Resonance problems for some non-autonomous ordinary differential equations

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Introduction

Recent years have seen a lot of activity in the study of quasilinear non-autonomous ordinary differential equations of the form

$$(\phi(u'))' = f(t, u, u') \tag{1}$$

where $\phi : (-a, a) \to (-b, b)$ is an increasing homeomorphism such that $\phi(0) = 0$ between the open sets intervals $(-a, a)$ and $(-b, b)$, with $0 < a, b \leq +\infty$. The situation generalizes the classical case where $a = b = +\infty$ and ϕ is the identity, and the well-studied case of the p-Laplacian ($p > 1$) where $a = b = +\infty$ and $\phi(s) = |s|^{p-2}s$.

In this last case, the Fredholm alternative for the solvability of

$$(|u'|^{p-2}u')' - \lambda|u|^{p-2}u = h(t) \tag{2}$$

with classical Dirichlet, Neumann or periodic boundary conditions on $[0, T]$

$$u(0) = 0 = u(T),\ u'(0) = 0 = u'(T),\ u(0) = u(T),\ u'(0) = u'(T) \tag{3}$$

is far to be fully understood, despite of recent interesting partial results.

Contemporary researches concern less standard situations where $\phi : (-a, a) \to \mathbb{R}$ (singular homeomorphism) and $\phi : \mathbb{R} \to (-a, a)$ (bounded homeomorphism). A model for the first case, namely $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, corresponds to acceleration in special relativity. A model for the second situation, namely $\phi(s) = \frac{s}{\sqrt{1+s^2}}$, corresponds to problem with curvature satisfying various conditions. In those case, both topological and variational methods give new complementary existence and multiplicity results. We will describe some of them.

Some attention will be given to the generalized forced pendulum equation

$$(\phi(u'))' + A \sin u = h(t) \tag{4}$$

when ϕ is singular or bounded.

The case of differential systems, with or without variational structure, will be considered as well.

1 Topological approach

1.1 Notations

We will consider quasilinear second order differential systems of the form

$$(\phi(u'))' = f(t, u, u'), \tag{5}$$

where $f : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is continuous, where $\phi : B(a) \to B(b)$ belong to a suitable class of homeomorphism with $B(\rho) \subset \mathbb{R}^n$ the open ball of center 0 and radius ρ, $B(+\infty) = \mathbb{R}^n$, $0 < a \leq +\infty$, $0 < b \leq +\infty$ and $a + b = +\infty$. We assume moreover that $\phi(0) = 0$. A *solution* of (5) on $[0,T]$ is a function $u \in C^1([0,T], \mathbb{R}^n)$ such that $u'(t) \in B(a)$ for all $t \in [0,T]$, $\phi \circ u' \in C^1([0,T], \mathbb{R}^n)$ and equation (5) holds. We have assumed that f is continuous for simplicity. The case of Carathéodory f can be treated as well.

The study of radial solutions on a ball or an annulus of some partial differential equations with Dirichlet or Neumann boundary conditions have led to differential equations or systems of the form

$$(r^{N-1}\phi(v'))' = r^{N-1} f(r, v, v')$$

for which existence and multiplicity results have been recently obtained in [8, 9, 10, 12, 13, 14, 15, 16, 17]. Those questions will not be considered here.

A homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is called *classical*, a homeomorphism $\phi : \mathbb{R}^n \to B(b)$ $(b < +\infty)$ *bounded*, and a homeomorphism $\phi : B(a) \to \mathbb{R}^n$ $(a < +\infty)$ *singular*. All those types were already considered in [48] in the scalar case and for periodic or Neumann problems with a nonlinearity depending only on the derivative. Standard examples of classical homeomorphisms correspond to $\phi(s) = s$, for which (5) is the semilinear system

$$u'' = f(t, u, u'), \tag{6}$$

or to

$$\phi(s) = \phi_p(s) := |s|^{p-2}s \quad (p > 1),$$

($|\cdot|$ the Euclidian norm in $\mathbb{R}^n$), for which (5) is the quasilinear system associated to the *p-Laplacian*

$$(|u'|^{p-2}u')' = f(t, u, u'). \tag{7}$$

An example of bounded homeomorphism corresponds to

$$\phi(s) = \phi_C(s) := \frac{s}{\sqrt{1 + |s|^2}},$$

for which (5) reduces for $n = 1$ to quasilinear equations associated to curvature or capillarity problems

$$\left(\frac{u'}{\sqrt{1 + u'^2}}\right)' = f(t, u, u'). \tag{8}$$

An example of singular homeomorphism corresponds to

$$\phi(s) = \phi_R(s) := \frac{s}{\sqrt{1-s^2}},$$

for which (5) reduces to quasilinear equations associated to relativistic acceleration

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = f(t,u,u'). \tag{9}$$

Notice that if ϕ is classical, the same is true for ϕ^{-1}, if ϕ is bounded, ϕ^{-1} is singular, and if ϕ is singular, ϕ^{-1} is bounded. In particular

$$\phi_p^{-1} = \phi_q \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and

$$\phi_C^{-1} = \phi_R.$$

Let $C = C([0,T],\mathbb{R}^n)$ with the uniform norm $|\cdot|_\infty$, $C^1 = C^1([0,T],\mathbb{R}^n)$ with the norm $\|u\|_1 = |u|_\infty + |u'|_\infty$, $L^1 = L^1(0,T;\mathbb{R}^n)$ with the usual norm $|\cdot|_1$, and define the mapping $N_f : C^1 \to C^1$ by

$$N_f(u)(t) = \int_0^t f[s,u(s),u'(s)]\,ds \quad (t \in [0,T]),$$

the integration operator $H : C \to C^1$ by

$$Hu(t) = \int_0^t u(s)\,ds \quad (t \in [0,T]),$$

and the linear projector $P : C \to \mathbb{R}^n \subset C$ by

$$Pu = u(0),$$

where $\mathbb{R}^n$ is identified with the subspace of C of constant mappings, In any vector space E, we denote by $B(\rho)$ the open ball of center 0 and radius $\rho > 0$. We write $B(+\infty) = E$.

1.2 A class of homeomorphisms and a nonlinear projector

A technical result is needed for the construction of the equivalent fixed point problems in the Dirichlet and periodic case. For simplicity, we only consider the cases where ϕ is singular, so that $\phi^{-1} : \mathbb{R}^n \to B(a)$ $(0 < a \le +\infty)$. The classical case has been considered elsewhere (see e.g. [63], and the bounded case requires some restrictions.

The class of homeomorphisms ϕ occuring in (5) is characterized by the following condition.

(H_Φ) *ϕ is a homeomorphism from $B(a) \subset \mathbb{R}^n$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi : \overline{B(a)} \to \mathbb{R}$ of class C^1 on $B(a)$, continuous strictly convex on $\overline{B(a)}$, and such that $\Phi(0) = 0$.*

So, ϕ is strictly monotone on $B(a)$, in the sense that

$$\langle \phi(u) - \phi(v), u - v\rangle > 0 \quad \text{for} \quad u \neq v,$$

and Φ reaches its minimum 0 at 0.

If $\Phi^* : \mathbb{R}^n \to \mathbb{R}$ is the Legendre-Fenchel transform of Φ defined by

$$\Phi^*(v) = \langle \phi^{-1}(v), v\rangle - \Phi[\phi^{-1}(v)] = \sup_{u \in \overline{B(a)}} \{\langle u, v\rangle - \Phi(u)\},$$

then Φ^* is also strictly convex, and, if $d := \max_{u\in\overline{B(a)}} \Phi(u)$,

$$\langle u, v\rangle - d \leq \langle u, v\rangle - \Phi(u) \leq \langle u, v\rangle \tag{10}$$

for all $u \in \overline{B(a)}$ and $v \in \mathbb{R}^n$. Consequently,

$$a|v| - d \leq \Phi^*(v) \leq a|v| \quad (v \in \mathbb{R}^n), \tag{11}$$

so that Φ^* is coercive on $\mathbb{R}^n$. Adapting the reasoning of Proposition 2.4 in [78], we obtain that Φ^* is of class C^1. Hence $\phi^{-1} = \nabla\Phi^*$, so that

$$v = \nabla\Phi(u) = \phi(u), \quad u \in B(a) \quad \Leftrightarrow \quad u = \phi^{-1}(v) = \nabla\Phi^*(v), \quad v \in \mathbb{R}^n.$$

Given $h \in C$ and $b \in \mathbb{R}^n$, let us define $\Gamma(b; h)$ by

$$\begin{aligned}\Gamma(b; h) &= \int_0^T \phi^{-1}[h(t) - b]\, dt = \int_0^T \nabla_b\Phi^*[h(t) - b]\, dt \\ &= \nabla_b \int_0^T \Phi^*[h(t) - b]\, dt = \nabla_b\gamma(b; h),\end{aligned}$$

where $\gamma(b; h)$ is defined by

$$\gamma(b; h) = \int_0^T \Phi^*[h(t) - b]\, dt.$$

The following Lemma is taken from [22].

Lemma 1 *If $\phi = \nabla\Phi$, with Φ verifying Assumption (H_Φ), then, for each $h \in C$, the system*

$$\int_0^T \phi^{-1}[h(t) - b]\, dt = 0 \tag{12}$$

has a unique solution $b := Q_\phi(h)$. Moreover, $Q_\phi : C \to \mathbb{R}^n$ is continuous, and Q_ϕ takes bounded sets of C into bounded sets of $\mathbb{R}^n$.

Proof. For each b, $c \in \mathbb{R}^n$ and any $\lambda \in]0,1[$, we have

$$\begin{aligned}\gamma[(1-\lambda)b+\lambda c] &= \int_0^T \Phi^*[(1-\lambda)(b-h(t))+\lambda(c-h(t))]\,dt \\ &< \int_0^T \{(1-\lambda)\Phi^*(b-h(t))+\lambda\Phi^*(c-h(t))\}\,dt \\ &\leq (1-\lambda)\gamma(b;h)+\lambda\gamma(c;h),\end{aligned}$$

so that $\gamma(\cdot;h)$ is strictly convex on $\mathbb{R}^n$ for each $h \in C$. Hence, $\Gamma(\cdot;h) = \nabla_b\gamma(\cdot;h)$ is strictly monotone on $\mathbb{R}^n$ for each $h \in C$. On the other hand, using (11), we get

$$aT|b| - \|h\|_1 - dT \leq \gamma(b;h) \leq Ta|b| + \|h\|_1, \tag{13}$$

so that, for each $h \in C$, $\gamma(b;h)$ is coercive. Consequently, for each $h \in C$, $\gamma(\cdot;h)$ admits a unique minimum $b := Q_\phi(h)$, which corresponds to the unique critical point of $\gamma(\cdot;h)$. This implies that, for each $h \in C$, the system $\Gamma(b;h) = 0$ has a unique solution $b := Q_\phi(h)$.

Let us now show that Q_ϕ is continuous. Let (h_n) be a sequence converging in C to $h \in C$. Then (h_n) is bounded. Let $b_n = Q_\phi(h_n)$. Then, by the convexity and coercivity of γ,

$$\gamma(0;h_n) \geq \gamma(b_n;h_n) - \langle\nabla_b\gamma(b_n;h_n), b_n\rangle = \gamma(b_n;h_n) \geq aT|b_n| - Td - \|h_n\|_1,$$

so that

$$|b_n| \leq (aT)^{-1}[\|h_n\|_1 + Td + \gamma(0;h_n)]$$

which shows that (b_n) is bounded. Going if necessary to a subsequence, we can assume that (b_n) converges to β. From the relations

$$\int_0^T \phi^{-1}[h_n(t) - b_n]\,dt = 0 \quad (n \in \mathbb{N}),$$

and the dominated convergence theorem, we deduce that

$$\int_0^T \phi^{-1}[h(t) - \beta]\,dt = 0,$$

i.e. by the uniquess of the solutions, $\beta = Q_\phi(h)$, a limit independent of the subsequence. Hence

$$Q_\phi(h) = \lim_{n\to\infty} Q_\phi(h_n),$$

and Q_ϕ is continuous. Notice also that $Q_\phi(0) = 0$.

Finally, to show that Q_ϕ takes bounded sets of C into bounded sets of $\mathbb{R}^n$, we use again convexity and (13) to obtain

$$\begin{aligned}\gamma(0;h) &\geq \gamma(Q_\phi(h);h) - \langle\nabla_b\gamma(Q_\phi(h);h), b\rangle \\ &= \gamma(Q_\phi(h);h) \geq aT|Q_\phi(h)| - Td - \|h\|_1,\end{aligned}$$

and hence, using again (13),

$$|Q_\phi(h)| \le (aT)^{-1}\left[(2\|h\|_1 + Td)\right].$$

■

Remark 1 Lemma 1 shows that the mapping Q_ϕ verifies the identity

$$Q \circ \phi^{-1} \circ (I - Q_\phi) \circ u = 0 \quad \text{for all} \quad u \in C. \tag{14}$$

Furthermore, from the homeomorphic character of ϕ and $\phi(0) = 0$, we have

$$Q_\phi(0) = 0. \tag{15}$$

Remark 2 It is easy to see that, for $n = 1$, Lemma 1 holds, with an elementary proof, for all increasing homeomorphisms $\phi : (-a, a) \to \mathbb{R}$ such that $\phi(0) = 0$. So, for $n = 1$, there is no need of assumption of the existence of the primitive of ϕ on $[-a, a]$.

Example 1 Let us consider the C^∞-mapping $\Phi : \overline{B(1)} \subset \mathbb{R}^n \to \mathbb{R}$, given by

$$\Phi(u) = 1 - \sqrt{1 - |u|^2} \quad (u \in \overline{B(1)}), \tag{16}$$

so that $0 \le \Phi(u) \le 1 \quad (u \in \overline{B(1)})$, and

$$\phi(u) = \nabla\Phi(u) = \frac{u}{\sqrt{1 - |u|^2}} \quad (u \in B(1)).$$

As $|\cdot|^2$ is strictly convex on $\mathbb{R}^n$, it follows that Φ is strictly convex on $\overline{B(1)}$. Furthermore, $\phi : B(1) \to \mathbb{R}^n$ is a homeomorphism such that, for any $v \in \mathbb{R}^n$.

$$\phi^{-1}(v) = \frac{v}{\sqrt{1 + |v|^2}} = \nabla\Phi^*(v),$$

where $\Phi^*(v) = \sqrt{1 + |v|^2}$ is strictly convex and of class C^∞ on $\mathbb{R}^n$. Hence, Assumption (H_Φ) with $a = 1$ holds for Φ given by (16).

1.3 Dirichlet problem

Let $f : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be continuous and consider for simplicity the homogeneous Dirichlet problem

$$(\phi(u'))' = f(t, u, u'), \quad u(0) = 0 = u(T). \tag{17}$$

The non-homogeneous case can be treated in a similar way, at the expense of a generalization of Lemma 1 (see [24] for details).

1.3.1 Equivalent fixed point problem

The following fixed point operator was introduced for classical ϕ in [46], for bounded ϕ in [19] and for singular ϕ in [20].

Theorem 1 *u is a solution of the Dirichlet problem (17) if and only if $u \in C^1$ is a fixed point of the operator S defined on C^1 by*

$$S(u) = H \circ \phi^{-1} \circ (I - Q_\phi) \circ N_f(u). \tag{18}$$

Proof. If u is a solution of (17), then

$$\phi(u') = c + N_f(u),$$

where $c = \phi(u'(0))$, and hence

$$u' = \phi^{-1} \circ [c + N_f(u)], \tag{19}$$

so that $u(T) = 0$ if

$$\int_0^T \phi^{-1}\{c + N_f(u)(t)\}\, dt = 0,$$

i.e., using Lemma 1, if

$$c = -Q_\phi[N_f(u)].$$

Hence, equation (19) and the boundary condition at 0 give

$$u = H \circ \phi^{-1} \circ (I - Q_\phi) \circ N_f(u) = S(u).$$

Conversely, if $u \in C^1$ is a fixed point of S, then taking respectively $t = 0$ and $t = T$ in $u = S(u)$, we get $u(0) = 0$ and

$$u(T) = \int_0^T \phi^{-1}[(I - Q_\phi)(N_f(u)(t))]\, dt = 0$$

by (14). On the other hand, differentiating the fixed point equation gives

$$u' = \phi^{-1} \circ (I - Q_\phi) \circ N_f(u)$$

i.e.

$$\phi(u') = (I - Q_\phi) \circ N_f(u),$$

and hence, $\phi(u') \in C^1$ and, differentiating again,

$$(\phi(u'))' = f(t, u, u').$$

■

It is standard to prove that $S : C^1 \to C^1$ is completely continuous.

1.3.2 Existence result for singular ϕ

With this reduction to a fixed point problem, the existence of a solution to (17) for any continuous f follows from Schauder's fixed point theorem. The result was first proved in [20].

Theorem 2 *For any continuous $f : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, the problem (17) has at least one solution.*

Proof. For any $u \in C^1$, we have

$$(S(u))' = \phi^{-1} \circ (I - Q_\phi) \circ N_f(u),$$

and hence

$$|(S(u))'|_\infty < a. \tag{20}$$

The second part of the proof of Theorem 1 also shows that, for any $u \in C^1$,

$$S(u)(0) = 0 = S(u)(T),$$

which, together with (20), gives

$$\|S(u)\|_1 = |S(u)|_\infty + |(S(u))'|_\infty \leq (T+1)a$$

for all $u \in C^1$. Thus S maps C^1 into the closed ball $\overline{B}((T+1)a)$ of C^1 and has a fixed point using Schauder's theorem. ■

A direct consequence of Theorem 17 if the following

Corollary 1 *Given any $h \in C$ and any continuous $g : \mathbb{R}^n \to \mathbb{R}^n$, the Dirichlet problem*

$$(\phi(u'))' + g(u) = h(t), \quad u(0) = 0 = u(T) \tag{21}$$

always has at least one solution.

If we recall that the classical linear Dirichlet problem

$$u'' + g(u) = h(t), \quad u(0) = 0 = u(T)$$

is usually called *non-resonant* if it is solvable for any $h \in C$, one can conclude that *the Dirichlet problem (21) is always non-resonant.*

1.4 Periodic problem

1.4.1 Equivalent fixed point problem for classical or singular ϕ

Let us consider now the periodic problem

$$(\phi(u'))' = f(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{22}$$

Notice first that the problem

$$(\phi(u'))' = 1, \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has no solution, because the existence of a solution would imply, by integration of both members of the differential equation and use of the boundary condition that $0 = T$. Hence we cannot expect an existence result like Theorem 17. This can be interpreted also by saying that the periodic problem

$$(\phi(u'))' = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

is *resonant*.

The following result was proved for classical ϕ in [63], for bounded ϕ in [19] and for singular ϕ in [20], in a slightly different setting.

Theorem 3 *u is a solution of the abstract periodic problem (22) if and only if $u \in C^1$ is a fixed point of the operator $M : C^1 \to C^1$ defined by*

$$M(u) = Pu - N_f(u)(T) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ N_f(u). \tag{23}$$

Furthermore, $|(M(u))'|_\infty < a$ for all $u \in C^1$ and M is completely continuous on C^1.

Proof. If u is a solution of problem (22), then $u \in C^1$, $\phi(u') \in C^1$ and, integrating both members of the differential equation over $[0, T]$ gives

$$N_f(u)(T) = 0. \tag{24}$$

The differential equation in (22) is equivalent to

$$\phi(u') = c + N_f(u)$$

and hence to

$$u' = \phi^{-1}[c + N_f(u)] \tag{25}$$

where $c = \phi(u'(0))$. The first boundary condition implies that c must be such that

$$\int_0^T \phi^{-1}[c + N_f(u)(t)]\, dt = 0$$

which, using Lemma 1, gives

$$c = -Q_\phi[N_f(u)].$$

Thus, equation (25) becomes

$$u' = \phi^{-1} \circ (I - Q_\phi) \circ N_f(u) \tag{26}$$

which is equivalent to the integrated form

$$u - Pu = H \circ \phi^{-1} \circ (I - Q_\phi) \circ N_f(u). \tag{27}$$

Finally, as (24) and (27) take values in supplementary subspaces of C, they can be written as the unique fixed point problem

$$u = Pu - N_f(u)(T) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ N_f(u). \tag{28}$$

Conversely, if u is a fixed point of M, i.e. a solution of equation (28), then taking $t = 0$ in (28) gives (24). Differentiating both members of (28) gives (26) which, integrated over $[0, T]$ gives

$$u(T) - u(0) = TQ \circ \phi^{-1} \circ (I - Q_\phi) \circ N_f(u) = 0$$

using (14), so that the first boundary condition is satisfied. Now (26) is equivalent to

$$\phi(u') = (I - Q_\phi) \circ N_f(u),$$

which gives, by differentiating both members

$$(\phi(u'))' = f(t, u, u'),$$

and hence, by integrating both members over $[0, T]$,

$$\phi[u'(T)] - \phi[u'(0)] = N_f(u)(T) = 0$$

which gives the second boundary condition. ∎

1.4.2 Existence theorem for singular ϕ

Again we concentrate on the case of a singular $\phi : B(a) \to \mathbb{R}^n$. In order to find conditions for the existence of fixed points of M using Leray-Schauder degree, we introduce the homotopy $\mathcal{M}$ defined on $C \times [0, 1]$ by

$$\mathcal{M}(u) = Pu - N_f(u)(T) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ \lambda N_f(u).$$

Notice that, for $\lambda \in (0, 1]$, an argument entirely similar to that of Theorem 3 shows that the fixed points of $\mathcal{M}(\cdot, \lambda)$ are the solutions of the periodic problem

$$(\phi(u')' = \lambda f(t, u, u'), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{29}$$

so that u is a solution of the equivalent problem For $\lambda = 0$, the fixed points of $\mathcal{M}(\cdot, 0)$ are the solutions of

$$u = Pu - N_f(u)(T),$$

so that they are constant u solutions of

$$N_f(u)(T) = 0.$$

We can now prove the following Leray-Schauder type existence result for problem (22. We denote by d_B the Brouwer degree for continuous mappings in $\mathbb{R}^n$.

