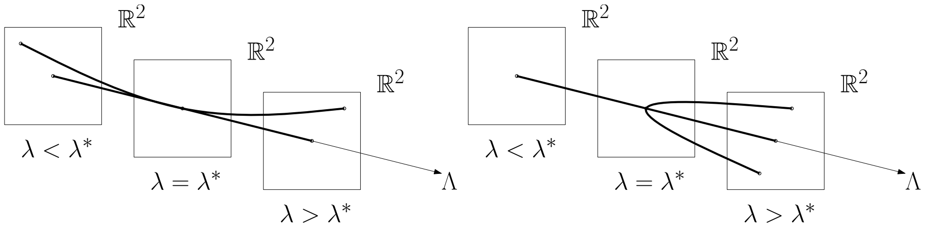

Figure 3: Left (transcritical): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\phi_\lambda(\cdot; 0, \eta)$ of (11) for different parameter values λ.
Right (supercritical pitchfork): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\phi_\lambda(\cdot; 0, \eta)$ of (12) for different parameter values λ

Corollary 1 (Transcritical bifurcation)**.** *Under the additional assumption*

$$g_{20} := \sum_{n \in \mathbb{Z}} \langle \Phi_{\lambda^*}(0, n+1)'\xi_1', D_1^2 f_n(0, \lambda^*)[\Phi_{\lambda^*}(n, 0)\xi_1]^2 \rangle \neq 0$$

one has $\dot{\lambda}(0) = -\dfrac{g_{20}}{2g_{11}}$ *and the following holds locally in* $U \times \Lambda_0$*:*

A difference equation (Δ_λ) *has a unique nontrivial entire bounded solution* ψ_λ *for* $\lambda \neq \lambda^*$ *and* 0 *is the* <u>*unique*</u> *entire bounded solution of* $(\Delta)_{\lambda^*}$*.*
Moreover, ψ_λ *is* <u>*homoclinic*</u> *to* 0*.*

Corollary 2 (Pitchfork bifurcation)**.** *For $m \geq 3$ and the additional assumptions*

$$\sum_{n\in\mathbb{Z}} \langle \Phi_{\lambda^*}(0,n+1)'\xi_1', D_1^2 f_n(0,\lambda^*)[\Phi_{\lambda^*}(n,0)\xi_1]^2\rangle = 0,$$
$$g_{30} := \sum_{n\in\mathbb{Z}} \langle \Phi_{\lambda^*}(0,n+1)'\xi_1', D_1^3 f_n(0,\lambda^*)[\Phi_{\lambda^*}(n,0)\xi_1]^3\rangle \neq 0$$

one has $\dot{\lambda}(0) = 0$, $\ddot{\lambda}(0) = -\frac{g_{30}}{3g_{11}}$ and the following holds locally in $U \times \Lambda_0$:

3) Subcritical case: If $g_{30}/g_{11} > 0$, then the unique entire bounded solution of (Δ_λ) is the trivial one for $\lambda \geq \lambda^*$ and (Δ_λ) has exactly two nontrivial entire solutions for $\lambda < \lambda^*$; both are homoclinic to 0.

4) Supercritical case: If $\frac{g_{30}}{g_{11}} < 0$, then the unique entire bounded solution of (Δ_λ) is the trivial one for $\lambda \leq \lambda^*$ and (Δ_λ) has exactly two nontrivial entire solutions for $\lambda > \lambda^*$; both are homoclinic to 0.

Example 8. *Consider the nonlinear difference equation*

$$x_{n+1} = f_n(x_n, \lambda) := \begin{pmatrix} b_n & 0 \\ \lambda & c_n \end{pmatrix} x_n + \begin{pmatrix} 0 \\ (x_n^1)^2 \end{pmatrix} \tag{11}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ *and sequences* b_n, c_n *defined in* (5).

As in the previous examples, the assumptions hold with $\lambda^* = 0$ *and*

$$g_{11} = \frac{4}{3} \neq 0, \qquad g_{20} = \frac{12}{7} \neq 0.$$

By Corollary 1 the trivial solution of (11) *has a transcritical bifurcation at* $\lambda = 0$.