Theorem 4 *Assume that there exists an open bounded set $\Omega \subset C$ such that the following conditions hold :*

1. *For each $\lambda \in (0, 1]$, there is no solution of problem*

$$(\phi(u'))' = \lambda f(t, u, u'), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{30}$$

such that $u \in \partial\Omega$.

2. *There is no solution on $\partial\Omega \cap \mathbb{R}^n$ of equation*

$$\overline{f}(u) := N_f(u)(T) = 0, \tag{31}$$

where $\mathbb{R}^n$ denotes the subspace of constant functions in C.

3. *$d_B[\overline{f}, \Omega \cap \mathbb{R}^n, 0] \neq 0$.*

Then problem (22) has at least one solution such that $u \in \{u \in C^1 : u \in \Omega,\ |u'|_\infty < a\}$, and, for the associated fixed point operator M, one has

$$d_{LS}[I - M, \Omega_\rho, 0] = d_B[\overline{f}, \Omega \cap \mathbb{R}, 0], \tag{32}$$

where $\rho \geq a$ and $\Omega_\rho \subset C^1$ is the open bounded set defined by

$$\Omega_\rho = \{u \in C^1 : u \in \Omega,\ |u'|_\infty < \rho\}.$$

Proof. Let u be a possible fixed point of $\mathcal{M}(\cdot, \lambda)$. If $\lambda \in (0, 1]$, then, by the reasoning above, u is a solution of problem (30), and $u \notin \partial\Omega$ by Assumption 1. Furthermore,

$$|u'|_\infty = |(\mathcal{M}(\cdot, \lambda))'|_\infty < a. \tag{33}$$

If $\lambda = 0$, then, by the reasoning above, u is a constant solution of (31). By Assumption 2, $u \notin \partial\Omega$. Consequently, for $\lambda \in [0, 1]$, and any $\rho \geq a$, $\mathcal{M}(\cdot, \lambda)$ has no fixed point on $\partial\Omega_\rho$. The homotopy invariance of Leray-Schauder degree implies that

$$d_{LS}[I - M, \Omega_\rho, 0] = d_{LS}[I - \mathcal{M}(\cdot, 0), \Omega_\rho, 0]. \tag{34}$$

Now,

$$\mathcal{M}(\cdot, 0) : C^1 \to \mathbb{R}^n$$

with $\mathbb{R}^n$ identified to the the subset of constant functions in C, and hence the reduction formula for Leray-Schauder degree gives

$$\begin{aligned} & d_{LS}[I - \mathcal{M}(\cdot, 0), \Omega_\rho, 0] = d_B[(I - \mathcal{M}(\cdot, 0))|_{\mathbb{R}^n}, \Omega_\rho \cap \mathbb{R}^n, 0] \\ = \quad & d_B[\overline{f}, \Omega \cap \mathbb{R}^n, 0] \neq 0, \end{aligned} \tag{35}$$

by Assumption 3. The result follows from relations (34), (35) and the existence property of Leray-Schauder degree. ■

1.4.3 An existence result for planar polynomial systems with singular ϕ

In this section, let us provide $\mathbb{R}^2$ with the multiplication structure of the complex plane $\mathbb{C}$, and consider the planar periodic problem

$$(\phi(z'))' = p(z) + h(t), \quad z(0) = z(T), \quad z'(0) = z'(T), \tag{36}$$

where $h \in C$ and $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of effective degree $N \geq 1$, namely

$$p(z) = \sum_{k=0}^{N} a_k z^k \quad (a_k \in \mathbb{C} \quad (k = 0, \ldots, N), \quad a_N \neq 0).$$

Theorem 5 *Problem (36) has at least one solution for every $h \in C$.*

Proof. In order to use Theorem 4, consider the family of problems

$$(\phi(z'))' = \lambda[p(z) + h(t)], \quad z(0) = z(T), \quad z'(0) = z'(T), \quad (\lambda \in (0, 1]), \tag{37}$$

and let z be a possible solution of (37). We know that

$$|z'|_\infty < a,$$

so that letting

$$z(t) = z_0 + \widehat{z}(t)$$

with $z_0 = z(0)$, we have

$$|\widehat{z}|_\infty < aT. \tag{38}$$

Integrating both members of (37) over $[0, T]$ and using the boundary conditions, we get

$$0 = \int_0^T [p(z_0 + \widehat{z}(t)) + h(t)]\, dt = 0,$$

i.e., explicitely,

$$0 = \int_0^T [\sum_{k=0}^{N} \sum_{j_k=0}^{k} \frac{k!}{j_j!(k - j_k)!} z_0^{j_k} \widehat{z}(t)^{k-j_k} + h(t)]\, dt.$$

This equation has the form

$$0 = \int_0^T [a_N z_0^N + \sum_{j=0}^{N-1} p_j[\widehat{z}(t)] z_0^j + h(t)]\, dt$$

and hence

$$|a_N||z_0|^N \leq \sum_{j=0}^{N-1} b_j |z_0|^j + T^{-1}|h|_1,$$

where

$$|b_j| = \max_{|u| \le aT} |p_j(u)| \quad (j = 0, 1, \ldots, N).$$

This implies the existence of $R_1 > 0$ such that $|z_0| < R$ and hence such that

$$\|z\|_1 < R_1 + a(T+1).$$

On the other hand, for any $\lambda \in [0,1]$ and any possible zero u of

$$\overline{F}(u, \lambda) := T \left[a_N u^N + \sum_{k=0}^{N-1} a_k u^k + \overline{h} \right]$$

we have

$$|a_N||u|^N \le \sum_{k=0}^{N-1} |a_k||u|^k + |\overline{h}|$$

and hence there exists $R_2 > 0$ such that $|u| < R_2$. Consequently, for any $R \ge R_2$,

$$d_B[\overline{f}, B(R), 0] = d_B[a_N z^N, B(R), 0] = N. \tag{39}$$

Taking

$$\Omega = \{z \in C^1 : |z|_\infty < B(R), \ |z'|_\infty < a\}$$

with $R \ge \max\{R_1 + a(T+1), R_2\}$, all the assumptions of Theorem 4 are satisfied.

■

Remark 3 Such a result does not holds in classical case, as shown by the example

$$z'' = -z + \sin t, \quad z(0) = z(2\pi), \quad z'(0) = z'(2\pi),$$

whose first term in right-hand side is a polynomial of degree one and which has no solution, as shown by multiplying each member by $\sin t$ and integrating the result over $[0, 2\pi]$. .

1.4.4 Asymptotic sign conditions for singular ϕ

We now restrict ourselve to scalar equations ($n = 1$). Let C^1_T be the space

$$C^1_T := \{u \in C^1 : u(0) = u(T), \quad u'(0) = u'(T)\}.$$

For $u \in C^1_T$, we write $u_L := \min_{[0,T]} u$, $u_M := \max_{[0,T]} u$. The following results were first proved in [20].

Lemma 2 *Assume that there exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$\begin{aligned} \epsilon \int_0^T f[t, u(t), u'(t)]\, dt &> 0 \quad \text{if} \quad u \in C^1_T, \quad u_L \ge R, \quad |u'|_\infty < a, \qquad (40) \\ \epsilon \int_0^T f(t, u(t), u'(t))\, dt &< 0 \quad \text{if} \quad u \in C^1_T, \quad u_M \le -R, \quad |u'|_\infty < a. \end{aligned}$$

Then problem (22) with $n = 1$ has at least one solution and, for the associated fixed point operator M, one has, with

$$\Omega_{\rho_1,\rho_2} = \{u \in C^1 : |u|_\infty < \rho_1,\ |u'|_\infty < \rho_2\},$$

$$d_{LS}[I - M, B(\rho_1) \times B(\rho_2), 0] = sgn\ \varepsilon. \tag{41}$$

for any $\rho_1 \geq R + aT$ and $\rho_2 \geq a$.

Proof. We construct an open set $\Omega \subset C^1$ having the properties requested by Theorem 4. Let $\lambda \in (0, 1]$ and u be a possible solution of (30). Then,

$$|u'|_\infty < a, \tag{42}$$

hence

$$|\widehat{u}(t)| = \left| \int_0^t u'(s)\, ds \right| \leq Ta$$

and therefore

$$|\widehat{u}|_\infty < Ta. \tag{43}$$

Integrating both members of the equation (30) on $[0, T]$ gives

$$\int_0^T f[s, u(s), u'(s)]\, ds = 0 \tag{44}$$

If $u_M \leq -R$ (resp. $u_L \geq R$) then, from (42) and (40), it follows that

$$\epsilon \int_0^T f[t, u(t), u'(t)]\, dt < 0 \quad (\text{resp.} \quad \epsilon \int_0^T f[t, u(t), u'(t)]\, dt > 0).$$

Using (44) we deduce that

$$u_M > -R \quad \text{and} \quad u_L < R. \tag{45}$$

From

$$u_M \leq u_L + \int_0^T |u'(t)|\, dt,$$

and relations (43), we obtain

$$-(R + aT) < u_L \leq u_M < R + aT.$$

and hence

$$|u|_\infty < R + aT. \tag{46}$$

Hence if we take, for any $\rho_1 \geq R + AT$ and $\rho_2 \geq a$,

$$\Omega = \{u \in C^1 : |u|_\infty < \rho_1,\ |u'|_\infty < \rho_2\},$$

in Theorem 4, its Assumption (1) and (2) hold. Using (34) and (35) and elementary results on the one-dimensional Brouwer degree, we get, for $\rho_1 \geq \rho$ and $\rho_2 \geq a$,

$$d_{LS}[I - M, \Omega, 0] = d_B[\overline{f}, (-\rho_1, \rho_1), 0] = \text{sign } \epsilon.$$

The result follows from Theorem 4. ■

Corollary 2 *Let $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, $k : \mathbb{R} \to \mathbb{R}$, and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous, with h is bounded on $[0,T] \times \mathbb{R} \times]-a,a[$ and g satisfies condition*

$$\lim_{u \to -\infty} g(t,u) = +\infty, \quad \lim_{u \to +\infty} g(t,u) = -\infty$$
$$(\textit{resp.} \quad \lim_{u \to -\infty} g(t,u) = -\infty, \quad \lim_{u \to +\infty} g(t,u) = +\infty)$$

uniformly in $t \in [0,T]$. Then the problem

$$(\phi(u'))' + k(u)u' + g(t,u) = h(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution.

Example 2 *If $c \in \mathbb{R} \setminus 0$, $d \in \mathbb{R}$, $q \geq 0$ and $p > 1$, the problem*

$$\left(\frac{u'}{\sqrt{1 - u'^2}} \right)' + d|u'|^q + c|u|^{p-1}u = e(t),$$
$$u(0) = u(T), \quad u'(0) = u'(T),$$

has at least one solution for all $e \in C$.

Corollary 3 *Let $k : \mathbb{R} \to \mathbb{R}$ and $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, with h bounded on $[0,T] \times \mathbb{R} \times]-a,a[$. Then, for each $\mu \neq 0$, the problem*

$$(\phi(u'))' + k(u)u' + \mu u = h(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution, and, for the associated fixed point operator M we have, for all sufficiently large $\rho_1 > 0$ and $\rho_2 > 0$ and

$$\Omega_{\rho_1,\rho_2} = \{u \in C^1 : |u|_\infty < \rho_1, \ |u'|_\infty < \rho_2\},$$

$$d_{LS}[I - M, \Omega_{\rho_1,\rho_2}, 0] = -sgn\ \mu.$$

When $h(t,u,w) = h(t)$ only depends upon t, Corollary 3 shows that, for $\mu \neq 0$, problem

$$(\phi(u'))' + \mu u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution for any $h \in C$. In this sense one can say that the problem is *non-resonant*. Consequently, $\mu = 0$ is the only value for which *resonance* occurs.

Another easy consequence is a *Landesman-Lazer-type existence condition* for the forced Liénard equation with singular ϕ.

Corollary 4 *Let $k, g : \mathbb{R} \to \mathbb{R}$ be continuous. Then problem*

$$(\phi(u'))' + k(u)u' + g(u) = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution for each $h \in C$ such that

$$\limsup_{u\to-\infty} g(u) < T^{-1}\int_0^T h(s)\,ds < \liminf_{u\to+\infty} g(u)$$

or such that

$$\limsup_{u\to+\infty} g(u) < T^{-1}\int_0^T h(s)\,ds < \liminf_{u\to-\infty} g(u).$$

Proof. Let us consider, say, the first case, the proof of the other one being similar. By Assumptions, there exists $R > 0$ such that

$$T^{-1}\int_0^T h(s)\,ds - g(u) < 0 \quad \text{for} \quad x \geq R,$$
$$T^{-1}\int_0^T h(s)\,ds - g(u) < 0 \quad \text{for} \quad x \leq -R.$$

Consequently, for all $u \in C^1_T$ with $u_L \geq R$ we have

$$\int_0^T [h(s) - k(u(s)u'(s)) - g(u(s))]\,ds < 0$$

and, for all $u \in C^1_T$ with $u_M \leq -R$ we have

$$\int_0^T [h(s) - k(u(s)u'(s)) - g(u(s))]\,ds > 0.$$

■

Example 3 *If $k : \mathbb{R} \to \mathbb{R}$ is continuous and $a \neq 0$, the problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + k(u)u' + a\arctan u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution if and only if

$$-\frac{|a|\pi}{2} < T^{-1}\int_0^T h(s)\,ds < \frac{|a|\pi}{2}.$$

If $k : \mathbb{R} \to \mathbb{R}$ is continuous and $a \neq 0$, the problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + k(u)u' + a\exp u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution if and only if

$$a\int_0^T h(s)\,ds > 0.$$

If $k : \mathbb{R} \to \mathbb{R}$ is continuous,$p > 1$ and $a \neq 0$, the problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + k(u)u' + a|u|^{p-2}u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution for all $h \in C$.

1.4.5 A localized sign condition and periodic nonlinearities for singular ϕ

We first prove, generalizing [11] an existence theorem for (22) which does not involve asymptotic conditions upon the nonlinearity. For any $u \in C$, we write $u(t) = \overline{u} + \widetilde{u}(t)$ with

$$\overline{u} = Qu := T^{-1} \int_0^T u(t)\, dt,$$

and we denote by $\widetilde{C}_T^1$ the set of $u \in C^1$ such that $u(0) = u(T), u'(0) = u'(T)$ and $\int_0^T u(t)\, dt = 0$.

Let $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $h \in C$.

Theorem 6 *Assume that there exists $r < s$ and $A \leq B$ such that*

$$\begin{aligned} T^{-1} \int_0^T g[t, r + \widetilde{u}(t), \widetilde{u}'(t)]\, dt \leq A \\ \text{and} \\ T^{-1} \int_0^T [g(t, s + \widetilde{u}(t), \widetilde{u}'(t)]\, dt \geq B \end{aligned} \tag{47}$$

or

$$\begin{aligned} T^{-1} \int_0^T g[t, r + \widetilde{u}(t), \widetilde{u}'(t)]\, dt \geq A \\ \text{and} \\ T^{-1} \int_0^T g[t, s + \widetilde{u}(t), \widetilde{u}'(t)]\, dt \leq B \end{aligned} \tag{48}$$

for any $\widetilde{u} \in \widetilde{C}_T^1$ satisfying $|\widetilde{u}|_\infty < \frac{aT}{2\sqrt{3}}$. If $A < B$, then for any $h \in C$ satisfying

$$A < \overline{h} < B, \tag{49}$$

problem

$$(\phi(u'))' + g(t, u, u') = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{50}$$

has at least one solution u such that $r < \overline{u} < s$. If $A \leq B$, then for any $h \in C$ satisfying

$$A \leq \overline{h} \leq B, \tag{51}$$

problem (50) has at least one solution u such that $r \leq \overline{u} \leq s$.

Proof. Assume that (47) holds, fix $\varepsilon > 0$ and, for any $\lambda \in [0,1]$, consider the periodic problem

$$\begin{aligned} (\phi(u'))' + \lambda g(t, u, u') + (1 - \lambda)\varepsilon \left(u - \frac{r+s}{2} \right) = \lambda h(t) \\ u(0) = u(T), \quad u'(0) = u'(T). \end{aligned} \tag{52}$$

We apply Leray-Schauder degree to the equivalent fixed point given by Lemma 3, providing a homotopy mapping $\mathcal{M} : C^1 \times [0,1] \to C^1$. Let us introduce the open subset of C^1

$$\Omega = \{u \in C^1 : \quad r < \overline{u} < s, \quad |\widetilde{u}|_\infty < \frac{aT}{2\sqrt{3}}\}.$$

We first show that if $A < B$ and condition (49) holds, then

$$u - \mathcal{M}(u,\lambda) \neq 0 \quad \text{for any} \quad \lambda \in (0,1] \quad \text{and} \quad u \in \partial\Omega, \tag{53}$$

and

$$u - \mathcal{M}(u,0) = 0 \quad \text{implies} \quad u \in \Omega.$$

So, let $u,\lambda) \in C^1 \times (0,1]$ be such that

$$u = \mathcal{M}(u,\lambda).$$

It follows that u is a solution of (52) and

$$|u'|_\infty < a. \tag{54}$$

Using (54) and the Sobolev inequality (see e.g. [78])

$$|\widetilde{u}|_\infty^2 \leq \frac{T}{12}\int_0^T |u'(t)|^2\,dt,$$

we infer that

$$|\widetilde{u}|_\infty < \frac{aT}{2\sqrt{3}}. \tag{55}$$

Integrating over $[0,T]$ the differential equation in (52), we obtain

$$(1-\lambda)\varepsilon\left(\overline{u} - \frac{r+s}{2}\right) + \lambda\left(T^{-1}\int_0^T g[t,\overline{u}+\widetilde{u}(t),\widetilde{u}'(t)]\,dt - \overline{h}\right) = 0. \tag{56}$$

On the other hand, from (47) and (49) it follows that

$$\begin{aligned}(1-\lambda)\varepsilon\left(r - \frac{r+s}{2}\right) &+ \lambda\left(T^{-1}\int_0^T g[t,r+\widetilde{u}(t),\widetilde{u}'(t)]\,dt - \overline{h}\right)\\ &\leq (1-\lambda)\varepsilon\frac{r-s}{2} + \lambda(A-\overline{h}) < 0;\\ (1-\lambda)\varepsilon\left(s - \frac{r+s}{2}\right) &+ \lambda\left(T^{-1}\int_0^T g[t,s+\widetilde{u}(t),\widetilde{u}'(t)]\,dt - \overline{h}\right)\\ &\geq (1-\lambda)\varepsilon\frac{s-r}{2} + \lambda(B-\overline{h}) > 0.\end{aligned} \tag{57}$$

Then, if $u \in \partial\Omega$, from (54) and (55) it follows that either $\overline{u} = r$ or $\overline{u} = s$. But $\overline{u}$ verifies (56), contradiction with (57). Consequently, (53) is proved. Thus, fixing any $\rho \geq a$, and letting

$$\Omega_\rho = \{u \in C^1 : u \in \Omega, |u'|_\infty < \rho\},$$

we obtain an open bounded set Ω_ρ such that no possible fixed point u of $\mathcal{M}(\cdot, \lambda)$ with $\lambda \in (0, 1]$ lies in $\partial\Omega_\rho$. Now, let u be such that

$$u = \mathcal{M}(u, 0).$$

We deduce that u verifies (54), (55) and (52) with $\lambda = 0$, namely

$$(\phi(u'))' + \varepsilon\left(u - \frac{r+s}{2}\right) = \lambda h(t), \quad u(0) = u(T), \quad u'(0) - u'(T).$$

Hence, $\overline{u} = \frac{r+s}{2}$ and $u \in \Omega_\rho$.

Then, using the invariance by homotopy, the excision property of the Leray-Schauder degree and Corollary 3, one has that

$$d_{LS}[I - \mathcal{M}(1, \cdot), \Omega_\rho, 0] = d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] = -1.$$

Hence, the existence property of the Leray-Schauder degree implies the existence of a fixed point $u \in \Omega_\rho$ of $\mathcal{M}(\cdot, 1)$ (in particular $r < \overline{u} < s$), which is also a solution of (50).

In the case where $A \leq B$ and condition (51) holds, with the same notations, it follows from (57) and the same reasoning that, for $\lambda \in [0, 1)$, $u \notin \partial\Omega$. The remaining of the proof is similar and the conclusion follows from the fact that either $\mathcal{M}(\cdot, 1)$ has a fixed point in $\partial\Omega$, in which case a solution exists on $\partial\Omega$, or we can apply the homotopy on $[0, 1]$ and a solution exists with $u \in \Omega$.

If condition (48) holds, one take $\varepsilon < 0$ and reason in a similar way. ■

An immediate consequence of Lemma 6 is the following result of [11], improving of earlier one of [107] (see also [108]).

Theorem 7 *Let $k : \mathbb{R} \to \mathbb{R}$ be continuous, $h \in C$ and $\mu > 0$. If*

$$aT < \pi\sqrt{3}, \quad |\overline{h}| < \mu \cos\left(\frac{aT}{2\sqrt{3}}\right),$$

then the periodic problem

$$(\phi(u'))' + k(u)u' + \mu \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{58}$$

has at least two solution u_1, u_2 such that $-\frac{\pi}{2} < \overline{u}_1 < \frac{\pi}{2} < \overline{u}_2 < \frac{3\pi}{2}$. If

$$aT = \pi\sqrt{3},$$

then problem (58) has at least one solution for any $h \in C$ with $\overline{h} = 0$.

Proof. A simple computation shows that we can apply Theorem 6 *(i)* with

$$r = -\frac{\pi}{2},\ s = \frac{\pi}{2} \quad and \quad A = \mu \sin\left(-\frac{\pi}{2} + \frac{aT}{2\sqrt{3}}\right) = -B;$$
$$r = \frac{\pi}{2},\ s = \frac{3\pi}{2} \quad and \quad A = \mu \sin\left(-\frac{\pi}{2} + \frac{aT}{2\sqrt{3}}\right) = \mu \sin\left(\frac{3\pi}{2} + \frac{aT}{2\sqrt{3}}\right) = -B.$$

■

1.4.6 A continuum containing the solution set for singular ϕ

When ϕ is singular, a variant of Theorem 3 is useful in various problems. Let us decompose any $u \in C^1$ in the form

$$u = u_0 + \widehat{u} \quad (u_0 = Pu = u(0), \quad \widehat{u}(0) = 0),$$

and let

$$\widehat{C}^1 = \{u \in C^1 : u(0) = 0\}.$$

and let $Q : C \to \mathbb{R}^n \subset C$ be the linear projector defined by

$$Qu = T^{-1} \int_0^T u(s)\, ds.$$

The following result was first proved in [20].

Lemma 3 *The set $\mathcal{S}$ of solutions $[u_0, \widehat{u}] \in \mathbb{R} \times \widehat{C}^1$ of the modified problem*

$$(\phi(\widehat{u}'))' = (I - Q)f(t, u_0 + \widehat{u}, \widehat{u}'), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{59}$$

contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and projection on $\widehat{C}^1$ is contained in the ball $B(a(T+1))$.

Proof. Using an argument similar to the one of Theorem 3, on can show that, for each fixed $u_0 \in \mathbb{R}$, problem (59) is equivalent to the fixed point problem in $\widehat{C}^1$

$$\widetilde{u} = H \circ \phi^{-1} \circ (I - Q_\phi) \circ N_{(I-Q)f}(u_0 + \widehat{u})] := \widehat{M}(u_0, \widehat{u}).$$

Again, $\widehat{M}$ is completely continuous on $\mathbb{R} \times \widehat{C}^1$, and, for each $[u_0, \widehat{u}] \in \mathbb{R} \times \widehat{C}^1$, we have

$$|(\widehat{M}(u_0, \widehat{u}))'|_\infty < a, \quad |\widehat{M}(u_0, \widehat{u})|_\infty < aT. \tag{60}$$

It follows from (60) that, for each $u_0 \in \mathbb{R}$, any possible fixed point $\widehat{u}$ of $\widehat{M}(u_0, \cdot)$ is such that

$$|\widehat{u}|_1 < a(T+1). \tag{61}$$

Furthermore, for each $\lambda \in [0,1]$, and each $\overline{u} \in \mathbb{R}$, any possible fixed point $\widehat{u}$ of

$$\widehat{\mathcal{M}}(u_0, \cdot, \lambda) := H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda N_{(I-Q)f}(u_0 + \cdot)]$$

satisfies, for the same reasons, inequality (61), which implies, for the Leray-Schauder degree d_{LS} [56] that

$$\begin{aligned} & d_{LS}[I - \widetilde{M}(0,\cdot), B_{a(T+1)}, 0] \\ = \; & d_{LS}[I - \widetilde{\mathcal{M}}(0,\cdot,1), B_{a(T+1)}, 0] \\ = \; & d_{LS}[I - \widetilde{\mathcal{M}}(0,\cdot,0), B_{a(T+1)}, 0] = d_{LS}[I, B_{a(T+1)}, 0] = 1. \end{aligned} \tag{62}$$

Conditions (61), (62) and Leray-Schauder theory [56, 65] then imply the existence of $\mathcal{C}$. ∎

1.4.7 Weak asymptotic sign conditions

The existence part of Lemma 2 can be obtained under non-strict asymptotic sign conditions upon f.

Theorem 8 *Assume that there exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$\begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t))\, dt \;&\geq\; 0 \quad \text{if} \quad u \in C^1_T, \quad u_L \geq R, \quad \|u'\|_\infty < a, \\ \epsilon \int_0^T f(t, u(t), u'(t))\, dt \;&\leq\; 0 \quad \text{if} \quad u \in C^1_T, \quad u_M \leq -R, \quad \|u'\|_\infty < a, \end{aligned} \tag{63}$$

Then problem (22) has at least one solution.

Proof. Consider the continuum $\mathcal{C}$ given by Lemma 3. If $[R + aT, \widehat{u}] \in \mathcal{C}$, then, for each $t \in [0, T]$,

$$R + aT + \widehat{u}(t) > R$$

and hence, using (63)

$$\epsilon \int_0^T f(t, R + aT + \widehat{u}(t), \widehat{u}'(t))\, dt \geq 0.$$

Similarly, if $[-R - aT, \widehat{u}] \in \mathcal{C}$, then

$$\epsilon \int_0^T f(t, -R - aT + \widehat{u}(t), \widehat{u}'(t))\, dt \leq 0.$$

The existence of $[u_0, \widehat{u}] \in \mathcal{C}$ such that

$$\epsilon \int_0^T f(t, u_0 + \widehat{u}(t), \widehat{u}'(t))\, dt = 0,$$

and hence such that $u = u_0 + \widehat{u}$ is a solution of (22) follows from the intermediate value theorem for a continuous function on a connected set. ∎

In the special case where $f(t,u,w) = h(t)$ only depends upon t, the condition (63) reduces to

$$\int_0^T h(t)\,dt = 0, \tag{64}$$

which is easily seen also to be necessary for the existence of a solution to the problem

$$(\phi(u'))' = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{65}$$

So we have the result

Corollary 5 *Problem (65) has at least one solution u if and only if condition (64) holds, in which case it has the one parameter family of solutions $c+u \quad (c \in \mathbb{R})$.*

Corollary 6 *Let $h \in C$, $k : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be continuous. If*

$$\int_0^T h(s)\,ds = 0$$

and if there exists $R > 0$ and $\epsilon \in \{-1, 1\}$ such that

$$\epsilon g(u)u \leq 0 \quad \textit{whenever} \quad |u| \geq R,$$

then the periodic problem for the Liénard equation with singular ϕ

$$(\phi(u'))' + k(u)u' + g(u) = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution.

Proof. Notice that, if $u \in C^1_T$, $u_L \geq R$ and $|u'|_\infty < a$, using the boundary conditions,

$$\epsilon \int_0^T \{h(t) - k[u(t)]u'(t) - g[u(t)]\}\,dt = -\epsilon \int_0^T g[u(t)]\,dt \geq 0,$$

and if $u \in C^1_T$, $u_M \leq R$ and $|u'|_\infty < a$,

$$\epsilon \int_0^T \{h(t) - k[u(t)]u'(t) - g[u(t)]\}\,dt = -\epsilon \int_0^T g[u(t)]\,dt \leq 0.$$

■

1.4.8 Lower and upper solutions for singular ϕ

In this subsection, we extend, following [20], the method of upper and lower solutions to the periodic boundary value problem (22) with $n = 1$ and ϕ singular.

Definition 1 *A* lower solution α *(resp.* upper solution β*) of (22) is a function* $\alpha \in C^1$ *such that* $|\alpha'|_\infty < a$, $\phi(\alpha') \in C^1$, $\alpha(0) = \alpha(T)$, $\alpha'(0) \geq \alpha'(T)$ *(resp.* $\beta \in C^1$, $|\beta'|_\infty < a$, $\phi(\beta') \in C^1$, $\beta(0) = \beta(T)$, $\beta'(0) \leq \beta'(T)$*) and*

$$(\phi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \quad (resp. \quad (\phi(\beta'(t)))' \leq f(t, \beta(t), \beta'(t))) \tag{66}$$

for all $t \in [0, T]$. *Such a lower or upper solution is called* strict *if the inequality (66) is strict for all* $t \in [0, T]$.

Theorem 9 *If (22) has a lower solution* α *and a upper solution* β *such that* $\alpha(t) \leq \beta(t)$ *for all* $t \in [0, T]$, *then problem (22) has a solution* u *such that* $\alpha(t) \leq u(t) \leq \beta(t)$ *for all* $t \in [0, T]$. *Moreover, if* α *and* β *are strict, then* $\alpha(t) < u(t) < \beta(t)$ *for all* $t \in [0, T]$, *and* $d_{LS}[I - M, \Omega_{\alpha,\beta}, 0] = 1$, *where*

$$\Omega_{\alpha,\beta} = \{u \in C^1 : \alpha(t) < u(t) < \beta(t) \quad for\ all \quad t \in [0, T], \quad |u'|_\infty < a\},$$

and M *is the fixed point operator associated to (22).*

Proof. Let $\gamma : [0, T] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(t, u) = \begin{cases} \beta(t) & if \quad u > \beta(t) \\ u & if \quad \alpha(t) \leq u \leq \beta(t) \\ \alpha(t) & if \quad u < \alpha(t), \end{cases}$$

and define $F : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ by $F(t, u, v) = f(t, \gamma(t, u), v)$. We consider the modified problem

$$(\phi(u'))' = F(t, u, u') + u - \gamma(t, u), \quad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{67}$$

and first show that if u is a solution of (67) then $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$, so that u is a solution of (22). Suppose by contradiction that there is some $t_0 \in [0, T]$ such that $[\alpha - u]_M = \alpha(t_0) - u(t_0) > 0$. If $t_0 \in]0, T[$; then $\alpha'(t_0) = u'(t_0)$ and there are sequences (t_k) in $[t_0 - \varepsilon, t_0[$ and (t_k') in $]t_0, t_0 + \varepsilon]$ converging to t_0 such that $\alpha'(t_k) - u'(t_k) \geq 0$ and $\alpha'(t_k') - u'(t_k') \leq 0$. As ϕ is an increasing homeomorphism, this implies $(\phi(\alpha'(t_0)))' \leq (\phi(u'(t_0)))'$. Hence, because α is a lower solution of (22) we obtain

$$\begin{aligned} (\phi(\alpha'(t_0)))' \leq (\phi(u'(t_0)))' &= f(t_0, \alpha(t_0), \alpha'(t_0)) + u(t_0) - \alpha(t_0)] \\ &< f(t_0, \alpha(t_0), \alpha'(t_0)) \leq (\phi(\alpha'(t_0)))', \end{aligned}$$

a contradiction. If $[\alpha - u]_M = \alpha(0) - u(0) = \alpha(T) - u(T)$, then $\alpha'(0) - u'(0) \leq 0, \alpha'(T) - u'(T) \geq 0$. Using that $\alpha'(0) \geq \alpha'(T)$, we deduce that $\alpha'(0) - u'(0) = 0 = \alpha'(T) - u'(T)$. This implies that $\phi(\alpha'(0)) = \phi(u'(0))$. On the other hand, $[\alpha - u]_M = \alpha(0) - u(0)$ implies, reasoning in a similarway as for $t_0 \in]0, T[$, that

$$(\phi(\alpha'(0)))' \leq (\phi(u'(0)))'.$$

Using the inequality above and $\alpha'(0) = u'(0)$, we can proceed as in the case $t_0 \in]0, T[$ to obtain again a contradiction. In consequence we have that $\alpha(t) \leq u(t)$ for all $t \in [0, T]$. Analogously, using the fact that β is a upper solution of (22), we can show that $u(t) \leq \beta(t)$ for all $t \in [0, T]$. We remark that if α, β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$.

We now apply Corollary 3 to the modified problem (67) to obtain the existence of a solution, and the relation

$$d_{LS}[I - \widetilde{M}, B(\rho), 0] = 1 \tag{68}$$

for the equivalent fixed point operator $\widetilde{M}$ and all sufficiently large $\rho > 0$.

Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$. If ρ is large enough, then, using (68) and the additivity-excision property of the Leray-Schauder degree, we have

$$d_{LS}[I - \widetilde{M}, \Omega_{\alpha,\beta}, 0] = d_{LS}[I - \widetilde{M}, B(\rho), 0] = 1.$$

On the other hand, as the completely continuous operator M associated to (22) is equal to $\widetilde{M}$ on $\overline{\Omega_{\alpha,\beta}}$, we deduce that $d_{LS}[I - M, \Omega_{\alpha,\beta}, 0] = 1$. ■

Remark 4 In contrast to the case of a classical ϕ, no Nagumo-type condition is required upon f in Theorem 9.

Remark 5 A careful analysis of the above proof implies that Theorem 9 holds also if $f : [0, T] \times]0, +\infty[\times \mathbb{R} \to \mathbb{R}$ is continuous.

We now show, using an argument of Amann-Ambrosetti-Mancini [2] for semilinear Dirichlet problems with bounded nonlinearity, that the existence conclusion in Theorem 9 also holds when the lower and upper solutions are not ordered.

Theorem 10 *If (22) has a lower solution α and an upper solution β, then problem (22) has at least one solution.*

Proof. Let $\mathcal{C}$ be given by Lemma 3. If there is some $[u_0, \widehat{u}] \in \mathcal{C}$ such that

$$\int_0^T f(t, u_0 + \widetilde{u}(t), \widehat{u}'(t))\, dt = 0,$$

then $u_0 + \widehat{u}$ solves (22). If

$$\int_0^T f(t, u_0 + \widehat{u}(t), \widehat{u}'(t))\, dt > 0$$

for all $[u_0, \widehat{u}] \in \mathcal{C}$, then, using (59), $u_0 + \widehat{u}$ is an upper solution for (22) for each $[u_0, \widehat{u}] \in \mathcal{C}$. Then, for $[\alpha_M + aT, \widehat{u}] \in \mathcal{C}$, $\alpha_M + aT + \widehat{u}(t) \geq \alpha(t)$ for all $t \in [0, T]$ is an upper solution and the existence of a solution to (22) follows from Theorem 9. Similarly, if

$$\int_0^T f(t, u_0 + \widehat{u}(t), \widehat{u}'(t))\, dt < 0$$

for all $[u_0, \widehat{u}] \in \mathcal{C}$, then $[\beta_L - aT, \widehat{u}] \in \mathcal{C}$ gives the lower solution $\beta_L - aT + \widehat{u}(t) \leq \beta(t)$ for all $t \in [0, T]$ and the existence of a solution. ■

The choice of constant lower and upper solutions in Theorems 9 and 10 leads to the following simple existence condition.

Corollary 7 *Problem (22) has at least one solution if there exist constants a and b such that*

$$f(t,a,0)\cdot f(t,b,0) \le 0$$

for all $t \in [0,T]$.

Another application of Theorems 9 and 10 gives necessary and sufficient conditions for the existence of a solution of problem

$$(\phi(u'))' = g(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{69}$$

when $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(t,\cdot)$ monotone for each fixed $t \in [0,T]$.

Corollary 8 *If $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(t,\cdot)$ is either non decreasing or non increasing for all $t \in [0,T]$, then problem (69) is solvable if and only if there exists $c \in \mathbb{R}$ such that*

$$\int_0^T g(s,c)\,ds = 0. \tag{70}$$

Proof. Necessity. If problem (69) has a solution u, then, integrating both members of the differential equation in (69) and using the boundary condition, it follows that

$$\int_0^T g(s,u(s))\,ds = 0. \tag{71}$$

Assuming for example that $g(s,\cdot)$ is non decreasing for every $s \in [0,T]$, we deduce from (71) that

$$\int_0^T g(s,u_L)\,ds \le 0 \le \int_0^T g(s,u_M)\,ds,$$

so that, by the intermediate value theorem, there exists some $c \in [u_L, u_M]$ satisfying (70). The reasoning is similar when $g(t,\cdot)$ is non decreasing for each $t \in [0,T]$.

Sufficiency. If $c \in \mathbb{R}$ satisfies (70), then the problem

$$(\phi(u'))' = g(t,c), \quad u(0 = u(T), \quad u'(0) = u'(T) \tag{72}$$

has a one-parameter family of solutions of the form $d + \widehat{u}(t)$ with $\widehat{u} \in \widehat{C}^1_T$. There exists $d_1 \le d_2$ such that, for all $t \in [0,T]$,

$$\alpha(t) := d_1 + \widehat{u}(t) \le c \le d_2 + \widehat{u}(t) =: \beta(t).$$

Hence, if $g(t,\cdot)$ is non decreasing for each $t \in [0,T]$,

$$(\phi(\alpha'(t)))' = \phi(\widehat{u}'(t)))' = g(t,c) \geq g(t,\alpha(t))$$

and α is a lower solution for (69). Similarly β is an upper solution for (69). A similar argument shows that, if $g(t,\cdot)$ is non increasing for every $t \in [0,T]$, α is an upper solution and β a lower solution for (69). So the result follows from Theorem 10. ■

Example 4 *For $h \in C$, $p > 0$ and $a \neq 0$, the problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + a(u^+)^p = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has a solution if and only if

$$a\int_0^T h(t)\,dt \geq 0.$$

1.4.9 Periodic nonlinearity and singular ϕ

We describe in this subsection some existence results of [11] which concern problems of the forced

$$(\phi(u'))' + k(u)u' + \mu \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \qquad (73)$$

where $k : \mathbb{R} \to \mathbb{R}$ is continuous, $\mu > 0$ and $h \in C$.

We first assume that $\phi : (-a,a) \to \mathbb{R}$ is singular.

Theorem 11 *Let $\mu > 0$ and assume that $h \in C$ satisfies*

$$|h|_\infty \leq \mu.$$

Then the periodic problem (73) has at least one solution. Moreover, if

$$|h|_\infty < \mu,$$

then (73) has at least two solutions not differing by a multiple of 2π.

Proof. Assume that $|h|_\infty \leq \mu$. Then $\alpha = -\frac{3\pi}{2}$ is a constant lower solution for (73) and $\beta = -\frac{\pi}{2}$ is a constant upper solution for (73) such that $\alpha < \beta$. Hence,using Theorem 9, it follows that (73) has a solution u_1 such that $\alpha \leq u_1 \leq \beta$. Note that if $|h|_\infty < \mu$, then α, β are strict and $\alpha < u_1 < \beta$. In this case, let M_μ be the fixed point operator associated to (73) and let

$$\Omega = \Omega_{-\frac{3\pi}{2},\frac{3\pi}{2}} \setminus (\overline{\Omega}_{-\frac{3\pi}{2},-\frac{\pi}{2}} \cup \overline{\Omega}_{\frac{\pi}{2},\frac{3\pi}{2}}).$$

Then using the additivity property of the Leray-Schauder degree and Theorem 9, we deduce that

$$d_{LS}[I - M_\mu, \Omega, 0] = -1.$$

Hence, the existence property of the Leray-Schauder degree yields the existence of a solution $u_2 \in \Omega$ of (73). If we assume that $u_2 = u_1 + 2j\pi$ for some $j \in \mathbb{Z}$ then, as $-3\pi/2 < u_1 < -\pi/2$, one has

$$-\frac{3\pi}{2} + 2j\pi < u_2 = u_1 + 2j\pi < -\frac{\pi}{2} + 2j\pi.$$

This leads to one of the contradictions : $u_2 \in \Omega_{\frac{\pi}{2},\frac{3\pi}{2}}$ if $j = 1$ or $u_2 = u_1 \in \Omega_{-\frac{3\pi}{2},-\frac{\pi}{2}}$ for $j = 0$. ■

1.4.10 Periodic nonlinearity and bounded ϕ

To obtain the same type of result for the case of bounded ϕ-Laplacians, i.e. when $\phi : \mathbb{R} \to (-a, a)$ is an increasing homeomorphism such that $\phi(0) = 0$, we use the following a priori estimate result.