The first component of the general solution $\phi_\lambda(\cdot; 0, \eta)$ *given by* (6) *is homoclinic, the second component fulfills*

$$\phi_\lambda^2(n; 0, \eta) = \begin{cases} 2^n \left(\eta_2 + \frac{4}{7}\eta_1^2 + \frac{2\lambda}{3}\eta_1\right) + o(1), & n \to \infty, \\ 2^{-n} \left(\eta_2 - \frac{2}{7}\eta_1^2 - \frac{2\lambda}{3}\eta_1\right) + o(1), & n \to -\infty. \end{cases}$$

One sees that $\phi_\lambda(\cdot; 0, \eta)$ *is bounded if and only if* $\eta = (0, 0)$ *or*

$$\eta_1 = -\frac{14}{9}\lambda, \qquad \eta_2 = \frac{28}{81}\lambda^2.$$

Hence, besides the zero solution there is a unique nontrivial entire solution passing through the initial point $\eta = (\eta_1, \eta_2)$ *at time* $n = 0$ *for* $\lambda \neq 0$.

This means the solution bifurcation pattern sketched in Figure 3 (left) holds.

Example 9. *Let δ be a fixed nonzero real number and consider the nonlinear difference equation*

$$x_{n+1} = f_n(x_n, \lambda) := \begin{pmatrix} b_n & 0 \\ \lambda & c_n \end{pmatrix} x_n + \delta \begin{pmatrix} 0 \\ (x_n^1)^3 \end{pmatrix} \tag{12}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and the b_n, c_n defined in (5).

As in Example 8, the assumptions of Corollary 2 are fulfilled with $\lambda^* = 0$.

The transversality condition reads as $g_{11} = \frac{4}{3} \neq 0$.

Moreover, $D_1^2 f_n(0,0) \equiv 0$ on $\mathbb{Z}$ implies $g_{20} = 0$, whereas the relation

$$D_1^3 f_n(0,0)\zeta^3 = \begin{pmatrix} 0 \\ 6\delta\zeta_1^3 \end{pmatrix}$$

for all $n \in \mathbb{Z}$, $\zeta \in \mathbb{R}^2$ leads to $g_{30} = 4\delta \neq 0$.

This gives the critical quotient $\dfrac{g_{30}}{g_{11}} = 3\delta$.

By Corollary 2 one deduces a subcritical (resp., supercritical) pitchfork bifurcation of the trivial solution to (12) *at* $\lambda = 0$*, provided* $\delta > 0$ *(resp.,* $\delta < 0$*).*

As above, the first component of the general solution $\phi_\lambda(\cdot; 0, \eta)$ *to* (12) *is given by* (6)*, which helps to compute for the second component that*

$$\phi_\lambda^2(n; 0, \eta) = \begin{cases} 2^n \left(\eta_2 + \frac{8\delta}{15}\eta_1^3 + \frac{2\lambda}{3}\eta_1\right) + o(1), & n \to \infty, \\ 2^{-n} \left(\eta_2 - \frac{2\delta}{15}\eta_1^3 - \frac{4\lambda}{3}\eta_1\right) + o(1), & n \to -\infty. \end{cases}$$

This asymptotic representation shows that $\phi_\lambda(\cdot; 0, \eta)$ *is homoclinic to* 0 *if and only if* $\eta = 0$ *or*

$$\eta_1^2 = -\frac{2}{\delta}\lambda, \qquad \eta_2 = \frac{4}{15}\frac{(5\delta + 16\lambda)}{\delta^2}\lambda^2.$$

There is a correspondence to the pitchfork solution bifurcation in Corollary 2.

See Figure 3 (right).

Lecture 6: Discrete-time random dynamical systems

Random dynamical systems on a state space X are nonautonomous by the very nature of the driving noise.

They can be formulated as skew-product systems with the driving system acting a probability sample space Ω rather than on a topological or metric parameter space P.

A major difference is that only measurability and not continuity w.r.t. the parameter can be assumed, which changes the types of results that can be proved.

In particular, the skew-product system does not form an autonomous semidynamical system on the product space $\Omega \times X$.

Nevertheless, there are many interesting parallels with the theory of deterministic nonautonomous dynamical systems.

Random difference equations

Let (X, d) be a complete metric space and consider a mapping $g : \Xi \times X \to X$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\xi_n, n \in \mathbb{Z}\}$ be a discrete-time stochastic process taking values in some space Ξ, i.e., a sequence of random variables or, equivalently, $\mathcal{F}$-measurable mappings $\xi_n : \Omega \to \Xi$ for $n \in \mathbb{Z}$.