Lemma 4 *Let $0 < b, c \leq \infty$, $\psi : (-b, b) \to (-c, c)$ be a homeomorphism such that $\psi(0) = 0$ and $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that there exists $e \in C$ such that $2\|e^-\|_1 < c$ and*

$$f(t, u, v) \geq e(t) \quad \textit{for all} \quad (t, u, v) \in [0, T] \times \mathbb{R}^2. \tag{74}$$

If u is a possible solution of problem

$$(\psi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{75}$$

then $\|u'\|_\infty \leq R_\psi$, where $R_\psi = \max(|\psi^{-1}(\pm 2\|e^-\|_1)|)$.

Proof. Let u be a solution of (75). This implies that

$$\int_0^T f(t, u(t), u'(t))\, dt = 0. \tag{76}$$

Using the fact that f is bounded from below by e, we deduce the inequality

$$|f(t, u, v)| \leq f(t, u, v) + 2e^-(t) \quad \text{for all} \quad (t, u, v) \in [0, T] \times \mathbb{R}^2. \tag{77}$$

From (75), (76) and (77) it follows that

$$|(\psi(u'))'|_1 = |f(\cdot, u(\cdot), u'(\cdot))|_1 \leq \int_0^T f(t, u(t), u'(t))\, dt + 2\|e^-\|_1 = 2\|e^-\|_1. \tag{78}$$

Because $u \in C^1$ is such that $u(0) = u(T)$, there exists $\xi \in [0, T]$ such that $u'(\xi) = 0$, which implies $\psi(u'(\xi)) = 0$ and

$$\psi(u'(t)) = \int_\xi^t (\psi(u'(s)))'\, ds \quad (t \in [0, T]).$$

Using the equality above and (78) we have that

$$|\psi(u'(t))| \leq 2\|e^-\|_1 \quad (t \in [0, T]),$$

and hence $\|u'\|_\infty \leq R_\psi$. ■

We can use this result to obtain a sufficient condition for the existence of multiple solution for a forced pendulum equation with bounded ϕ.

Corollary 9 *Assume that $\psi : \mathbb{R} \to (-c, c)$ $(0 < c \leq \infty)$ is an increasing homeomorphism such that $\psi(0) = 0$ and*

$$|[h - \mu]^-|_1 < c/2.$$

If $|h|_\infty \leq \mu$ then problem

$$(\psi(u'))' + \mu \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{79}$$

has at least one solution. Moreover, if $|h|_\infty < \mu$, then problem (79) has at least two solutions not differing by a multiple of 2π.

Proof. Let R_ψ be the constant given in Lemma 4 with $e = h - \mu$. Let $b = R_\psi + 1$ and consider an increasing homeomorphism $\phi : (-b, b) \to \mathbb{R}$ such that $\phi = \psi$ on $[-R_\psi, R_\psi]$. It follows that $R_\psi = R_\phi$ and applying Lemma 4, we deduce that u is a solution of (73) with $k \equiv 0$ if and only if u is a solution of (79). Now the result follows from Theorem 11. ■

Example 5 *If $h \in C$ is such that*

$$\overline{h} = 0, \quad |h|_\infty < \mu < \frac{1}{2T},$$

then the problem

$$\Big(\frac{u'}{\sqrt{1 + u'^2}}\Big)' + \mu \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

has at least two solutions not differing by a multiple of 2π.

1.4.11 Ambrosetti-Prodi problem with coercive restoring force and singular ϕ

In this subsection, we consider, following [20], periodic problems of the type

$$(\phi(u'))' + k(u)u' + f(t, u, u') = s, \quad u(0) = u(T), \quad u'(0) - u'(T) \tag{80}$$

when $s \in \mathbb{R}$, $k : \mathbb{R} \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and g satisfies the *coercivity condition*

$$(H_f) \quad g(t, u, v) \to +\infty \quad \text{if} \quad |u| \to \infty \quad \text{uniformly in} \quad [0, T]\times]-a, a[. \tag{81}$$

We are interested in studying the existence and multiplicity of the solutions of (80) in terms of the value of the parameter s *(Ambrosetti-Prodi problem [3, 71])*.

We first obtain an *a priori* bound for the set of possible solutions.

Lemma 5 *For each $b \in \mathbb{R}$, there exists $\rho = \rho(b) > 0$ such that any possible solution u of (80) with $s \leq b$ belongs to the open ball $B(\rho)$.*

Proof. Let $s \leq b$ and u be a solution of (80). This implies that u satisfies

$$|u'|_\infty < a \tag{82}$$

and

$$T^{-1} \int_0^T g(t, u(t), u'(t))\, dt = s. \tag{83}$$

Using (81) we can find $R > 0$ such that

$$f(t, u, v) > b \quad \text{if} \quad |u| \geq R, \quad (t, v) \in [0, T] \times]-a, a[. \tag{84}$$

If $u_L \geq R$, then using (82) and (84), we deduce that

$$T^{-1} \int_0^T g(t, u(t), u'(t))\, dt > b,$$

which, together with (83) gives $s > b$, a contradiction. So we have $u_L < R$. Analogously we can show that $u_M > -R$. Then using the inequality

$$u_M \leq u_L + \int_0^T |u'(\tau)| d\tau,$$

we obtain $|u|_\infty < R + Ta$. We can take any $\rho \geq R + (T + 1)a$. ■

Theorem 12 *If g satisfies condition (81), there exists $s_1 \in \mathbb{R}$ such that problem (80) has zero, at least one or at least two solutions according to $s < s_1, s = s_1$ or $s > s_1$.*

Proof. Let

$$S_j = \{s \in \mathbb{R} : (80) \text{ has at least j solutions } \} \quad (j \geq 1).$$

(a) $S_1 \neq \varnothing$.
Take $s^* > \max_{t \in [0,T]} g(t, 0, 0)$ and use (81) to find $R_+^* > 0$ such that

$$\max_{t \in [0,T]} g(t, R_+^*, 0) > s^*.$$

Then $\beta \equiv R_+^* > 0$ is a strict upper solution and $\alpha \equiv 0$ is a strict lower solution for (80) with $s = s^*$. Hence, using Theorem 9, $s^* \in S_1$.
(b) If $\widetilde{s} \in S_1$ and $s > \widetilde{s}$ then $s \in S_1$.
Let $\widetilde{u}$ be a solution of (80) with $s = \widetilde{s}$, and let $s > \widetilde{s}$. Then $\widetilde{u}$ is a strict upper solution for (80). Take now $R_- < \widetilde{u}_L$ such that $\min_{t \in [0,T]} g(t, R_-, 0) > s$: $\alpha \equiv R_-$ is a strict lower solution for (80). From Theorem 9, $s \in S_1$.
(c) $s_1 = \inf S_1$ is finite and $S_1 \supset]s_1, \infty[$.

Let $s \in \mathbb{R}$ and suppose that (80) has a solution u. Then (82) and (83) hold, implying that $s \geq c$, with $c = \inf_{[0,T]\times\mathbb{R}\times\,]-a,a[}\, g$. To obtain the second part of claim (c), we apply (b).

(d) $S_2 \supset\,]s_1, \infty[$.

Let $s_3 < s_1 < s_2$. For each $s \in \mathbb{R}$, let $\mathcal{M}(s,\cdot)$ be the fixed point operator in C^1 associated to problem (80). Using Lemma 5 we find ρ such that each possible zero of $I - \mathcal{M}(s,\cdot)$ with $s \in [s_3, s_2]$ is such that $u \in B(\rho)$. Consequently, the Leray-Schauder degree $d_{LS}[I-\mathcal{M}(s,\cdot), B(\rho), 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $u - \mathcal{M}(s_3, u) \neq 0$ for all $u \in C^1$. This implies that $d_{LS}[I - \mathcal{M}(s_3,\cdot), B(\rho), 0] = 0$, so that $d_{LS}[I - \mathcal{M}(s_2,\cdot), B(\rho), 0] = 0$ and, by excision property of Leray-Schauder degree, $d_{LS}[I-\mathcal{M}(s_2,\cdot), B(\rho'), 0] = 0$ if $\rho' > \rho$. Let $s \in\,]s_1, s_2[$ and $\widehat{u}$ be a solution of (80) (using (c)). Then $\widehat{u}$ is a strict upper solution of (80) with $s = s_2$. Let $R < \widehat{u}_L$ be such that $\min_{t\in[0,T]} g(t, R, 0) > s_2$. Then R is a strict lower solution of (80) with $s = s_2$. Consequently, using Theorem 9, problem (80) with $s = s_2$ has a solution in $\Omega_{R,\widehat{u}}$ and $d_{LS}[I-\mathcal{M}(s_2,\cdot), \Omega_{R,\widehat{u}}, 0] = -1$. Taking ρ' sufficiently large, we deduce from the additivity property of Leray-Schauder degree that

$$\begin{aligned} d_{LS}[I - \mathcal{M}(s_2,\cdot), B(\rho')\backslash\overline{\Omega}_{R,\widehat{u}}, 0] &= d_{LS}[I - \mathcal{M}(s_2,\cdot), B(\rho'), 0] \\ -d_{LS}[I - \mathcal{M}(s_2,\cdot), \Omega_{R,\widehat{u}}, 0] &= -d_{LS}[I - \mathcal{M}(s_2,\cdot), \Omega_{R,\widehat{u}}, 0] = -1, \end{aligned}$$

and (80) with $s = s_2$ has a second solution in $B(\rho') \setminus \overline{\Omega}_{R,\widehat{u}}$.

(e) $s_1 \in S_1$.

Let (τ_k) be a sequence in $]s_1, +\infty[$ converging to s_1, and let u_k be a solution of (80) with $s = \tau_k$ given by (*c*). Using Theorem 3 we deduce that

$$u_k = \mathcal{M}(\tau_k, u_k). \tag{85}$$

From Lemma 5, there exists $\rho > 0$ such that $\|u_k\|_1 < \rho$ for all $k \geq 1$. The complete continuity of $\mathcal{M}$ implies that, up to a subsequence, the right-hand member of (85) converges in C^1, and hence (u_k) converges to some $u \in C^1$ such that $u = \mathcal{M}(s_1, u)$, i.e. to a solution of (80) with $s = s_1$. ■

A similar proof provides the following dual Ambrosetti-Prodi condition.

Theorem 13 *If g satisfies the anticoercivity condition*

$$g(t,u,v) \to -\infty \quad \textit{if} \quad |u| \to \infty \quad \textit{uniformly in} \quad [0,T]\times\,]-a,a[. \tag{86}$$

there exists $s_1 \in \mathbb{R}$ such that problem (80) has zero, at least one or at least two solutions according to $s > s_1, s = s_1$ or $s < s_1$.

Corollary 10 *Let $e \in C$, $k, g : \mathbb{R} \to \mathbb{R}$ be continuous be such that*

$$g(u) \to +\infty \quad (\textit{resp.} -\infty) \quad \textit{if} \quad |u| \to \infty.$$

Then, there exists $s_1 \in \mathbb{R}$ such that the problem

$$(\phi(u'))' + k(u)u' + g(u) = s + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has no solution if $s < s_1$ (resp. $s > s_1$), at least one solution if $s = s_1$ and at least two solutions if $s > s_1$ (resp. $s < s_1$).

Example 6 *For each $e \in C$, $k : \mathbb{R} \to \mathbb{R}$ continuous, $p > 0$ and $c > 0$ (resp. $c < 0$), there exists $s_1 \in \mathbb{R}$ such that the problem*

$$\left(\frac{u'}{\sqrt{1 - u'^2}} \right)' + f(u)u' + c|u|^p = s + e(t),$$
$$u(0) = u(T), \quad u'(0) = u'(T)$$

has no solution if $s < s_1$ (resp. $s > s_1$), at least one solution if $s = s_1$ and at least two solutions if $s > s_1$ (resp. $s < s_1$).

1.4.12 Ambrosetti-Prodi problem with bounded restoring force and singular ϕ

The coercivity condition upon g can be replaced by a boundedness condition without loosing the Ambrosetti-Prodi type conclusion. Consider periodic boundary value problems of the form

$$(\phi(u'))' + k(u)u' + g(u) = e(t) + s, \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{87}$$

where $s \in \mathbb{R}$ is a parameter, $e \in C$, $k : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are continuous and the following conditions hold :

(H1) $\int_0^T e(t)dt = 0$.

(H2) $g(u) > 0$ for all $u \in \mathbb{R}$.

(H3) $\lim_{u \to \pm\infty} g(u) = 0$.

We write $g_M := \max_{\mathbb{R}} g$. The classical problem

$$u'' + g(u) = e(t) + s$$

was considered by Ward in [111] using a variational method and under the supplementary condition that the indefinite integral G is g is bounded. The case of singular ϕ presented here was given in [23].

Consider

$$S_j = \{s \in \mathbb{R} : (87) \text{ has at least j solutions } \} \quad (j \geq 1).$$

Lemma 6 *If $s \in S_1$, then $0 < s \leq g_M$.*

Proof. Assumptions (H2) and (H3) imply that g is bounded and $0 < g(u) \leq g_M$ for all $u \in \mathbb{R}$. Hence, if u is a solution of (87) then, using (H1), it follows that

$$T^{-1} \int_0^T g(u(t))dt = s, \tag{88}$$

and $0 < s \leq g_M$. ■

For $s \in \mathbb{R}$, we define the continuous operator $N_f : C^1 \times \mathbb{R} \to C^1$ by

$$N_f(u,s)(t) = \int_0^t [e(\tau) + s - k(u(\tau))u'(\tau) - g(u(\tau))]\, d\tau \quad (t \in [0,T]).$$

Using Theorem 3, it follows that $u \in C^1$ is a solution of (87) if and only if

$$u = Pu - N_s(u)T) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ N_f(u,s) := \mathcal{G}(u,s),$$

and the nonlinear operator $\mathcal{G}(\cdot, s) : C^1 \to C^1$ is completely continuous.

Let $n : C^1 \to C$ be the continuous mapping defined by

$$n(u)(t) = e(t) - k(u(t))u'(t) - g(u(t)) \quad (t \in [0,T]),$$

and $\widehat{M} : \mathbb{R} \times \widehat{C}^1 \to \widehat{C}^1$ be the completely continuous operator defined by

$$\widehat{M}(u_0, \widehat{u}) = H \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ (I - Q)n(u_0 + \widehat{u}).$$

If u is a solution of (87), then (88) holds and $\widehat{u} = \widehat{M}(u_0, \widehat{u})$. Reciprocally, if $[u_0, \widehat{u}] \in \mathbb{R} \times \widehat{C}^1$ is such that $\widehat{u} = \widehat{M}(u_0, \widehat{u})$, then $u = u_0 + \widehat{u}$ is a solution of (87) with $s = T^{-1} \int_0^T g(u(t))dt$.

Using Lemma 3, we deduce the following useful result.

Lemma 7 *The set of the solutions $[u_0, \widehat{u}] \in \mathbb{R} \times \widehat{C}^1$ of problem*

$$(\phi(\widetilde{u}'))' + k(u_0 + \widehat{u})\widehat{u}' + g(u_0 + \widehat{u}) = e(t) + T^{-1} \int_0^T g(u_0 + \widehat{u}(t), \widehat{u}'(t))\, dt \quad (89)$$

contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and projection on $\widehat{C}^1$ is contained in an open ball $B(\rho)$.

Let $\gamma : \mathbb{R} \times \widehat{C}^1 \to \mathbb{R}$ be the continuous function defined by

$$\gamma(u_0, \widehat{u}) = T^{-1} \int_0^T g(u_0 + \widehat{u}(t))\, dt.$$

Lemma 8 $S_1 \neq \varnothing$.

Proof. Let $[u_0, \widehat{u}] \in \mathcal{C}$. Then $u = u_0 + \widehat{u}$ is a solution of (87) with $s = \gamma(u_0, \widehat{u})$. ■

Let us consider

$$s^*(e) = \sup S_1.$$

Lemma 9 *We have that $0 < s^*(e) \leq g_M$ and $s^*(e) \in S_1$.*

Proof. The first assertion follows from Lemma 6. Let $\{s_n\}$ be a sequence belonging to S_1 which converges to $s^*(e)$. Let $u_n = u_{n,0} + \widehat{u}_n$ be a solution of (87) with $s = s_n = \gamma(u_{n,0}, \widehat{u}_n)$. It follows that $\widehat{u}_n = \widehat{M}(u_{n,0}, \widehat{u}_n)$ and $\{\widehat{u}_n\}$ belongs to $B(\rho)$. Hence, if up to a subsequence $u_{n,0} \to \pm\infty$, then using (H3) it follows that $\gamma(u_{n,0}, \widehat{u}_n) \to 0$, which means that $s^*(e) = 0$, a contradiction. We have proved that $\{(u_{n,0}, \widehat{u}_n)\}$ is a bounded sequence in $\mathbb{R} \times \widetilde{C}^1$. Because $\widetilde{M}$ is completely continuous, we can assume, passing to a subsequence, that $\widehat{M}(u_{n,0}, \widehat{u}_n) \to \widehat{u}$ and $u_{n,0} \to u_0$. We deduce that $\widehat{u} = \widehat{M}(u_0, \widehat{u})$, $\gamma(u_0, \widehat{u}) = s^*(e)$ and $u = u_0 + \widehat{u}$ is a solution of (87) with $s = s^*(e)$. ∎

Arguing as in the proof of Lemma 9 we deduce the following a priori estimate result.

Lemma 10 *Let $0 < s_1 < s^*(e)$. Then, there is $\rho' > 0$ such that any possible solution u of (87) with $s \in [s_1, s^*(e)]$ belongs to $B(\rho')$.*

Lemma 11 *We have $(0, s^*(e)) \subset S_2$.*

Proof. Let $s_1, s_2 \in \mathbb{R}$ such that $0 < s_1 < s^*(e) < s_2$. Using Lemma 6, Lemma 10 and the invariance property of Leray-Schauder degree, it follows that there is $\rho' > 0$ sufficiently large such that $d_{LS}[I - \mathcal{G}(s, \cdot), B(\rho'), 0]$ is well defined and independent of $s \in [s_1, s_2]$. However, using Lemma 6 we deduce that $u - \mathcal{G}(s_2, u) \neq 0$ for all $u \in C^1$. This implies that $d_{LS}[I - \mathcal{G}(s_2, \cdot), B(\rho'), 0] = 0$, so that $d_{LS}[I - \mathcal{G}(s_1, \cdot), B(\rho'), 0] = 0$ and, by excision property of Leray-Schauder degree,

$$d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''}, 0] = 0 \quad \text{if} \quad \rho'' \geq \rho'. \tag{90}$$

Let u_* be a solution of (87) with $s = s^*(e)$ (using Lemma 9). Then, u_* is a strict lower solution of (87) with $s = s_1$. Using Lemma 7 and (H3), there is $[u_0^*, \widehat{u}^*] \in \mathcal{C}$ such that $u^* = u_0^* + \widehat{u}^* > u_*$ on $[0, T]$ and $\gamma(u_0^*, \widehat{u}^*) < s_1$. It follows that u^* is an upper solution of (87) with $s = s_1$. So, using Theorem 9, we have that

$$d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0] = 1, \tag{91}$$

for some $r > 0$, and (87) has a solution in $\Omega^r_{u_*, u^*}$. Taking ρ'' sufficiently large and using (90) and (91), we deduce from the additivity property of Leray-Schauder degree that

$$\begin{aligned} d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''} \backslash \overline{\Omega^r}_{u_*, u^*}, 0] &= d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''}, 0] \\ -d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0] &= -d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0] = 1, \end{aligned}$$

and (87) with $s = s_1$ has a second solution in $B_{\rho''} \setminus \overline{\Omega^r}_{u_*, u^*}$. ∎

Theorem 14 *If conditions (H1)-(H3) hold, there exists $s^*(e) \in (0, \sup_R g]$ such that problem (87) has zero, at least one or at least two solutions according to $s \notin (0, s^*(e)]$, $s = s^*(e)$ or $s \in (0, s^*(e))$.*

Proof. The conclusion of Theorem 14 follows from Lemmas 6, 9 and 11. ∎

Example 7 Let $e \in C$ be such that $\int_0^T e(t)dt = 0$. If $b > 0$ and $c \geq 0$ then there is $s^* > 0$ such that the periodic boundary value problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \frac{cu'^4 + b}{1+|u|} = e(t) + s, \quad u(0) = u(T), \quad u'(0) - u'(T),$$

has zero, at least one or at least two solutions according to $s \notin (0, s^*]$, $s = s^*$ or $s \in (0, s^*)$.

1.4.13 Equations with singular restoring forces and singular ϕ

In this subsection we prove, following [20], the existence of *positive* solutions for the following periodic problems with *singular attractive* restoring force

$$(\phi(u'))' + g(u) = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{92}$$

or with *singular repulsive* restoring force

$$(\phi(u'))' - g(u) = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{93}$$

where $e \in C$ and $g :]0, +\infty[\to]0, +\infty[$ is continuous and such that

$$g(u) \to +\infty \quad \text{as} \quad u \to 0+, \tag{94}$$

$$g(u) \to 0 \quad \text{as} \quad u \to +\infty. \tag{95}$$

The classical case where $\phi(s) \equiv s$ was first considered by Lazer and Solimini in [55], and the case of a p-Laplacian in [52].

Theorem 15 *Suppose that g satisfies conditions (94) and (95). Then (92) has at least one solution if and only if $\int_0^T e(t)\,dt > 0$.*

Proof. If u is a solution of (92), then $Qe = Qg(u) > 0$ because g is positive. Conversely, suppose that $Qe > 0$. Using (94), there exists $\epsilon > 0$ such that $g(\epsilon) > e(t)$ for all $t \in [0, T]$. Hence, $\alpha \equiv \epsilon$ is a strict lower solution for (92). On the other hand, using Corollary 5, there exists $w \in C^1_T$ such that $(\phi(w'))' = e(t) - Qe$. Using (95), there exists some $\delta > 0$ such that $\beta(t) = \delta + w(t) > \alpha(t)$ and $g(\beta(t)) < Qe$ for all $t \in [0, T]$. Then, β is a strict upper solution for (92) and Theorem 9 implies the result. ■

Example 8 *If $\mu > 0$ and $e \in C$, the problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \frac{1}{u^\mu} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one solution if and only if $\int_0^T e(t)\,dt > 0$.

To solve (93) we need the following supplementary condition

$$\int_0^1 g(u)\,du = +\infty. \tag{96}$$

Lemma 12 *Suppose that g satisfies conditions (94), (95), and (96). There exists $\epsilon > 0$ such that if $\lambda \in [0,1]$ and u is any positive solution of*

$$\begin{aligned}(\phi(u'))' = (1-\lambda)[Qg(u) + Qe] + \lambda g(u) + \lambda e(t),\\ u(0) = u(T), \quad u'(0) = u'(T),\end{aligned} \tag{97}$$

then $u(t) > \epsilon$ for all $t \in [0,T]$.

Proof. Let $\lambda \in [0,1]$ and u be a possible positive solution of (97). Then

$$Qg(u) + Qe = 0 \tag{98}$$

and hence, if $\lambda \in]0,1]$, (97) is equivalent to

$$(\phi(u'))' = \lambda g(u) + \lambda e(t), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{99}$$

Using the positivity of g, we deduce that

$$|g(u) + e(t)| \le g(u) + |e(t)| = g(u) + e(t) + 2e^-(t) \tag{100}$$

for all $(t,u) \in [0,T] \times \mathbb{R}$. From (99), (98) and (100) it follows that (with $|\cdot|_1$ the L^1-norm on $[0,T]$)

$$|(\phi(u'))'|_1 = \lambda \|N_g(u) + e\|_1 \le 2\lambda |e^-|_1. \tag{101}$$

Because $u \in C^1$ is such that $u(0) = u(T)$, there exists $\eta \in [0,T]$ such that $u'(\eta) = 0$, which implies $\phi(u'(\eta)) = 0$ and

$$\phi(u'(t)) = \int_\eta^t (\phi(u'(s)))'\,ds \quad (t \in [0,T]).$$

Using the equality above and (101) we have that

$$|\phi(u'(t))| \le 2\lambda |e^-|_1 \quad (t \in [0,T]), \tag{102}$$

Using (94), there exists $\xi > 0$ such that

$$g(u) > -Qe \quad \text{for all} \quad 0 < u \le \xi. \tag{103}$$

and therefore, by (103) and (98), there exists $t_1 \in [0,T]$ such that $u(t_1) > \xi$. Now, let

$$v(t) = \phi(u'(t)) \tag{104}$$

which implies

$$u'(t) = \phi^{-1}(v(t)) \tag{105}$$

for all $t \in [0, T]$. Introducing (104) in (99) we obtain

$$v'(t) - \lambda g(u(t)) = \lambda e(t) \tag{106}$$

for all $t \in [0, T]$. Multiplying (105) by $v'(t)$ and (106) by $u'(t)$ and subtracting we get

$$v'(t)\phi^{-1}(v(t)) - \lambda g(u(t))u'(t) = \lambda e(t)u'(t)$$

i.e.

$$\left(\int_0^{v(t)} \phi^{-1}(s)\, ds \right)' - \lambda g(u(t))u'(t) = \lambda e(t)u'(t)$$

for all $t \in [0, T]$. This implies that

$$\int_0^{v(t)} \phi^{-1}(s)\, ds - \int_0^{x(t_1)} \phi^{-1}(s)\, ds - \lambda \int_{u(t_1)}^{u(t)} g(s) ds = \lambda \int_{t_1}^{t} e(s)u'(s) ds$$

for all $t \in [0, T]$. Using the fact that $\int_0^v \phi^{-1}(s)\, ds \geq 0$ for all $v \in \mathbb{R}$, we deduce that

$$\lambda \int_{u(t)}^{u(t_1)} g(s) ds \leq \int_0^{v(t_1)} \phi^{-1}(s)\, ds + \lambda \int_{t_1}^{t} e(s)u'(s) ds \tag{107}$$

for all $t \in [0, T]$. Using (102), (104) and (107), we obtain

$$\begin{aligned} \int_{u(t)}^{u(t_1)} g(s) ds &\leq \frac{1}{\lambda} \left(\int_0^{2\lambda |e^-|_1} \phi^{-1}(s)\, ds + \int_0^{-2\lambda |e^-|_1} \phi^{-1}(s)\, ds \right) + a\|e\|_1 \\ &\leq \max_{[-2|e^-|_1, 2|e^-|_1]} 2|\phi^{-1}||e^-|_1 + a|e|_1 := c \end{aligned} \tag{108}$$

for all $t \in [0, T]$. Using (96) we can find $0 < \epsilon < \xi$ such that

$$\int_\epsilon^\xi g(t) dt > c. \tag{109}$$

Since $u(t_1) > \xi$ and g is positive, from (108) and (109) one gets $u(t) > \epsilon$ for all $t \in [0, T]$. Now, for $\lambda = 0$, the solutions of (97) are the constant functions u solutions of

$$g(u) + Qe = 0$$

and they satisfy $u > \xi > \epsilon$. ■

Theorem 16 *Suppose that $e \in C$ and g satisfies conditions (94), (95), and (96). Then (93) has at least one positive solution if and only if $\int_0^T e(t)\,dt < 0$.*

Proof. If u is a solution, then $Qe = -Qg(u) < 0$. For sufficiency, we use the homotopy (97) and the corresponding homotopy for the associated family of fixed point operators $\mathcal{M}(\lambda, \cdot)$ defined in (23) with $f = g + e$. Let $\lambda \in [0, 1]$ and u be a possible positive solution of (97). We already know from Lemma 12 that $u(t) > \epsilon$ for some $\epsilon > 0$ and all $t \in [0, T]$. From assumption (95) follows easily the existence of $R > 0$ such that $g(u) < -Qe$ if $u \geq R$. Hence, because of (98), there exists $t_2 \in [0, T]$ such that $u(t_2) < R$, which implies $u(t) < R + aT \quad (t \in [0, T])$. Hence, all the possible positive solutions of problem (97) are contained in the open bounded set

$$\Omega := \{u \in C^1 : \epsilon < u(t) < R + aT \ (0 \leq t \leq T),\ |u'|_\infty < a\}.$$

From the homotopy invariance of Leray-Schauder degree, we obtain

$$\begin{aligned} & d_{LS}[I - \mathcal{M}(1, \cdot), \Omega, 0] = d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] \\ &= d_B[g + Qe, \Omega \cap \mathbb{R}, 0] = d_B[g + Qe,]\epsilon, R[, 0] = -1, \end{aligned}$$

so that that $\mathcal{M}(1, \cdot)$ has a fixed point. ■

Example 9 *If $\mu \geq 1$ and $e \in C$, problem*

$$\left(\frac{u'}{\sqrt{1 - u'^2}}\right)' - \frac{1}{u^\mu} = e(t), \quad u(0) = u(T), \quad u'(0) - u'(T)$$

has at least one positive solution if and only if $\int_0^T e(t)\,dt < 0$.

1.5 Neumann problem for singular ϕ

1.5.1 Equivalent fixed point problem

Let us consider the homogeneous Neumann problem for continuous $f : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$

$$(\phi(u'))' = f(t, u, u'), \quad u'(0) = 0 = u'(T). \tag{110}$$

The proofs of the following results are similar to the corresponding ones for the periodic problem and left to the reader as an exercise. The result already appeared in [18] together with an application to sign conditions for exsitence.

Theorem 17 *u is a solution of the Neumann problem (110) if and only if $u \in C^1$ is a fixed point of the operator K defined on C^1 by*

$$K(u) = Pu - N_f(u)(T) + H \circ \phi^{-1} \circ N_f(u). \tag{111}$$

Furthermore, $|(K(u))'|_\infty < a$ for all $u \in C^1$ and K is completely continuous on C^1.

Theorem 18 *Assume that there exists an open bounded set $\Omega \subset C$ such that the following conditions hold :*

1. *for each $\lambda \in (0,1]$, there is no solution of problem*

$$(\phi(u')' = \lambda f(t,u,u'), \quad u'(0) = 0 = u'(T) \tag{112}$$

such that $u \in \partial\Omega$.

2. *there is no solution on $\partial\Omega \cap \mathbb{R}^n$ of equation*

$$\overline{f}(u) := N_f(u)(T) = 0,$$

where $\mathbb{R}^n$ denotes the subspace of constant functions in C.

3. $d_B[\overline{f}, \Omega \cap \mathbb{R}^n, 0] \neq 0$.

Then problem (110) has at least one solution such that $u \in \Omega$, and, for the associated fixed point operator K, one has

$$d_{LS}[I - K, \Omega_\rho, 0] = d_B[\overline{f}, \Omega \cap \mathbb{R}, 0],$$

where $\rho \geq a$ and $\Omega_\rho \subset C^1$ is the open bounded set defined by

$$\Omega_1 = \{u \in C^1 : u \in \Omega,\ |u'|_\infty < \rho\}.$$

Theorem 19 *The set $\mathcal{S}$ of the solutions $(\overline{u}, \widehat{u}) \in \mathbb{R} \times \widetilde{C}^1$ of problem*

$$(\phi(\widehat{u}'))' = (I - Q)F(\overline{u} + \widehat{u}), \quad u'(0) = 0 = u'(T) \tag{113}$$

contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and projection on $\widehat{C}^1$ is contained in the ball $B(a(T+1))$.

1.5.2 Existence theorems

Let C^1_N be the space

$$C^1_N := \{u \in C^1 : \quad u'(0) = 0 = u'(T)\}.$$

For $u \in C^1_N$, we write $u_L := \min_{[0,T]} u$, $u_M := \max_{[0,T]} u$, $\widehat{u}(t) = u(t) - u(0$.

One can prove, in a similar way as in the periodic case, the Neumann version of the Theorem 8 with C^1_T replaced by C^1_N, so that condition (64) is also necessary and sufficient for the existence of a solution to the problem

$$(\phi(u'))' = h(t), \quad u'(0) = 0 = u'(T). \tag{114}$$

Similarly, the conclusion of Corollary 2 holds, with the same assumptions, for problem

$$(\phi(u'))' + g(t,u) = h(t,u,u'), \quad u'(0) = 0 = u'(T)$$

and the conclusion of Corollary 3 holds, with the same assumptions, for problem

$$(\phi(u'))' + \mu u = h(t, u, u'), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

In particular, problem

$$(\phi(u'))' + \mu u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution for any $h \in C$, and one can say again that the system is *non-resonant.* Consequently, $\mu = 0$ is the only value for which *resonance* occurs for Neumann boundary conditions.

One can easily check also that the conclusions of Corollaries 6 and 4 hold, with the same assumptions, but for 'relativistic Duffing equations' only

$$(\phi(u'))' + g(u) = h(t), \quad u'(0) = 0 = u'(T),$$

instead of the more general class of Liénard equations.

The results about lower and upper solutions, Ambrosetti-Prodi-type results, singular nonlinearities and periodic nonlinearities have their counterpart, with similar proofs, for the Neumann problem.

2 Lagrangian variational approach for periodic solutions

2.1 Introduction

In this section, we consider the existence of solutions of the problem

$$(\phi(u'))' = \nabla_u F(t, u) + h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{115}$$

where $\phi : B(a) \subset \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism satisfying Assumption (H_Φ), $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is such that $\nabla_u F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ exists and satisfies Carathéodory conditions, and $h \in L^1(0, T; \mathbb{R}^n)$.

For the classical case

$$u'' = \nabla_u F(t, u) + h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{116}$$

existence results were proved through the direct method of the calculus of variations, by Berger and Schechter [26, 27], when $F(t, u) - \langle h(t), u\rangle$ is coercive in u uniformly in $t \in [0, T]$, and in [78] when $\int_0^T [F(t, u) - \langle h(t), u\rangle]\, dx$ is coercive and, either $\nabla_u F(t, u)$ is bounded or $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$. Many extensions have been given for wider classes of potentials F and for u'' replaced by the p-Laplacian $(|u'|^{p-2}u)'$ (see [59],[93]-[106],[110]-[114] and their references). The case of a bounded ϕ has been considered in [30], whose results are described in this Section using a variant of the method of [30] introduced in [14].

An existence theorem was proved for (116) in [77, 78], when F is periodic in each variable u_i for a.e. $t \in [0, T]$ and h has mean value zero, easily extended to the case of the p-Laplacian as shown in [72]. When u'' is replaced by a 'relativistic' differential operator $(\phi(u'))'$ like above, the scalar case was considered in [29] and the case of system (115), under conditions upon F of the type covered in [27] and [78]. No proof based upon topological methods of the results given here is known by now.

2.2 The functional framework

2.2.1 The functional

The following variational setting for dealing with equations or systems of type (115) was first introduced in [14].

In $\mathbb{R}^n$, we denote the usual inner product by $\langle \cdot, \cdot \rangle$ and the corresponding Euclidian norm by $|\cdot|$. We denote the usual norm in $L^p := L^p := L^p(0,T;\mathbb{R}^n)$ $(1 \le p \le \infty)$ by $\|\cdot\|_p$. We set $C := C([0,T],\mathbb{R}^n)$ and $W^{1,\infty} := W^{1,\infty}([0,T],\mathbb{R}^n)$. The usual norm $\|\cdot\|_\infty$ is considered on C and L^∞. The space $W^{1,\infty}$ is endowed with the norm

$$\|v\| = \|v\|_\infty + \|v'\|_\infty, \quad v \in W^{1,\infty}.$$

each $v \in L^1$ can be written $v(t) = \overline{v} + \widetilde{v}(t)$, with

$$\overline{v} := T^{-1} \int_0^T v(t)\, dt, \quad \int_0^T \tilde{v}(t)\, dt = 0.$$

If $v \in W^{1,\infty}$ then each component $\tilde{v}_j$ vanishes at some $t_j \in [0,T]$ and

$$|\tilde{v}_j(t)| = |\tilde{v}_j(r) - \tilde{v}_j(t_j)| \le \int_0^T |v_j'(t)|\, dt \le T\|v_j'\|_\infty,$$

so, one has that

$$||\widetilde{v}||_\infty \le T\|v'\|_\infty. \tag{117}$$

Putting

$$K := \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \le a, \quad v(0) = v(T)\},$$

it is clear that K is a convex subset of $W^{1,\infty}$.

Let $\Psi : C \to (-\infty, +\infty]$ be defined by

$$\Psi(v) = \begin{cases} \varphi(v), & \text{if } v \in K, \\ \\ +\infty, & \text{otherwise,} \end{cases} \tag{118}$$

with $\varphi : K \to \mathbb{R}$ given by

$$\varphi(v) = \int_0^T \Phi(v'(t))\, dt, \quad v \in K. \tag{119}$$

Obviously, Ψ is proper and convex. The following Lemma allows to prove the lower semicontinuity of Ψ.

Lemma 13 *If $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(t) \to u(t)$ for all $t \in [0,T]$, then*

(i) $u \in K$;

(ii) $u_n' \to u'$ in the w^*–topology $\sigma(L^\infty, L^1)$.

Proof. From the relation

$$|u_n(t_1) - u_n(t_2)| = \left| \int_{t_2}^{t_1} u_n'(t)\,dt \right| \leq a|t_1 - t_2|,$$

letting $n \to \infty$, we get

$$|u(t_1) - u(t_2)| \leq a|t_1 - t_2| \quad (t_1, t_2 \in [0,T]),$$

which yields $u \in K$.

Next, we show that that if $\{u_k'\}$ is a subsequence of $\{u_n'\}$ with $u_k' \to v \in L^\infty$ in the w^*–topology $\sigma(L^\infty, L^1)$ then

$$v = u' \quad \text{a.e. on } [0,T]. \tag{120}$$

Indeed, as

$$\int_0^T u_k'(t) f(t)\,dt \to \int_0^T v(t) f(t)\,dt \quad \text{for all} \quad f \in L^1(0,T;\mathbb{R}),$$

taking $f \equiv \chi_{t_1,t_2}$, the characteristic function of the interval having the endpoints $t_1, t_2 \in [0,T]$, it follows

$$\int_{t_1}^{t_2} u_k'(t)\,dt \to \int_{t_1}^{T} t_2 v(t)\,dt \quad (t_1, t_2 \in [0,T]).$$

Then, letting $k \to \infty$ in

$$u_k(t_2) - u_k(t_1) = \int_{t_1}^{t_2} u_k'(t)\,dt$$

we obtain

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} v(t)\,dt \quad (t_1, t_2 \in [0,T])$$

which, clearly implies (120).

Now, to prove *(ii)* it suffices to show that if $\{u_j'\}$ is an arbitrary subsequence of $\{u_n'\}$, then it contains itself a subsequence $\{u_k'\}$ such that $u_k' \to u'$ in the w^*–topology $\sigma(L^\infty, L^1)$. Since L^1 is separable and $\{u_j'\}$ is bounded in $L^\infty = (L^1)^*$, we know that it has a subsequence $\{u_k'\}$ convergent to some $v \in L^\infty$ in the w^*–topology $\sigma(L^\infty, L^1)$. Then, as shown before (see (120)), we have $v = u'$. ■

Consequently, if $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(t) \to u(t)$ for all $t \in [0,T]$, then $u \in K$ and

$$\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n). \tag{121}$$

This implies that Ψ is lower semicontinuous on C. Also, note that K is closed in C.

Next, let $\mathcal{G} : C \to \mathbb{R}$ be defined by

$$\mathcal{G}(u) = \int_0^T [F(t,u(t)) + \langle h(t), u(t)\rangle]\, dt, \quad u \in C.$$

A standard reasoning shows that $\mathcal{G}$ is of class C^1 on C and its derivative is given by

$$\mathcal{G}'(u)(v) = \int_0^T \langle \nabla F(t,u(t)) + h(t), v(t)\rangle\, dt, \quad u, v \in C.$$

2.2.2 Critical points and solutions of differential systems

The functional $I : C \to (-\infty, +\infty]$ defined by

$$I = \Psi + \mathcal{G} \tag{122}$$

has the structure required by Szulkin's critical point theory [97], namely the sum of a proper convex lower semicontinuous function and of a function of class C^1.

Accordingly, a function $u \in C$ is *a critical point* of I if $u \in K$ and satisfies the inequality

$$\Psi(v) - \Psi(u) + \mathcal{G}'(u)(v-u) \geq 0 \quad \text{for all } v \in C, \tag{123}$$

or, equivalently

$$\int_0^T [\Phi(v'(t)) - \Phi(u'(t))]\, dt + \int_0^T \langle \nabla F(t,u(t)) + h(t), v(t) - u(t)\rangle\, dt \geq 0 \quad \text{for all } v \in K.$$

The following simple result is given in [97].