The equation

$$\boxed{x_{n+1}(\omega) = g\left(\xi_n(\omega), x_n(\omega)\right) \quad \text{for all } n \in \mathbb{Z},\ \omega \in \Omega, \quad (1)}$$

is a random difference equation on X driven by the stochastic process ξ_n.

Greater generality is achieved by representing the driving noise process by a metrical dynamical system (i.e., measure theoretic) θ on some canonical sample space Ω, i.e., a group of $\mathcal{F}$-measurable mappings $\{\theta_n, n \in \mathbb{Z}\}$ under composition formed by iterating a measurable mapping $\theta : \Omega \to \Omega$ and its measurable inverse mapping θ^{-1} $: \Omega \to \Omega$, i.e., with $\theta_0 = \mathrm{id}_\Omega$ and

$$\theta_{n+1} := \theta \circ \theta_n, \qquad \theta_{-n-1} := \theta^{-1} \circ \theta_{-n} \quad \text{for all } n \in \mathbb{N},$$

where $\theta_{-1} := \theta^{-1}$.

It is usually assumed that θ generates an ergodic process on Ω.

Let $f : \Omega \times X \to X$ be $\mathcal{F} \times \mathcal{B}(X)$-measurable, where $\mathcal{B}(X)$ is the Borel σ-algebra on X.

Then the random difference equation has the form

$$\boxed{x_{n+1}(\omega) = f\left(\theta^n(\omega), x_n(\omega)\right) \quad \text{for all } n \in \mathbb{Z},\ \omega \in \Omega. \quad (2)}$$

Define recursively a solution mapping $\varphi : \mathbb{Z}^+ \times \Omega \times X \to X$ for the random difference equation (2) by $\varphi(0, \omega, x) := x$ and

$$\varphi(n+1, \omega, x) = f(\theta^n(\omega), \varphi(n, \omega, x)) \quad \text{for all } n \in \mathbb{N},\ x \in X$$

and $\omega \in \Omega$.

Then, φ satisfies the discrete-time cocycle property w.r.t. θ, i.e.,

$$\varphi(n+m, \omega, x) = \varphi\left(n, \theta_m(\omega), \varphi(m, \omega, x_0)\right) \quad \text{for all } m, n \in \mathbb{Z}^+,$$

for $x \in X$ and $\omega \in \Omega$. The mapping φ is called a cocycle mapping.

The random difference equation (2) generates a discrete-time random dynamical cocycle mapping φ on the state space X.

Definition 1. *A (discrete-time) random dynamical system* (θ, φ) *on* $\Omega \times X$ *consists of a metrical dynamical system* θ *on* Ω*, i.e., a group of measure preserving mappings* $\theta_n : \Omega \to \Omega$*,* $n \in \mathbb{Z}$*, such that*

i) $\theta_0 = id_\Omega$ *and* $\theta_n \circ \theta_m = \theta_{n+m}$ *for all* n*,* $m \in \mathbb{Z}$*,*

ii) the map $\omega \mapsto \theta_n(\omega)$ *is measurable and invariant w.r.t.* $\mathbb{P}$ *in the sense that* $\theta_n(\mathbb{P}) = \mathbb{P}$ *for each* $n \in \mathbb{Z}$*,*

and a cocycle mapping $\varphi : \mathbb{Z}^+ \times \Omega \times X \to X$ *such that*

a) $\varphi(0, \omega, x_0) = \varphi_0$ *for all* $x_0 \in X$ *and* $\omega \in \Omega$*,*

b) $\varphi(n + m, \omega, x_0) = \varphi\left(n, \theta_m(\omega), \varphi(m, \omega, x_0)\right)$ *for all* n*,*$m \in \mathbb{Z}^+$*,* $x_0 \in X$ *and* $\omega \in \Omega$*,*

c) $(x_0) \mapsto \varphi(n, \omega, x_0)$ *is continuous for each* $(n, \omega) \in \mathbb{Z}^+ \times \Omega$*,*

d) $\omega \mapsto \varphi(n, \omega, x_0)$ *is* $\mathcal{F}$*-measurable for all* $(n, x_0) \in \mathbb{Z}^+ \times X$*.*

The notation $\theta_n(\mathbb{P}) = \mathbb{P}$ for the measure preserving property of θ_n w.r.t. $\mathbb{P}$ is just a compact way of writing

$$\mathbb{P}(\theta_n(A)) = \mathbb{P}(A) \quad \text{for all } n \in \mathbb{Z}, A \in \mathcal{F}\,.$$

- Note that $\pi = (\theta, \phi)$ has a skew-product structure on $\Omega \times X$, but it is not an autonomous semidynamical system on $\Omega \times X$ since no topological structure is assumed on Ω.