Lemma 14 *Each local minimum of I is a critical point of I.*

Proof. Let u be a local minimum of I. By the convexity of Ψ, given $v \in C$, we have, for all $t \in (0,1]$ sufficiently small,

$$\begin{aligned} 0 &\leq \lambda^{-1}\{I[(1-\lambda)u + \lambda v] - I(u)\} \\ &= \lambda^{-1}\{\Psi[(1-\lambda)u + \lambda v] - \Psi(u) + \mathcal{G}[u + \lambda(v-u)] - \mathcal{G}(u)\} \\ &\leq \Psi(v) - \Psi(u) + \lambda^{-1}\{\mathcal{G}[u + \lambda(v-u)] - \mathcal{G}(u)\} \end{aligned}$$

which gives (123) by letting $\lambda \to 0$. ∎

Now, we consider the periodic boundary value problem (115) under the basic hypothesis (H_Φ). Recall that by a *solution* of (115) we mean a function $u \in C^1$, such that $\|u'\|_\infty < a$, $\phi(u')$ is differentiable a.e. and (115) is satisfied a.e. The following elementary lemma will be useful is relating the critical points of I to the solutions of (115).

Lemma 15 *For every $f \in C$, problem*

$$(\phi(u'))' = \overline{u} + f, \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{124}$$

has a unique solution u_f, which is also the unique solution of the variational inequality

$$\int_0^T [\Phi(v'(t)) - \Phi(u'(t)) + \langle \overline{u}(t), \overline{v}(t) - \overline{u}(t)\rangle + \langle f(t), v(t) - u(t)\rangle]\, dt \geq 0$$

$$\text{for all } v \in K, \tag{125}$$

and the unique minimum over K of the strictly convex functional J defined by

$$J(u) = \int_0^T \left[\Phi(u'(t)) + \frac{|\overline{u}|^2}{2} + \langle f(t), u(t)\rangle\right] dt. \tag{126}$$

Proof. Problem (124) is equivalent to finding $u = \overline{u} + \widetilde{u}$ with $\overline{u}$ and $\widetilde{u}$ solutions of

$$\begin{cases} (\phi(\widetilde{u}'))' = \widetilde{f}, \quad \widetilde{u}(0) = \widetilde{u}(T), \quad \widetilde{u}'(0) = \widetilde{u}'(T), \\ \overline{u} = -\overline{f}, \quad \int_0^T \widetilde{u}(t)\, dt = 0. \end{cases} \tag{127}$$

Now the first equation gives, with $\widetilde{V}$ denoting the unique primitive of $\widetilde{v} \in \widetilde{C}$ belonging to $\widetilde{C}$,

$$\phi(\widetilde{u}'(t)) - \overline{\phi(\widetilde{u}')} = \widetilde{F}$$

giving, if we let $c = \overline{\phi(\widetilde{u}')}$,

$$\widetilde{u}'(t) = \phi^{-1}[c + \widetilde{F}(t)].$$

Now u will satisfy the first boundary condition if c is such that

$$\int_0^T \phi^{-1}[c + \widetilde{F}(t)]\, dt = 0.$$

Lemma 1 implies the existence and uniqueness of c, and hence the unique solvability of problem (124).

Now, if u is a solution of (124), then, taking $v \in K$, taking the inner product of each member of the differential system by $v - u$, integrating over $[0, T]$, and using integration by parts and the boundary conditions, we get

$$\int_0^T [\langle \phi(u'(t)), v'(t) - u'(t)\rangle + \langle \overline{u}, \overline{v} - \overline{u}\rangle + \langle f(t), v(t) - u(t)\rangle]\, dt = 0,$$

which gives (125) if we use the convexity inequality for Φ

$$\Phi(v') - \Phi(u') \geq \langle \phi(u'), v' - u' \rangle.$$

The convexity inequality

$$\frac{|\overline{v}|^2}{2} - \frac{|\overline{u}|^2}{2} \geq \langle \overline{u}, \overline{v} - \overline{u} \rangle$$

introduced in (125) implies that

$$\int_0^T \left[\Phi(v'(t)) - \Phi(u'(t)) + \frac{|\overline{v}|^2}{2} + \langle f(t), v(t) \rangle - \frac{|\overline{u}|^2}{2} - \langle f(t), u(t) \rangle \right] dt \geq 0 \quad \text{for all } v \in K,$$

which shows that J has a minimum on K at u. Conversely if it is the case, then, for all $\lambda \in (0, 1]$ and all $v \in K$, we get

$$\begin{aligned} & \int_0^T \{ \Phi[(1-\lambda)u'(t) + \lambda v'(t)] + \frac{|(1-\lambda)\overline{u} + \lambda \overline{v}|^2}{2} \\ + \quad & \langle f(t), (1-\lambda)u(t) + \lambda v(t) \rangle \} \, dt \\ \geq \quad & \int_0^T [\Phi(u'(t)) + \frac{|\overline{u}|^2}{2} + \langle f(t), u(t) \rangle] \, dt, \end{aligned}$$

which, using the convexity of Φ, simplifying, dividing both members by λ and letting $\lambda \to 0$, gives the variational inequality (125). Thus solving (125) is equivalent to minimizing (126) over K. Now, it is straightforward to check that J is strictly convex over K and therefore has a unique minimum there, which gives the required uniqueness conclusions of Lemma 59. ■

The idea of proof of the result below first occured in [29].

Proposition 1 *If u is a critical point of I, then u is a solution of problem (115).*

Proof. For u a critical point of I, we set

$$f_u := \nabla F(\cdot, u) + h - \overline{u} \in C$$

and consider the problem

$$(\phi(w'))' = \overline{w} + f_u(t), \quad w(0) = w(T), \quad 'w'(0) = w'(T). \tag{128}$$

By virtue of Lemma 59, problem (128) has an unique solution $\widehat{u}$ and it is also the unique solution of the variational inequality

$$\int_0^T [\Phi(v'(t)) - \Phi(\widehat{u}'(t)) + \langle \overline{\widehat{u}}, (\overline{v} - \overline{\widehat{u}} \rangle + \langle f_u(t), v(t) - \widehat{u}(t)] \, dt \geq 0 \quad \text{for all } v \in K. \tag{129}$$

Since u is a critical point of I, we infer that

$$\int_0^T [\Phi(v'(t)) - \Phi(u'(t)) + \langle \overline{u}, \overline{v} - \overline{u} \rangle + \langle f_u(t), v(t) - u(t) \rangle] \, dt \geq 0 \quad \text{for all } v \in K. \tag{130}$$

It follows by uniqueness that $u = \widehat{u}$. Hence, u solves problem (115). ■

2.3 Ground state solutions

The following results come from [30] and [15].

2.3.1 A sufficient condition for minimization

We begin by a lemma which is the main tool for the minimization problems in this subsection. With this aim, for any $\rho > 0$, set

$$\widehat{K}_\rho := \{u \in K \ : \ |\overline{u}| \leq \rho\}.$$

Lemma 16 *Assume that there is some $\rho > 0$ such that*

$$\inf_{\widehat{K}_\rho} I = \inf_K I. \tag{131}$$

Then I is bounded from below on C and attains its infimum at some $u \in \widehat{K}_\rho$, which solves problem (115).

Proof. By virtue of (131) and $\inf_C I = \inf_K I$, it suffices to prove that there is some $u \in \widehat{K}_\rho$ such that

$$I(u) = \inf_{\widehat{K}_\rho} I. \tag{132}$$

Then, we get that u is a minimum point of I on C, so, on account of Lemma 14, u is a critical point of I, and by virtue of Proposition 1, a solution of (115).

If $v \in \widehat{K}_\rho$ then, using (117) we obtain

$$|v(t)| \leq |\overline{v}| + |\widetilde{v}(r)| \leq \rho + Ta.$$

This, together with $\|v'\|_\infty \leq a$ show that $\widehat{K}_\rho$ is bounded in $W^{1,\infty}$ and, by the compactness of the embedding $W^{1,\infty} \subset C$, the set $\widehat{K}_\rho$ is relatively compact in C. Let $\{u_n\} \subset \widehat{K}_\rho$ be a minimizing sequence for I. Passing to a subsequence if necessary and using Lemma 13, we may assume that $\{u_n\}$ converges uniformly to some $u \in K$. It is easily seen that actually $u \in \widehat{K}_\rho$. From (121) and the continuity of $\mathcal{F}$ on C, we obtain

$$I(u) \leq \liminf_{n \to \infty} I(u_n) = \lim_{n \to \infty} I(u_n) = \inf_{\widehat{K}_\rho} I,$$

showing that (132) holds true. ■

2.3.2 Periodic nonlinearities

The following result deals with problem (115) with periodic nonlinearities. It was first proved in the scalar case in [29] and in the vector case in [30] using different arguments.

Theorem 20 *If there are some* $\omega_1 > 0, \ldots \omega_n > 0$ *such that*

$$F(t,u) = F(t, u + \omega_j e_j)$$

for all $(t,u) \in [0,T] \times \mathbb{R}^n$*, then, for any* $h \in C$ *with* $\overline{h} = 0$*, the problem*

$$(\phi(u'))' = \nabla_u F(t,u) + h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution $u \in \widehat{K}_\omega$*, which iminimizes* I *on* C *(or* K*).*

Proof. Let $\omega = n^{1/2} \max_{1 \le j \le n} \omega_j$, so that

$$[0, \omega_1] \times \ldots \times [0, \omega_n] \subset \overline{B}(\omega).$$

Due to the periodicity of $F(t, \cdot)$ and because of $\overline{h} = 0$, it holds

$$I(v + j_1\omega_1 e_1 + \ldots + j_n \omega_n e_n) = I(v)$$

for all $v \in K$ and $(j_1, \ldots, j_n) \in \mathbb{Z}^n$. Then, the conclusion follows from the equality

$$\{I(v) \, : \, v \in K\} = \{I(v) \, : \, v \in \widehat{K}_\omega\}$$

and Lemma 16. ■

For the use in examples, let us introduce the continuous mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S(u) := (\sin u_1, \sin u_2, \ldots, \sin u_n),$$

so that

$$S(u) = \nabla c(u) \text{ with } c(u) := (-\sum_{j=1}^{n} \cos u_j).$$

Example 10 *For any* $A \in \mathbb{R}$ *and any* $h \in L_1^n$ *such that* $\overline{h} = 0$*, the periodic problem*

$$\left(\frac{u'}{\sqrt{1 - |u'|^2}} \right)' + AS(u) = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution.

This corresponds to $F(t,u) = -Ac(u)$ such that $F(t, u + 2\pi e_j) = F(t,u)$ for all $t \in [0,T]$, all $u \in \mathbb{R}^n$, and all $j = 1, \ldots, n$.

In particular, in the scalar case, *the forced relativistic pendulum problem*

$$\left(\frac{u'}{\sqrt{1 - u'^2}} \right)' + A \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution whenever $\overline{h} = 0$.

2.3.3 Asymptotically positive potential

Theorem 21 *If*

$$\liminf_{|u|\to\infty}[F(t,u)+\langle h(t),u\rangle]>0 \quad \text{uniformly in} \quad t\in[0,T], \tag{133}$$

then (115) has at least one solution which minimizes I on C.

Proof. Using (117) and (133) it follows that there exists $\rho>0$ such that

$$F(t,u((t))+\langle h(t),u(t)\rangle>0$$

for any $u\in K$ such that $|\overline{u}|>\rho$. It follows that $I(u)>0$ provided that $u\in K$ and $|\overline{u}|>\rho$. The proof follows from Lemma 16, as $I(0)=0$. ■

Example 11 *The problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)'=\frac{u+h(t)}{1+[u+h(t)]^2}+\cos u, \quad u(0)=u(T), \quad u'(0)=u'(T)$$

has at least one solution for all $h\in C$.

Example 12 *If $b\in L^1$ and essinf $b>0$, the problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)'=b(t)\frac{u}{\sqrt{1+|u|^2}}+S(u)+h(t), \tag{134}$$
$$u(0)=u(T),\ u'(0)=u'(T)$$

has at least one solution for all $h\in L^n_\infty$ such that $|h|_\infty<$ essinf b.

Indeed,

$$F(t,u)=b(t)\sqrt{1+|u|^2}+c(u)$$

and

$$\begin{aligned}&\liminf_{|u|\to\infty}[b(t)\sqrt{1+|u|^2}+c(u)-\langle h(t),u\rangle]\\ \geq\ &\liminf_{|u|\to\infty}[|u|(\text{essinf } b|u|-|h|_\infty)-n]>0.\end{aligned}$$

Example 13 *For any $b\in L^1$ such that $b(t)\geq 0$ for a.e. $t\in[0,T]$ with $\overline{b}>0$, and any $h\in L^n_\infty$, the periodic problem*

$$\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)'=b(t)e^{|u|^2}u+S(u)+h(t), \quad u(0)=u(T),\ u'(0)=u'(T)$$

has at least one solution for all $h\in C$.

Indeed,

$$F(t,u)=\frac{1}{2}b(t)e^{|u|^2}+c(u)+\langle h(t),u\rangle$$

and

$$\begin{aligned}&\liminf_{|u|\to\infty}[(1/2)b(t)e^{|u|^2}+c(u)+\langle h(t),u\rangle\\ \geq\ &\liminf_{|u|\to\infty}[(1/2)(\text{essinf } b)e^{|u|^2}-n-|h|_\infty|u|]>0.\end{aligned}$$

2.3.4 Nonlinearities with polynomial growth

Let us consider the case of problem (115) with a nonlinearity $\nabla_u F$ having a polynomial growth in u and a potential F satisfying a semi-coercivity condition of the Ahmad-Lazer-Paul type [1]. The growth condition upon $\nabla_u F$ is :

(H_P) *There exists* $\alpha \geq 0$, $g \in L^1$ *nonnegative and* $k \in L^1$ *nonnegative such that, for a.e.* $t \in [0,T]$ *and all* $u \in \mathbb{R}^n$, *one has*

$$|\nabla_u F(t,u)| \leq g(t)|u|^\alpha + k(t).$$

For the classical problem with $(\phi(u'))'$ replaced by u'', the case where $\alpha = 0$ was considered in [78], and the case where $\alpha \in [0,1)$ in [102]. Define the mapping $\overline{F} : \mathbb{R}^n \to \mathbb{R}$ by

$$\overline{F}(u) = T^{-1} \int_0^T F(t,u)\, dt.$$

Theorem 22 *Assume that Assumptions* (H_Φ), (H_F) *and* (H_P) *hold. Then, for all* $h \in L_1$ *such that*

$$\lim_{|u|\to\infty} |u|^{-\alpha} \left[\overline{F}(u) - \langle \overline{h}, u\rangle\right] = +\infty, \tag{135}$$

problem (115) has at least one solution which minimizes I *over* C *(or* K*).*

Proof. Using the elementary inequality in $\mathbb{R}^n$

$$|y+z|^\alpha \leq 2^\alpha(|y|^\alpha + |z|^\alpha),$$

we have, for all $u \in K$,

$$\begin{aligned}
I(u) &= \int_0^T [\Phi(u'(t)) + F(t,\overline{u}) + F(t,u(t)) - F(t,\overline{u}) + \langle h(t), u(t)\rangle]\, dt \\
&\geq T \min \Phi + \int_0^T [F(t,\overline{u}) + \langle h(t), \overline{u}\rangle]\, dt \\
&+ \int_0^T \langle \int_0^1 \nabla_u F(t, \overline{u} + s\widetilde{u}(t))\, ds + \widetilde{h}(t), \widetilde{u}(t)\rangle\, dt \\
&\geq T \min \Phi + T[\overline{F}(\overline{u}) + \langle \overline{h}, \overline{u}\rangle] \\
&- \int_0^T \int_0^1 [g(t)|\overline{u} + s\widetilde{u}(t)|^\alpha + k(t)]\, |\widetilde{u}|\, ds\, dt - \|h\|_1 \|\widetilde{u}\|_\infty \\
&\geq T \min \Phi + T[\overline{F}(\overline{u}) + \langle \overline{h}, \overline{u}\rangle] \\
&- \|g\|_1 2^\alpha [|\overline{u}|^\alpha + (Ta)^\alpha] Ta - (\|k\|_1 + \|h\|_1) Ta,
\end{aligned}$$

where we have used Sobolev inequality. Hence

$$\begin{aligned}
I(u) &\geq T \min \Phi + |\overline{u}|^\alpha \left\{ T|\overline{u}|^{-\alpha}[\overline{F}(\overline{u}) + \langle \overline{h}, \overline{u}\rangle] - \|g\|_1 2^\alpha Ta \right\} \\
&- [(2Ta)^\alpha \|g\|_1 + \|k\|_1 + \|h\|_1] Ta
\end{aligned} \tag{136}$$

Assumption (135) implies the existence of some $\rho > 0$ such that the second term in the right-hand member of (136) is positive for $|\overline{u}| \geq \rho$. As $I(0) = 0$, the result follows from Lemma 16. ■

Remark 6 In the classical case where $(\phi(u'))'$ is replaced by u'', one needs, as shown in [102], $\alpha \in [0,1)$ and Assumption (135) is replaced by the stronger condition

$$|u|^{-2\alpha}\left[\overline{F}(u) - \langle \overline{h}, u\rangle\right] \to +\infty \quad as \quad |u| \to \infty.$$

Example 14 *For any $b \in L_1$ such that $\overline{b} > 0$, the problem (134) has at least one solution for all $h \in L_1^n$ such that $|\overline{h}| < \overline{b}$.*

Indeed, we have in this case $F(x,u) = b(x)\sqrt{1+|u|^2} + c(u)$, $\alpha = 0$, and, for any $v \in \mathbb{R}^n \setminus \{0\}$,

$$\overline{b}\sqrt{1+|v|^2} + c(v) + \langle \overline{h}, v\rangle \geq |v|[\overline{b} - |\overline{h}|] - n,$$

with the right-hand member tending to $+\infty$ when $|v| \to \infty$.

With respect to Example 12, the use of the boundedness conditions allows weakening the condition upon b and h from $\operatorname{essinf} b > 0$ and $\|h\|_\infty < \operatorname{essinf} b$ to $\overline{b} > 0$ and $|\overline{h}| < \overline{b}$.

In particular, in the scalar case, *for every $h \in L^1$ such that $-\frac{\pi}{2} < \overline{h} < \frac{\pi}{2}$, the problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' - \arctan u - \cos u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution. Corollary 4 does not apply to this example.

Remark 7 When $\alpha = 0$, condition (135) is of the type introduced by Ahmad-Lazer-Paul [1] for the Laplacian with Dirichlet conditions. The reader will observe that for $\alpha = 0$, the conclusion of Theorem 22 still remains true if (135) is replaced by the weaker but more technical condition

$$\liminf_{|u|\to\infty}[\overline{F}(u) + \langle \overline{h}, u\rangle]\, dt > aT\int_0^T [l(t) + |\widetilde{h}(t)|]\, dt$$

with $l(t) = g(t) + k(t)$.

2.3.5 Convex potentials

Theorem 23 *Let $F(t,\cdot)$ be convex for all $t \in [0,T]$. Then, problem (115) has at least one solution which minimizes I on C (or K) if condition (135) with $\alpha = 0$ holds.*

Proof. By Assumption (135) with $\alpha = 0$, the real function $\overline{F} + \langle \overline{h}, \cdot\rangle$ achieves a minimum at some point $\overline{v} \in \mathbb{R}^n$, for which

$$\nabla\overline{F}(\overline{v}) + \overline{h} = 0. \tag{137}$$

Now, by the convexity of $F(t, \cdot)$,

$$
\begin{aligned}
I(u) &= \int_0^T [\Phi(u'(t)) + F(t,\overline{v}) + \langle h(t), \overline{v}\rangle]\, dt \\
&+ \int_0^T [F(t,u(t)) - F(t,\overline{v}) + \langle h(t), u(t) - \overline{v}\rangle]\, dt \\
&\geq T \min \Phi + T[\overline{F}(\overline{v}) + \langle \overline{h}, \overline{v}\rangle] + \int_0^T \langle \nabla_u F(t,\overline{v}) + h(t), u(t) - \overline{v}\rangle\, dt \\
&= T \min \Phi + T[\overline{F}(\overline{v}) + \langle \overline{h}, \overline{v}\rangle] + \int_0^T \langle \nabla_u F(t,\overline{v}) + h(t), \widetilde{u}(t)\rangle\, dt \\
&\geq T \min \Phi + T[\overline{F}(\overline{v}) + \langle \overline{h}, \overline{v}\rangle] - aT\|\nabla_u F(\cdot,\overline{v}) + h\|_1. \qquad (138)
\end{aligned}
$$

From (135) we can find $\rho > 0$ such that $I(u) > 0$ provided that $|\overline{u}| \geq \rho$. Furthermore $I(0) = 0$. Therefore (131) is fulfilled and the result follows from Lemma 16. ■

Example 15 *For any $b \in L^1$ such that $b(t) \geq 0$ for a.e. $t \in [0,T]$ and $\overline{b} > 0$, and any $h \in L^n_1$, the periodic problem*

$$
\left(\frac{u'}{\sqrt{1 - |u'|^2}}\right)' = b(t)e^{|u|^2}u + h(t), \quad u(0) = u(T),\ u'(0) = u'(T)
$$

has at least one solution.