Random attractors

A systematic treatment of the theory of random dynamical systems, both continuous and discrete-time, is presented in the book by Ludwig Arnold.

Unlike a deterministic skew-product system, a random dynamical system (θ, φ) on $\Omega \times X$ is not an autonomous semidynamical system on $\Omega \times X$.

Nevertheless, skew-product deterministic systems and random dynamical systems have many analogous properties, and concepts and results for one can often be used with appropriate modifications for the other.

The most significant modification concerns measurability and the nonautonomous sets under consideration are random sets.

Let (X, d) be a Polish space, i.e., a complete and separable metric space.

Definition 2. *A family $\mathcal{D} = \{D_\omega, \omega \in \Omega\}$ of nonempty subsets of X is called a random set if the mapping $\omega \mapsto \mathrm{dist}(x, D_\omega)$ is $\mathcal{F}$-measurable for all $x \in X$.*

A random set $\mathcal{D}$ is called a random closed set if D_ω is closed for each $\omega \in \Omega$ and is called a random compact set if D_ω is compact for each $\omega \in \Omega$.

Random sets are called tempered if their growth w.r.t. the driving system θ is sub-exponential.

Definition 3. *A random set $\mathcal{D} = \{D_\omega, \omega \in \Omega\}$ in X is said to be tempered if there exists a $x_0 \in X$ such that*

$$D_\omega \subset \{x \in X \,:\, d(x, x_0) \leq r(\omega)\} \quad \textit{for all } \omega \in \Omega\,,$$

where the random variable $r(\omega) > 0$ is tempered, i.e.,

$$\sup_{n\in\mathbb{Z}} \{r(\theta_n(\omega))e^{-\gamma|n|}\} < \infty \quad \textit{for all } \omega \in \Omega,\ \gamma > 0.$$

The collection of all tempered random sets in X will be denoted by $\mathfrak{D}$.

A random attractor of a random dynamical system is a random set which is a pullback attractor in the pathwise sense w.r.t. the attracting basin of tempered random sets.

Definition 4. *A random closed set $\mathcal{A} = (A_\omega)_{\omega\in\Omega}$ from $\mathfrak{D}$ is called a random attractor of a random dynamical system (θ, φ) on $\Omega \times X$ in $\mathcal{D}$ if $\mathcal{A}$ is a φ-invariant set, i.e.,*

$$\varphi(n, \omega, A_\omega) = A_{\theta_n(\omega)} \quad \textit{for all } n \in \mathbb{Z}^+, \omega \in \Omega\,,$$

and pathwise pullback attracting in $\mathfrak{D}$, i.e.,

$$\lim_{n\to\infty} \operatorname{dist}\big(\varphi\big(n, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big), A_\omega\big) = 0 \quad \textit{for all } \omega \in \Omega,\ \mathcal{D} \in \mathfrak{D}.$$

If the random attractor consists of singleton sets, i.e., $A_\omega = \{Z^*(\omega)\}$ for some random variable Z^* with $Z^*(\omega) \in X$, then $\bar{Z}_n(\omega) := Z^*(\theta_n(\omega))$ is a stationary stochastic process on X.

The existence of a random attractor is ensured by that of a a pullback absorbing set.

The tempered random set $\mathcal{B} = \{B_\omega, \omega \in \Omega\}$ in the following theorem is called a pullback absorbing random set.

Theorem 1. *Let (θ, φ) be a random dynamical system on $\Omega \times X$ such that $\varphi(n, \omega, \cdot) : X \to X$ is a compact operator for each fixed $n > 0$ and $\omega \in \Omega$. Suppose there exists a tempered random set $\mathcal{B} = \{B_\omega, \omega \in \Omega\}$ with closed and bounded component sets and an $N_{\mathcal{D},\omega} \geq 0$ such that*

$$\varphi\big(t, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big) \subset B_\omega \quad \text{for all } n \geq N_{\mathcal{D},\omega}, \tag{3}$$

and every tempered random set $\mathcal{D} = \{D_\omega, \omega \in \Omega\}$.
Then the random dynamical system (θ, φ) has a unique random pullback attractor $\mathcal{A} = \{A_\omega, \omega \in \Omega\}$ with component sets defined by

$$A_\omega = \bigcap_{m>0} \overline{\bigcup_{n \geq m} \varphi(n, \theta_{-n}(\omega), B(\theta_{-n}(\omega))} \quad \text{for all } \omega \in \Omega. \tag{4}$$

The proof of Theorem 1 is essentially the same as its counterparts for deterministic skew-product.