Indeed, $F(t,u) = \frac{b(t)}{2}e^{|u|^2}$ is convex in u for a.e. $t \in [0,T]$,

$$
\overline{F}(u) + \langle \overline{h}, u\rangle \geq \frac{\overline{b}}{2}e^{|u|^2} - |\overline{h}||u|,
$$

and the right-hand member tends to $+\infty$ as $|u| \to \infty$.

With respect to Example 13, the use of the convexity of F allows the replace the assumption essinf $b > 0$ by the weaker one $b(t) \geq 0$ and $\overline{b} > 0$. But the oscillatory term $S(u)$ has to be dropped.

2.4 Saddle Point solutions for bounded nonlinearities

The following results are inspired from [15].

2.4.1 Palais-Smale condition

Towards the application of the minimax results obtained by Szulkin in [97] to the functional I defined by (122) we have to know when I satisfies the compactness *Palais-Smale* (in short, (PS)) *condition.*

We say that a sequence $\{u_n\} \subset K$ is a *(PS)–sequence* if $I(u_n) \to c \in \mathbb{R}$ and

$$\int_0^T [\Phi(v'(t)) - \Phi(u_n'(t)) + \langle \nabla_u F(t, u_n(t)) + h(t), v(t) - u_n(t)\rangle]\, dt$$
$$\geq -\varepsilon_n \|v - u_n\|_\infty \quad \text{for all} \quad v \in K, \qquad (139)$$

where $\varepsilon_n \to 0+$. According to [97], the functional I is said to satisfy the *(PS) condition* if any (PS)–sequence has a convergent subsequence in C.

The lemma below provides useful properties of the *(PS)*–sequences.

Lemma 17 *Let $\{u_n\}$ be a (PS)–sequence. Then the following hold true :*

(i) the sequence $\left\{\int_0^T F(t, u_n(t)) + \langle h(t), u_n(t)\rangle\, dt\right\}$ *is bounded;*

(ii) if $\{\overline{u}_n\}$ is bounded, then $\{u_n\}$ has a convergent subsequence in C;

(iii) one has that

$$\left|\int_0^T \langle \nabla_u F(t, u_n(t)) + h(t), u_n(t)\rangle\, dt\right| \leq \varepsilon_n \quad \textit{for all} \quad n \in \mathbb{N}. \qquad (140)$$

Proof. *(i)* This is immediate from the fact that $\{I(u_n)\}$ and Φ are bounded.

(ii) From (117) and $u_n \in K$, the sequence $\{\widetilde{u}_n\}$ is bounded in $W^{1,\infty}$. By the compactness of the embedding $W^{1,\infty} \subset C$, we deduce that $\{\widetilde{u}_n\}$ has a convergent subsequence in C. Using then the boundedness of $\{\overline{u}_n\} \subset \mathbb{R}$ it follows that $\{u_n\}$ has a convergent subsequence in C.

(iii) Taking $v = u_n + w$ with $w \in \mathbb{R}^n$ such that $|w| = 1$ in (139) one obtains

$$\langle [\int_0^T \nabla_u F(t, u_n(t)) + h(t)\, dt], w\rangle \geq -\varepsilon_n$$

for all $w \in \mathbb{R}^n$ with $|w| = 1$, and hence (140). ■

2.4.2 Bounded nonlinearities with anti-coercive potential

Theorem 24 *Let $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ and $l \in L^1$ be such that condition (H_P) with $\alpha = 0$ is satisfied. If*

$$\lim_{|x|\to\infty} [\overline{F}(x) + \langle \overline{h}, x\rangle] = -\infty, \qquad (141)$$

then problem (115) has at least one solution.

Proof. We shall apply the Saddle Point Theorem for functionals of Szulkin's type [97, Theorem 3.5].

From (141) the functional I is not bounded from below. Indeed, if $v = c \in \mathbb{R}^n$ is a constant function then

$$I(c) = \int_0^T [F(t,c) + \langle h(t), c\rangle]\, dt \to -\infty \quad \text{as} \quad |c| \to \infty. \tag{142}$$

We split $C = \mathbb{R}^n \bigoplus X$, where $X = \{v \in C \ : \ \overline{v} = 0\}$. Note that

$$I(v) \geq \int_0^T [F(t, \widetilde{v}(t)) + \langle h(t), \widetilde{v}(t)\rangle]\, dt \quad \text{for all } v \in K \cap X,$$

which together with (117) imply that there is a constant $\alpha \in \mathbb{R}$ such that

$$I(v) \geq \alpha \quad \text{for all } v \in X. \tag{143}$$

Using (142) and (143) we can find some $R > 0$ so that

$$\sup_{S_R} I < \inf_X I,$$

where $S_R = \{c \in \mathbb{R}^n : |c| = R\}$.

It remains to show that I satisfies the (PS) condition. Let $\{u_n\} \subset K$ be a (PS)–sequence. Since $\{I(u_n)\}$, $\{\varphi(u_n)\}$ are bounded and, by (H_P) with $\alpha = 0$, we have, letting $l(t) = g(t) + k(t)$,

$$\begin{aligned} & \left| \int_0^T [F(t, u_n(t)) + \langle h(t), u_n(t)\rangle - F(t, \overline{u}_n) - \langle h(t), \overline{u}_n\rangle]\, dt \right| \\ \leq & \int_0^T \int_0^1 |\langle \nabla_u F(t, \overline{u}_n(t) + s\widetilde{u}_n(t)) + h(t), \widetilde{u}_n(t)\rangle|\, ds\, dt \\ \leq & \ aT \int_0^T l(t)\, dt. \end{aligned}$$

From

$$\begin{aligned} I(u_n) &= \varphi(u_n) + \int_0^T [F(t, \overline{u}_n) + \langle h(t), \overline{u}_n\rangle]\, dt \\ &+ \int_0^T [F(t, u_n(t)) - F(t, \overline{u}_n) + \langle h(t), \widetilde{u}_n(t)\rangle]\, dt \end{aligned}$$

it follows that there exists a constant $\beta \in \mathbb{R}$ such that

$$\int_0^T [F(t, \overline{u}_n) + \langle h(t), \overline{u}_n\rangle]\, dt \geq \beta.$$

Then by (141) the sequence $\{\overline{u}_n\}$ is bounded and Lemma 17 *(ii)* ensures that $\{u_n\}$ has a convergent subsequence in C. Consequently, I satisfies the (PS) condition and the conclusion follows from [97, Theorem 3.5] and Proposition 1. ■

Remark 8 Condition (141), also of the Ahmad-Lazer-Paul type [1] is, in some sense, 'dual' to condition (135).

Example 16 *For any $b \in L_1$ such that $\overline{b} < 0$, the periodic problem (134) has at least one solution for all $h \in L_1^n$ such that $|\overline{h}| < \overline{b}$.*

Indeed, we have in this case $F(t,u) = b(t)\sqrt{1+|u|^2} + c(u)$, and, for any $v \in \mathbb{R}^n \setminus \{0\}$,

$$\overline{b}\sqrt{1+|v|^2} + c(v) + \langle \overline{h}, v \rangle \leq |v| \left[\overline{b}\sqrt{1+|v|^{-2}} + n|v|^{-1} + |\overline{h}| \right],$$

with the right-hand member tending to $-\infty$ when $|v| \to \infty$.

Theorem 25 *If*

$$\lim_{|x|\to\infty} [F(t,x) + \langle h(t), x \rangle] = -\infty, \quad \text{uniformly in} \quad t \in [0,T], \tag{144}$$

then (115) has at least one solution.

Proof. We keep the notations introduced in the proof of Theorem 24. Clearly, (144) implies (141) and from the proof of Theorem 24 it follows that I has the geometry required by the Saddle Point Theorem. To show that I satisfies the (PS) condition, let $\{u_n\} \subset K$ be a (PS)–sequence. If $\{|\overline{u}_n|\}$ is not bounded, we may assume going if necessary to a subsequence, that $|\overline{u}_n| \to \infty$. Using (117) and (144) we deduce that

$$F(t, u_n(t)) + \langle h(t), u_n(t) \rangle \to -\infty, \quad \text{uniformly in } t \in [0,T].$$

This implies

$$\int_0^T [F(t, u_n(t)) - \langle h(t), u_n(t) \rangle] \, dt \to -\infty,$$

contradicting Lemma 17 *(i)*. Hence, $\{\overline{u}_n\}$ is bounded and by Lemma 17 *(ii)*, the sequence $\{u_n\}$ has a convergent subsequence in C. Therefore, I satisfies the (PS) condition. The proof is complete. ■

Remark 9 No result corresponding to Theorem 25 holds for the classical case where $(\phi(u'))'$ is replaced by u''. Indeed, if λ_k is a positive eigenvalue of $-u''$ on $[0,T]$ with periodic boundary conditions, and φ_k a corresponding eigenfunction, the problem

$$u'' = -\lambda_k u + \varphi_k(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has no solution, but $-\lambda_k \dfrac{u^2}{2} + \varphi_k(x)u \to -\infty$ uniformly in $[0,T]$ when $|u| \to \infty$.

Example 17 *The problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}} \right)' + \frac{u+h(t)}{1+[u+h(t)]^2} = \cos u, \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has at least one solution for all $h \in C$.

2.5 Multiple solutions near resonance

This subsection, which presents some of the results of [17], is devoted to the existence of multiple solutions for the scalar periodic problems of the form

$$(\phi(u'))' = \lambda|u|^{m-2}u - g(t,u) + h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \quad (145)$$

where $\phi : (-a,a) \to \mathbb{R}$ satisfies Assumption (H_Φ) for $n = 1$, $m \geq 2$, $\lambda > 0$, $\overline{h} = 0$ and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies suitable conditions. We show in particular the existence of at least three solutions when $G(t,x) = \int_0^x g(t,s)\,ds$ has a polynomial growth of order strictly smaller than $m2$, satisfies a Ahmad-Lazer-Paul condition holds and $\lambda > 0$ is sufficiently small. Results of this type, called *multiplicity results near resonance*, have been initiated in the classical case where $(\phi(u'))'$ is replaced by u'' by Schmitt and the author in [75], using bifurcation from infinity and Leray-Schauder degree theory. A variational approach was introduced by Sanchez in [92] to deal with such multiplicity problems, and conditions of type (i)' and (ii)' were introduced by Ma, Ramos and Sanchez in [90, 61] for semilinear and quasilinear Dirichlet problems involving the p-Laplacian. See also [62, 60, 79, 37, 84] for a similar variational treatment of various semilinear or quasilinear equations, systems or inequalities with Dirichlet conditions, [80] for perturbations of p-Laplacian with Neumann boundary conditions, and [58] for periodic solutions of perturbations of the one-dimensional p-Laplacian. The existence of at least two solutions near resonance at a non-principal eigenvalue was first obtained in [74] using a topological approach and then for semilinear or quasilinear problems using critical point theory in [38, 53, 96]. This question seems to be meaningless for the singular ϕ considered here because resonance only occurs at $\lambda = 0$.

The main used tools are some abstract local minimization results combined with mountain pass techniques in the frame of the Szulkin's critical point theory [97].

2.5.1 A localized minimum principle

Let $(X, \|\cdot\|)$ be a real Banach space and I be a functional of Szulkin's type

$$I = \mathcal{F} + \psi,$$

where $\psi : X \to (-\infty, +\infty]$ is proper (i.e., $D(\psi) := \{v \in X; : \ \psi(v) < +\infty\} \neq \emptyset$), convex, lower semicontinuous (in short, l.s.c.) and $\mathcal{F} \in C^1(X;\mathbb{R})$.

Proposition 2 *Suppose that I satisfies the (PS) condition and there exists an open set U such that*

$$-\infty < \inf_{\overline{U}} I < \inf_{\partial U} I. \quad (146)$$

Then I has at least one critical point $u \in U$ such that $I(u) = \inf_{U} I$.

Proof. Let

$$c_0 = \inf_{\overline{U}} I \tag{147}$$

and $\{\varepsilon_n\}$ be a sequence with $\varepsilon_n \to 0$ and

$$0 < \varepsilon_n < \inf_{\partial U} I - c_0 \quad \text{for all } n \in \mathbb{N}. \tag{148}$$

Using Ekeland's variational principle [40, 78], applied to $I|_{\overline{U}}$, for each $n \in \mathbb{N}$, we can find $v_n \in \overline{U}$ such that

$$I(v_n) \leq c_0 + \varepsilon_n \tag{149}$$

and

$$I(v) \geq I(v_n) - \varepsilon_n \|v - v_n\| \quad \text{for all } v \in \overline{U}. \tag{150}$$

From (148) and (149) it follows $I(v_n) < \inf_{\partial U} I$, which ensures that $v_n \in U$, for all $n \in \mathbb{N}$. Let $v \in X$, $n \in \mathbb{N}$ be arbitrarily chosen and $t_0 := t_0(v, n) \in (0, 1)$ be so that $v_n + t(v - v_n) \in U$, for all $t \in (0, t_0)$. Using (150) and the convexity of ψ, we get

$$\frac{\mathcal{F}(v_n + t(v - v_n)) - \mathcal{F}(v_n)}{t} + \psi(v) - \psi(v_n) \geq -\varepsilon_n \|v - v_n\|$$

and, letting $t \to 0+$, one obtains

$$\mathcal{F}'(v_n)(v - v_n)\rangle + \psi(v) - \psi(v_n) \geq -\varepsilon_n \|v - v_n\| \quad \text{for all } v \in X. \tag{151}$$

On the other hand, from (149) it is clear that

$$I(v_n) \to c_0. \tag{152}$$

Since I satisfies the (PS) condition, (151) and (152) ensure that $\{v_n\}$ contains a subsequence, still denoted by $\{v_n\}$, convergent to some $u \in \overline{U}$.

By the lower semicontinuity of ψ it holds

$$\psi(u) \leq \liminf_{n\to\infty} \psi(v_n) \tag{153}$$

and, on account of $\mathcal{F} \in C^1(X; \mathbb{R})$, one obtains

$$\lim_{n\to\infty} \mathcal{F}'(v_n)(v - v_n) = \mathcal{F}'(u)(v - u) \quad \text{for all } v \in X. \tag{154}$$

From (151), (153) and (154) we deduce

$$\mathcal{F}'(u)(v - u) + \psi(v) - \psi(u) \geq 0 \quad \text{for all } v \in X. \tag{155}$$

Also, from (147), (152) and (153) we have

$$c_0 \leq I(u) \leq \lim_{n\to\infty} \mathcal{F}(v_n) + \liminf_{n\to\infty} \psi(v_n) = \liminf_{n\to\infty} I(v_n) = c_0,$$

hence $I(u) = c_0$ and from (146), $u \in U$. This together with (155) shows that c_0 is a critical value of I. ∎

2.5.2 Hypotheses and the functional framework

Throughout this subsection we assume that the following hypothesis upon g and h hold true.
(H_f) *The functions* $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, $b, h : [0,T] \to \mathbb{R}$ *are continuous; the constants* $\alpha > 0$, $m \geq 2$ *are fixed and* λ *is a real positive parameter.*

We denote by G the indefinite integral of g with respect to the second variable defined by

$$G(t,x) := \int_0^x g(t,\xi)\, d\xi, \quad (t,x) \in [0,T] \times \mathbb{R},$$

and assume that G satisfies the following hypotheses :

(i)′ *there exists* $k_1, k_2 > 0$ *and* $0 < \sigma < m$ *such that*

$$-l(t) \leq G(t,x) \leq k_1|x|^\sigma + k_2, \quad \textit{for all} \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{156}$$

where $0 \leq l$ *and* $l \in L^1$ *;*

(ii)′ *one has that either*

$$\lim_{|x|\to\infty} \int_0^T G(t,x)\, dt = +\infty, \tag{157}$$

or the limits $G_\pm(t) = \lim_{x\to\pm\infty} G(t,x)$ *exist for all* $t \in [0,T]$ *and*

$$\begin{aligned} G(t,x) < G_+(t), &\quad \textit{fo rall} \quad t \in [0,T],\ x \geq 0, \\ G(t,x) < G_-(t), &\quad \textit{for all} \quad t \in [0,T],\ x \leq 0; \end{aligned} \tag{158}$$

(iii)′ *it holds*

$$\int_0^T h(t)\, dt = 0,$$

We define $\widehat{\mathcal{F}}_\lambda : C \to \mathbb{R}$ by

$$\widehat{\mathcal{F}}_\lambda(u) = \int_0^T \left[\frac{\lambda}{m}|u(t)|^m - F(t,u(t)) + h(t)u(t)\right] dt, \quad u \in C.$$

A standard reasoning shows that $\widehat{\mathcal{F}}_\lambda$ is of class C^1 on C and

$$\widehat{\mathcal{F}}'_\lambda(u)(v) = \int_0^T \left[\lambda|u(t)|^{m-2}u(t) - g(t,u(t)) + h(t)\right] v(t)\, dt, \quad u, v \in C.$$

Then it is clear that $\widehat{I}_\lambda : C \to (-\infty, +\infty]$ defined by

$$\widehat{I}_\lambda = \widehat{\mathcal{F}}_\lambda + \Psi,$$

where Ψ is defined in (118) with $n = 1$, has the structure required by Szulkin's critical point theory. By the results of the beginning of the Chapter, the search of solutions of problem (145) reduces to finding critical points of the energy functional $\widehat{I}_\lambda$.

We also need in the proof the following inequalities for $u \in K$.

Lemma 18 *Let $p \geq 1$ be a real number. Then*

$$|u(t)|^p \geq |\overline{u}|^p - paT|\overline{u}|^{p-1} \quad \textit{for all} \quad u \in K \quad \textit{and all} \quad t \in [0,T] \tag{159}$$

and there are constants $\alpha_1,\ \alpha_2 \geq 0$ such that

$$\begin{aligned} |u(t)|^p &\leq |\overline{u}|^p + \alpha_1|\overline{u}|^{p-1} + \alpha_2 \\ \textit{for all} \quad u \in K \quad \textit{with} \quad |\overline{u}| &\geq 1 \qquad \textit{and all} \quad t \in [0,T]. \end{aligned} \tag{160}$$

Proof. The result is trivial for $p = 1$. If $p > 1$, $u \in K$ and $t \in [0,T]$, then, using the convexity of the differentiable function $s \mapsto |s|^p$, we get

$$\begin{aligned} |u(t)|^p &= |\overline{u} + \widetilde{u}(t)|^p \geq |\overline{u}|^p + p|\overline{u}|^{p-2}\overline{u}\widetilde{u}(t) \\ &\geq |\overline{u}|^p - p|\overline{u}|^{p-1}Ta. \end{aligned}$$

On the other hand, denoting by $\widetilde{p}$ the smallest integer larger or equal to p and letting $M := aT$, we have, for all $t \in [0,T]$,

$$\begin{aligned} |u(t)|^p &= |\overline{u} + \widetilde{u}(t)|^p \leq (|\overline{u}| + M)^p = |\overline{u}|^p \left(1 + \frac{M}{|\overline{u}|}\right)^p \\ &\leq |\overline{u}|^p \left(1 + \frac{M}{|\overline{u}|}\right)^{\widetilde{p}} = |\overline{u}|^p \left(1 + \sum_{k=1}^{\widetilde{p}} \frac{\widetilde{p}!}{k!(\widetilde{p}-k)!} \frac{M^k}{|\overline{u}|^k}\right) \\ &= |\overline{u}|^p + \sum_{k=1}^{\widetilde{p}} \frac{\widetilde{p}!}{k!(\widetilde{p}-k)!} M^k |\overline{u}|^{p-k}, \end{aligned}$$

and (160) follows easily. ∎

2.5.3 Existence of three periodic solutions

The following existence result, inspired from [90, 61] provides a useful tool in obtaining multiple solutions.

Lemma 19 *Assume that $\overline{h} = 0$ and that there exists $k_1, k_2 > 0$ and $0 < \sigma < m$ such that*

$$-l(t) \leq G(t,x) \leq k_1|x|^\sigma + k_2 \quad \textit{for all} \quad (t,x) \in [0,T] \times \mathbb{R}_+, \tag{161}$$

with some $l \in L^1$, $l \geq 0$. If either

$$\lim_{x \to +\infty} \int_0^T G(t,x)\,dt = +\infty, \tag{162}$$

or $G_+(t) := \lim_{x\to+\infty} G(t,x)$ exists for all $t \in [0,T]$ and

$$G(t,x) < G_+(r), \quad \textit{for allt} \in [0,T],\ x \geq 0, \tag{163}$$

then there exists $\lambda_+ > 0$ such that problem (145) has at least one solution $u_\lambda > 0$ for any $0 < \lambda < \lambda_+$ which minimize $\widehat{I}_\lambda$ on $C^+ = \{v \in C : v \geq 0\}$. Moreover, u_λ is a local minimum for $\widehat{I}_\lambda$.