The only new feature is that of measurability, i.e., to show that $\mathcal{A} = \{A_\omega), \omega \in \Omega\}$ is a random set.

This follows since the set-valued mappings $\omega \mapsto \varphi\big(n, \theta_{-n}(\omega), B(\theta_{-n}(\omega))\big)$ are measurable for each $n \in \mathbb{Z}^+$.

Arnold & Schmalfuß showed that a random attractor is also a forward attractor in the weaker sense of convergence in probability, i.e.,

$$\lim_{n\to\infty} \int_\Omega \operatorname{dist}\left(\varphi(n,\omega,D_\omega), A_{\theta_n(\omega)}\right) \, \mathbb{P}(d\omega) = 0$$

for all $\mathcal{D} \in \mathfrak{D}$.

This allows individual sample paths to have large deviations from the attractor, but for all paths to converge in this probabilistic sense.

Random Markov chains

Discrete-time finite state Markov chains with a tridiagonal structure are common in biological applications.

They have a transition matrix $[I_N + \Delta Q]$, where I_N is the $N \times N$ identity matrix and Q is the tridiagonal $N \times N$-matrix

$$Q = \begin{bmatrix} -q_1 & q_2 & & & & \bigcirc \\ q_1 & -(q_2+q_3) & q_4 & & & \\ & \ddots & \ddots \quad \ddots & \ddots & \ddots & \\ & & & q_{2N-5} & -(q_{2N-4}+q_{2N-3}) & q_{2N-2} \\ \bigcirc & & & & q_{2N-3} & -q_{2N-2} \end{bmatrix} \tag{5}$$

where the q_j are positive constants.

Such a Markov chain is a first order linear difference equation

$$\mathbf{p}^{(n+1)} = [I_N + \Delta Q]\,\mathbf{p}^{(n)} \tag{6}$$

on the probability simplex Σ_N in $\mathbb{R}^N$ defined by

$$\Sigma_N = \left\{ \mathbf{p} = (p_1, \cdots, p_N)^T : \sum_{j=1}^{N} p_j = 1,\ p_1, \ldots, p_N \in [0,1] \right\}.$$

The Perron-Frobenius theorem applies to the matrix $L_\Delta := I_N + \Delta Q$, provided $\Delta > 0$ is chosen sufficiently small.

In particular, the transition matrix has an eigenvalue $\lambda = 1$ and there is a positive eigenvector $\bar{\mathbf{x}}$, which can be normalizsed (in the $\|\cdot\|_1$ norm) to give a probability vector $\bar{\mathbf{p}}$, i.e., $[I_N + \Delta Q]\bar{\mathbf{p}} = \bar{\mathbf{p}}$, so $Q\bar{\mathbf{p}} = 0$.

Specifically, the probability vector

$$\bar{\mathbf{p}}_1 = \frac{1}{\|\bar{\mathbf{x}}\|_1}, \quad \bar{\mathbf{p}}_{j+1} = \frac{1}{\|\bar{\mathbf{x}}\|_1} \prod_{i=1}^{j} \frac{q_{2i-1}}{q_{2i}} \quad \text{for all } j = 1, \ldots, N-1,$$

where

$$\|\bar{\mathbf{x}}\|_1 = \sum_{j=1}^{N} \bar{\mathbf{x}}_j = 1 + \sum_{j=1}^{N-1} \prod_{i=1}^{j} \frac{q_{2i-1}}{q_{2i}}.$$

The following result is well known.

Theorem 2. *The probability eigenvector* $\bar{\mathbf{p}}$ *is an asymptotically stable steady state of the difference equation* (6) *on the simplex* Σ_N*.*

In a random environment, e.g., with randomly varying food supply, the transition probabilities may be random,

i.e., *the band entries* q_i *of the matric* Q *may depend on the sample space parameter* $\omega \in \Omega$.

Let $\mathcal{L}$ be a set of linear operators $L_\omega : \mathbb{R}^N \to \mathbb{R}^N$ parametrised by the parameter ω taking values in Ω and let $\{\theta_n, n \in \mathbb{Z}\}$ be a group of maps of Ω onto itself.

The maps $x \mapsto L_\omega x$ serve as the generator of a linear cocycle $F_{\mathcal{L}}(n, \omega)$.

Then $(\theta, F_{\mathcal{L}})$ is a random dynamical system on $\Omega \times \Sigma_N$.

Assumption 1. *There exist numbers* $0 < \alpha \leq \beta < \infty$ *such that the uniform estimates hold*

$$\alpha \leq q_i(\omega) \leq \beta \quad \text{for all } \omega \in \Omega,\ i = 1, 2, \ldots, 2N-2. \tag{7}$$

Theorem 3. *(Kloeden & Kozyakin)* *Let $F_{\mathcal{L}}(n,\omega)x$ be the linear cocycle*

$$F_{\mathcal{L}}(n,\omega)x = L_{\theta_{n-1}\omega} \cdots L_{\theta_1\omega} L_{\theta_0\omega} x.$$

with matrices $L_\omega := I_N + \Delta Q(\omega)$, where the tridiagonal matrices $Q(\omega)$ are of the form (5) with the entries $q_i = q_i(\omega)$ satisfying the uniform estimates (7) in Assumption 1. In addition, suppose that $0 < \Delta < \frac{1}{2\beta}$.
Then, the simplex Σ_N is positively invariant under $F_{\mathcal{L}}(n,\omega)$, i.e.,

$$F_{\mathcal{L}}(n,\omega)\Sigma_N \subseteq \Sigma_N \quad \textit{for all } \omega \in \Omega.$$

Moreover, for n large enough, the restriction of $F_{\mathcal{L}}(n,\omega)x$ to the set Σ_N is a uniformly dissipative and uniformly contractive cocycle (w.r.t. the Hilbert metric), which has a random attractor $\mathcal{A} = \{A_\omega, \omega \in \Omega\}$ such that each set A_ω, $\omega \in \Omega$, consists of a single point.

The proof of Theorem 3 involves positive matrices and the Hilbert projective metric on positive cones in $\mathbb{R}^N$.

The random attractor here is an entire random sequence $\{a_{\theta_n \omega}, n \in \mathbb{Z}\}$ in $\Sigma_N(\gamma) \subset \overset{\circ}{\Sigma}_N$, where

$$\Sigma_N(\gamma) = \left\{ x = (x_1, x_2, \ldots, x_N) : \sum_{i=1}^{N} x_i = 1, \; x_1, x_2, \ldots, x_N \geq \gamma^{N-1} \right\}.$$

with $\gamma := \min\{\Delta\alpha, 1 - 2\Delta\beta\} > 0$.

Write $A_\omega = \{a_\omega\}$ for the singleton component subsets of the random attractor $\mathcal{A}$.

It attracts other iterates of the random Markov chain in the pullback sense.

Pullback convergence is generally, not the same as forward convergence in the sense usually understood in dynamical systems, but in this case it is the same due to the uniform boundedness of the contractive rate w.r.t. ω.

Corollary 1. *For any norm* $\|\cdot\|$ *on* $\mathbb{R}^N$, $\mathbf{p}^{(0)} \in \Sigma_N$ *and* $\omega \in \Omega$

$$\left\| \mathbf{p}^{(n)}(\omega) - a_{\theta_n \omega} \right\| \to 0 \quad \text{as} \quad n \to \infty.$$

The random attractor is, in fact, asymptotic Lyapunov stable in the conventional forward sense.

Approximating invariant measures

Consider a compact metric space (X, d). A random difference equation (2) on X driven by the noise process θ generates a random dynamical system (θ, φ). It can be reformulated as a difference equation with a triangular or skew-product structure

$$(\omega, x) \mapsto F(\omega, x) := \begin{pmatrix} \theta(\omega) \\ f(\omega, x) \end{pmatrix}$$

An invariant measure μ of $F = (\theta, \varphi)$ on $\Omega \times X$ defined by $\mu = F^*\mu$ can be decomposed as

$$\mu(\omega, B) = \mu_\omega(B)\,\mathbb{P}(d\omega) \quad \text{for all } B \in \mathcal{B}(X),$$

where the measures μ_ω on X are θ-invariant w.r.t. f, i.e.,

$$\mu_{\theta(\omega)} = \mu_\omega\left(f^{-1}(\omega, B)\right) \quad \text{for all } B \in \mathcal{B}(X),\ \omega \in \Omega.$$

This decomposition is very important since only the state space X, but not the sample space Ω, can be discretised.

To compute a given invariant measure μ consider a sequence of finite subsets X_N of X given by

$$X_N = \left\{x_1^{(N)}, \cdots, x_N^{(N)}\right\} \subset X,$$

for $N \in \mathbb{N}$ with maximal step size

$$h_N = \sup_{x \in X} \operatorname{dist}(x, X_N)$$

such that $h_N \to 0$ as $N \to \infty$.

An invariant μ will be approximated by a sequence of invariant stochastic vectors associated with random Markov chains describing transitions between the states of the discretised state spaces X_N.

These involves random $N \times N$ matrices, i.e., measurable mappings

$$P_N : \Omega \to \mathcal{S}_N,$$

where $\mathcal{S}_N$ denotes the set of $N \times N$ (nonrandom) stochastic matrices, with

$$P_N^n(\theta^m(\omega))P_N^m(\omega) = P_N^{m+n}(\omega), \quad \text{for all } m, n \in \mathbb{Z}_+. \tag{8}$$

Recall that a stochastic matrix has non-negative entries with the columns summing to 1.

Consider a random Markov chain $\{P_N(\omega), \omega \in \Omega\}$ and a random probability vector $\{\mathbf{p}_N(\omega), \omega \in \Omega\}$ on the deterministic grid X_N. Then

$$\mathbf{p}_{N,n+1}(\theta^{n+1}(\omega)) \;=\; \mathbf{p}_{N,n}(\theta^n(\omega)) P_N(\theta^n(\omega))$$

and an equilibrium probability vector is defined by

$$\bar{\mathbf{p}}_N(\theta(\omega)) \;=\; \bar{\mathbf{p}}_N(\omega) P_N(\omega) \quad \text{for all } \omega \in \Omega.$$

It can be represented trivially as a random measure $\mu_{N,\omega}$ on X.

• The distance between random probability measures will be given with the Prokhorov metric ρ.

• The distance distance of a random Markov chain $P : \Omega \to \mathcal{S}_N$ and the generating mapping f of the random dynamical system is defined by

$$D(P(\omega), f) = \sum_{i,j=1}^{N} \left(p_{i,j}(\omega)\, \mathrm{dist}_{X\times X}((x_i^{(N)}, x_j^{(N)}), \mathrm{Gr} f(\omega, \cdot)) \right), \tag{9}$$

where the distance to the random graph is given by

$$\mathrm{dist}_{X\times X}((x,y), \mathrm{Gr} f(\omega, \cdot)) = \inf_{z\in X} \max\{d(x,z), d(y, f(\omega,z))\} \quad \text{for all } x, y \in X.$$

The necessary and sufficient result below holds if θ-semi-invariant rather than θ-invariant families of decomposed probability measures are used.

Definition 5. *A family of probability measures μ_ω on X is called θ-semi-invariant w.r.t. f, if*

$$\mu_{\theta(\omega)} \leq \mu_\omega \left(f^{-1}(\omega, B)\right) \quad \text{for all } B \in \mathcal{B}(X),\ \omega \in \Omega.$$

Such θ-semi-invariant families are θ-invariant when the mappings $x \mapsto f(\omega, x)$ are continuous.

Theorem 4. *(Imkeller & Kloeden) A random probability measure $\{\mu_\omega, \omega \in \Omega\}$ is θ-semi-invariant w.r.t. f on X if and only if it is randomly stochastically approachable, i.e., for each N there exist*

i) a grid X_N with fineness $h_N \to 0$ as $N \to \infty$

ii) a random Markov chain $\{P_N(\omega), \omega \in \Omega\}$ on X_N

iii) random probability measure $\{\mu_{N,\omega}, \omega \in \Omega\}$ on X corresponding to a random equilibrium probability vector $\{\bar{\mathbf{p}}_N(\omega), \omega \in \Omega\}$ of $\{P_N(\omega), \omega \in \Omega\}$ on X_N

with the convergences of the expected distances

$$\mathbb{E}D\left(P_N(\omega), f(\omega, \cdot)\right) \to 0, \qquad \mathbb{E}\rho\left(\mu_{N,\omega}, \mu_\omega\right) \to 0 \quad \text{as } n \to \infty.$$

The double terminology random stochastic seems to be an overkill, but just think of a Markov chain for which the transition probabilities are not fixed, but can vary randomly in time.