Proof. First, notice that, using (117), one has

$$\overline{u} - aT \leq u(t) \leq \overline{u} + aT \quad \text{for all} \quad u \in K, \tag{164}$$

hence

$$\overline{u} \to +\infty \quad \text{as} \quad |u|_\infty \to \infty \quad \text{if} \quad u \in C^+ \cap K. \tag{165}$$

Also, it is clear that

$$|u(t)| \leq |\overline{u}| + aT \quad \text{for all} \quad u \in K \quad \text{and} \quad t \in [0,T]. \tag{166}$$

From (161) it follows that

$$\widehat{I}_\lambda(u) \geq \int_0^T \left[\frac{\lambda}{m}|u(t)|^m - k_1|u(t)|^\sigma - k_2 - |h|_\infty|u(t)|\right] dt,$$

for all $u \in C^+$. Hence, using (159), (166), (165) and $\sigma < m$, we deduce immediately that

$$\widehat{I}_\lambda(u) \to +\infty \quad \text{whenever} \quad |u|_\infty \to \infty \quad \text{in} \quad C^+, \tag{167}$$

that is $\widehat{I}_\lambda$ is coercive on C^+, and hence bounded from below on C^+. Now, let $\{u_n\} \subset C^+ \cap K$ be a minimizing sequence, i.e. $\widehat{I}_\lambda(u_n) \to \inf_{C^+} \widehat{I}_\lambda$ as $n \to \infty$. From (167), $\{u_n\}$ is bounded in C, and using the fact that $\{u_n\} \subset K$, we infer that $\{u_n\}$ is bounded in $W^{1,\infty}$. $W^{1,\infty}$ being compactly embedded in C, $\{u_n\}$ has a subsequence converging in C to some $u_\lambda \in C^+ \cap K$. By the lower semicontinuity of $\widehat{I}_\lambda$ it follows

$$\widehat{I}_\lambda(u_\lambda) = \inf_{C^+} \widehat{I}_\lambda.$$

We *claim* that

$$\overline{u}_\lambda \to +\infty \quad \text{as} \quad \lambda \to 0. \tag{168}$$

Assuming this for the moment, it follows from (164) and (168) that there exists $\lambda_+ > 0$ such that $u_\lambda > 0$ for any $0 < \lambda < \lambda_+$, implying that u_λ is a local minimum for $\widehat{I}_\lambda$. Consequently, u_λ is a critical point of $\widehat{I}_\lambda$, and hence a solution of (145) for any $0 < \lambda < \lambda_+$.

In order to prove the claim, assume first that (162) holds true. Then, consider $M > 0$ and $x_M > 0$ such that

$$\int_0^T G(t, x_M)\, dt > 2M. \tag{169}$$

On the other hand, as $\overline{h} = 0$, one has that for all $\lambda > 0$,

$$\widehat{I}_\lambda(x) = \frac{\lambda T}{m}|x|^m - \int_0^T G(t, x)\, dt \quad (x \in \mathbb{R}). \tag{170}$$

Choosing $\lambda_M > 0$ such that

$$\frac{\lambda_M T}{m} x_M^m < M,$$

and using (169), (170), it follows that

$$\widehat{I}_\lambda(x_M) < -M \quad \text{for all} \quad 0 < \lambda < \lambda_M.$$

Consequently,

$$\inf_{C^+} \widehat{I}_\lambda \to -\infty \quad \text{as} \quad \lambda \to 0,$$

which, together with (164) implies (168), as claimed.

Now, let (163) holds true, and assume also by contradiction that there exists $\lambda_n \to 0$ such that $\{\overline{u}_{\lambda_n}\}$ is bounded. On account of (164) and of the compactness of the embedding of $W^{1,\infty}$ in C, one can assume, going if necessary to a subsequence, that $\{u_{\lambda_n}\}$ converges in C to some $u \in C^+$. Using (163) and Fatou's lemma it follows that

$$\int_0^T G(t, u(t))\, dt < \int_0^T G_+(t)\, dt \le \liminf_{s \to \infty} \int_0^T G(t, s + \widetilde{u}(t))\, dt,$$

which implies the existence of $s_0 > 0$ sufficiently large, with $s_0 + \widetilde{v} \in C^+$ for all $v \in K$, and of $\rho > 0$ such that

$$\int_0^T [G(t, u(t)) - G(t, s_0 + \widetilde{u}(t))]\, dt < -\rho.$$

So, for n sufficiently large, we have

$$\int_0^T [G(t, u_{\lambda_n}(t)) - G(t, s_0 + \widetilde{u}_{\lambda_n}(t))]\, dt < -\rho. \tag{171}$$

On the other hand, using (164), we get

$$\int_0^T \frac{\lambda_n}{m} [|s_0 + \widetilde{u}_{\lambda_n}(t)|^m - |u_{\lambda_n}(t)|^m]\, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{172}$$

Notice that, as $\overline{h} = 0$, for all $\lambda > 0$ and $s \in \mathbb{R}$, one has

$$\widehat{I}_\lambda(s + \widetilde{u}_\lambda) = \int_0^T \Phi(u'_\lambda)(t)\,dt + \int_0^T \frac{\lambda}{m}|s + \widetilde{u}_\lambda(t)|^m\,dt$$
$$- \int_0^T G(t, s + \widetilde{u}_\lambda(t))\,dt - \int_0^T h(t)\widetilde{u}_\lambda(t)\,dt.$$

Then, by (171) and (172) we obtain

$$\widehat{I}_{\lambda_n}(s_0 + \widetilde{u}_{\lambda_n}) < \widehat{I}_{\lambda_n}(u_{\lambda_n}),$$

for n sufficiently large, contradicting the definition of u_{λ_n}. This proves the claim and the proof is complete. ■

The multiplicity result will follow from Lemma 19 and the following version of the mountain pass lemma given in [97].

Lemma 20 *If $I = \Psi + G$ satisfies (PS)-condition, 0 is a local minimum of I and if $I(e) \leq I(0)$ for some $e \neq 0$, then I has a critical point different from 0 and e. In particular, if I has two local minima, then it has at least a third critical point.*

Theorem 26 *Assume that conditions $\overline{h} = 0$, (156) and either (157) or (158) hold true. Then there exists $\lambda_0 > 0$ such that problem (145) has at least three solutions for any $\lambda \in (0, \lambda_0)$.*

Proof. From Lemma 19, it follows that there exists $\lambda_+ > 0$ such that $\widehat{I}_\lambda$ has a local minimum at some $u_{\lambda,1} > 0$ for any $0 < \lambda < \lambda_+$. Using exactly the same strategy, we can find $\lambda_- > 0$ such that $\widehat{I}_\lambda$ has a local minimum at some $u_{\lambda,2} < 0$ for any $0 < \lambda < \lambda_-$. Taking $\lambda_0 = \min\{\lambda_-, \lambda_+\}$ it follows that $\widehat{I}_\lambda$ has two local minima for any $\lambda \in (0, \lambda_0)$. On the other hand, from the proof of Lemma 19, it is easy to see that $\widehat{I}_\lambda$ is coercive on C, implying that $\widehat{I}_\lambda$ satisfies the (PS) condition for any $\lambda > 0$. Hence, from Lemma 20, we infer that $\widehat{I}_\lambda$ has at least three critical points for all $\lambda \in (0, \lambda_0)$ which are solutions of (145). ■

Remark 10 (i) When g is bounded, it is well known [1] that the Ahmad-Lazer-Paul condition (157) generalizes the Landesman-Lazer condition

$$\int_0^T g^-(t)\,dt < 0 < \int_0^T g_+(t)\,dt,$$

where $g^-(t) = \limsup_{x \to -\infty} g(t, x)$ and $g_+(t) = \liminf_{x \to +\infty} g(t, x)$.

(ii) Condition (158) holds true whenever one has the sign condition

$$xg(t, x) > 0 \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad x \neq 0.$$

(iii) The condition :

there exists $0 < \theta < m$ *such that*

$$xg(t,x) - \theta G(t,x) \to -\infty \quad \text{as} \quad |x| \to \infty, \text{ uniformly in } \quad t \in [0,T],$$

introduced in [92, 62], together with the sign condition

$$xg(t,x) > 0 \text{ for all } t \in [0,T] \text{ and } |x| \geq x_0$$

for some $x_0 > 0$, imply (156) and (157).

Example 18 *Let* $m \in \mathbb{N}$ *be even and* $h \in C$ *be with* $\overline{h} = 0$*. Then there exists* $\lambda_0 > 0$ *such that the periodic problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = \lambda |u|^{m-2}u - \frac{u^{m-1}}{1+u^m} + h(t),$$
$$u(0) = u(T), \quad u'(0) = u'(T)$$

has at least three solutions for all $\lambda \in (0, \lambda_0)$.

2.6 BV periodic solutions of the forced pendulum with curvature operator

In this section we sketch Obersnel-Omari's recent proof [83] of the existence of at least two solutions for problems of the form

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = f(t,x) + h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{173}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is supposed, for simplicity, continuous, and satisfies the following periodicity condition, where $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the indefinite integral of f defined by

$$F(t,u) = \int_0^u f(t,x)\,dx \quad (t \in [0,T],\ u \in \mathbb{R}).$$

(H_ω) *The function* F *satisfies the* ω*-periodicity condition*

$$F(t, x+\omega) = F(t,x) \quad \textit{for all} \quad t \in [0,T],\ x \in \mathbb{R}. \tag{174}$$

This condition is in particular satisfied, with $\omega = 2\pi$, for the *forced pendulum equation with curvature operator*, i.e. for the problem

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' + A \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{175}$$

We assume in addition that $h \in C$ has mean value zero, namely

$$\int_0^T h(t)\,dt = 0. \tag{176}$$

2.6.1 The action functional

The action functional $\mathcal{E}$ associated to problem (175) is given by

$$u \mapsto \int_0^T [\sqrt{1+u'^2(t)} + F(t,u(t)) + h(t)u(t)]\,dt \tag{177}$$

This functional is well defined on the space

$$W_T^{1,1} = \{u \in W^{1,1}([0,T]) : u(0) = u(T)\}$$

and, using the Assumptions (H_ω) and (176), and Sobolev inequality, it is not difficult to see that $\mathcal{E}$ is bounded from below. Furthermore, standard reasonings imply that $\mathcal{E}$ is continuous, Gateaux-differentiable and lower semi-continuous. Consequently, $\mathcal{E}$ admits a bounded minimizing sequence but, because $W_T^{1,1}$ is not reflexive, the usual property of extraction of a weakly converging subsequence fails. Indeed, one can construct an example of such sequences having no subsequence converging to some element of $W_T^{1,1}$ even when $f \equiv 0$ (see [83]).

Various considerations lead to the choice of the larger space $BV = BV(0,T)$ of functions having finite total variation, namely such that

$$\int_0^T |Dv| = \sup\{\int_0^T v(t)w'(t)\,dt : w \in C_0^1((0,T)) \quad \text{and} \quad |w|_\infty \leq 1\} < +\infty.$$

Here

$$C_0^1((0,T)) = \{u \in C^1((0,T)) \quad \text{with compact support in} \quad (0,T)\}.$$

BV is a Banach space with respect to the norm

$$|v|_{BV} = \int_0^T |Dv| + |v|_q$$

for any $q \in [1,+\infty)$ fixed. To take in account the periodic boundary condition, the relaxed functional $\mathcal{I} : BV \to \mathbb{R}$ is defined by

$$\mathcal{I}(u) = \int_0^T \sqrt{1+|Du|^2} + \int_0^T [F(t,u(t)) + h(t)u(t)]\,dt + |u(T^-) - u(0^+)|,$$

where

$$\begin{aligned} &\int_0^T \sqrt{1+|Du|^2}\,dt \\ = \quad &\sup\{\int_0^T [v(t)w_1'(t) + w_2(t)]\,dt : w_1,\, w_2 \in C^1 \text{and} \quad |w_1^2 + w_2^2|_\infty \leq 1\}. \end{aligned}$$

Define $\mathcal{J} : BV \to \mathbb{R}$ by

$$\mathcal{J}(u) = \int_0^T \sqrt{1+|Du|^2} + |u(T^-) - u(0^+)|.$$

It is a nontrivial fact to prove [83] that $\mathcal{J}$ is convex, Lipschitz continuous, and lower semicontinuous with respect to the L^1-convergence. As $\mathcal{F} : BV \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_0^T [F(t, u(t)) + h(t)u(t)]\, dt$$

is of class C^1, we are again in the setting of Szulkin's critical point theory [98] and a critical point of $\mathcal{I}$ is some $u \in BV$ satisfying the differential inequality

$$\mathcal{J}(v) - \mathcal{J}(u) + \int_0^T g(t, u(t))(v(t) - u(t))\, dt \geq 0$$

for all $v \in BV$. Any solution u of this variational inequality belonging to $W_T^{1,1}$ will be a weak solution of (173). The proof of the lower semicontinuity property of $\mathcal{J}$ depends upon the following lemma essentially due to Anzellotti [5].

Lemma 21 *For any given $v \in BV$, there exists a sequence (v_n) in $W_T^{1,1}$ such that*

$$\begin{aligned}
\lim_{n\to\infty} v_n &= v \quad \text{in} \quad L^1, \\
\lim_{n\to\infty} \int_0^T |v_n'(t)|\, dt &= \int_0^T [|Dv| + |v(T^-) - v(0^+)|, \\
\lim_{n\to\infty} \int_0^T \sqrt{1 + |v_n'(t)|^2}\, dt &= \int_0^T \sqrt{1 + |Dv|^2} + |v(T^-) - v(0^+)|.
\end{aligned}$$

2.6.2 Existence of two BV solutions

The proof existence of a solution depends on the following BV-version of Wirtinger inequality [83].

Lemma 22 *For every $v \in BV$ such that $\int_0^T v(t)\, dt = 0$, one has*

$$|v|_1 \leq \frac{T}{4} \left(\int_0^T |Dv| + |v(T^-) - v(0^+)| \right)$$

and the constant $T/4$ is sharp.

Hence one can prove the following multiplicity result [83].

Theorem 27 *Assume that $h \in L^p$ for some $p > 1$, that Assumptions (H_ω), (176) hold and that*

$$\sup\{\int_0^T h(t)w(t)\, dt : \qquad (178)$$

$$w \in BV, \int_0^T |Dw| + |w(T^-) - w(0^+)| \leq 1\} < 1.$$

Then problem (173) has at least two geometrically distinct solutions.

Proof. (sketched). The first solution u_0 is obtained as a global minimum of $\mathcal{I}$ in a rather standard way. Letting $u_1 = u_0 + \omega$ (also a global minimum of $\mathcal{I}$), the second solution is obtained through a modified problem constructed in such a way that any critical point of the association functional $\mathcal{H}$ in BV lies between u_0 and u_1 The functional $\mathcal{H}$ is then extended to a functional $\mathcal{M}$ on L^q where $\frac{1}{p} + \frac{1}{q} = 1$ which is shown to be bounded from below and coercive. Furthermore, $\mathcal{M}$ satisfies Palais-Smale condition in Szulkin's sense. Hence $\mathcal{M}$ has a global minimum u, shown to be a critical point of $\mathcal{H}$ too, so that $u_0(t) \leq u(t) \leq u_1(t)$. From this information, the existence of a second geometrically distinct critical point follows. ■

Corollary 11 *Assumption (178) holds if either*

$$|h|_\infty < \frac{4}{T}$$

or h has a primitive H such that

$$|H|_\infty < 1.$$

3 Hamiltonian variational approach for periodic solutions

3.1 Introduction

Using Lusternik-Schnirelman theory in Hilbert manifolds [85] or variants of it, Chang [31], Rabinowitz [87] and the author [67] have independently obtained results which imply that the problem

$$q'' = \nabla_q F(t,q) + h(t), \quad q(0) = q(T), \quad q'(0) = q'(T) \tag{179}$$

with $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying Assumption

(HF) *F is continuous, ω_i-periodic with respect to each q_i $(i = 1, 2, \ldots, n)$ and such that $\nabla_q F$ exists and is continuous on $[0,T] \times \mathbb{R}^n$,*

has at least $n+1$ geometrically distinct solutions for every $h \in L^2$ verifying the Assumption

(Hh) $\int_0^T h(t)\,dt = 0.$

Because of the periodicity property of F, if $q(t)$ is a solution of (179), the same is true for $(q_1(t)+j_1\omega_1, q_2(t)+j_2\omega_2, \ldots, q_n(t)+j_n\omega_n)$ for any $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$, and hence two solutions q and $\widehat{q}$ of (179) are called *geometrically distinct* if

$$q \not\equiv \widehat{q} \,(\mathrm{mod}\ \omega_j e_j, j = 1, 2, \ldots, n).$$

This result is an extension of an earlier one of the author and Willem [77] who proved, under the same conditions, the existence of at least two geometrically distinct solutions, using the same variant of the mountain pass lemma introduced in [76] to treat the special case where $n = 1$, and in particular the *forced pendulum problem*

$$q'' + A \sin q = h(t), \quad q(0) = q(T), \quad q'(0) = q'(T).$$

See [70] for a survey of this problem.

The corresponding existence result for the *relativistic forced pendulum equation*

$$\left(\frac{q'}{\sqrt{1 - q'^2}} \right)' + A \sin q = h(t), \quad q(0) = q(T), \quad q'(0) = q'(T) \tag{180}$$

has been recently considered by Brezis and the author [29], who proved the existence of at least one solution of (180) when Assumption (Hh) holds, by minimizing the corresponding action functional

$$u \mapsto \int_0^T [1 - \sqrt{1 - q'(t)^2} + A \cos q(t) - h(t)q(t)] \, dt$$

over the closed convex subset made of functions in $W^{1,\infty}(0,T)$ such that $q(0) = q(T)$ and $|q'|_\infty \leq 1$. The main difficulty consisted in showing that such a minimum indeed satisfies (180). The result is obtained in [29] for the more general problem

$$(\phi(q'))' = \partial_q F(t,q) + h(t), \quad q(0) = q(T), \quad q'(0) = q'(T), \tag{181}$$

where $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, F is ω_1-periodic in q, continuous, $\partial_q F$ is continuous and h verifies Assumption (Hh). The same authors in [30] have extended their existence result to the corresponding n-dimensional problem

$$(\phi(q'))' = \nabla_q F(t,q) + h(t), \quad q(0) = q(T), \quad q'(0) = q'(T), \tag{182}$$

when Assumptions (HF) and (Hh) hold, and ϕ satisfies condition (H_Φ). Very recently, Bereanu and Torres [25] have extended the mountain pass approach of [76] to obtain the existence of at least two geometrically distinct solutions for problem (181). It is not clear if their approach is applicable to system (181) and, would it be the case, the existence of two solutions only would be insured.

The aim of this chapter, which describes some results of [73], is to prove that, under Assumptions (H_Φ), (HF) and (Hh), problem (22) has at least $n+1$ geometrically distinct solutions. To do this, we reduce problem (22) to an equivalent *Hamiltonian* system, and apply an abstract result of Szulkin [98] to this system. The advantage of the Hamiltonian formulation with respect to the Lagrangian one used in [29, 30] and presented in Chapter 3 is that the Hamiltonian action functional is defined on the whole space, so that the Hamiltonian system is trivially its Euler-Lagrange, and many standard techniques of critical point theory can be directly applied. The price to pay in the Hamiltonian formalism is that the Hamiltonian action functional is now indefinite, excluding the obtention of existence results by minimization and of multiplicity results through classical Lusternik-Schnirelman category. Although its final result is stated in terms of the classical cuplength of a finite-dimensional manifold, the underlying technique in Szulkin's paper [98] (see also variants in [57, 43]) is a more sophisticated concept of relative category inspired by [91, 44, 45].

3.2 An equivalent Hamiltonian system and its action

3.2.1 Equivalent Hamiltonian system

Like in Chapter 1, we introduce the change of variables

$$\nabla\Phi(q') = p$$

which is equivalent to

$$q' = \nabla\Phi^*(p),$$

to transform the problem (22) is the equivalent one

$$q' = \nabla\Phi^*(p), \quad p' = \nabla_q F(t,q) + h(t), \quad q(0) = q(T), \quad p(0) = p(T). \tag{183}$$

With the Hamiltonian function $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$H(t,p,q) = \Phi^*(p) - F(t,q) - \langle h(t), q\rangle,$$

problem (183) takes the Hamiltonian form

$$p' = -\nabla_q H(t,p,q), \quad q' = \nabla_p H(t,p,q), \quad q(0) = q(T), \quad p(0) = p(T),$$

or, in a more concise way, letting $z = (p,q)$ and introducing the $2n \times 2n$ symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

$$Jz' = \nabla_z H(t,z), \quad z(0) = z(T). \tag{184}$$

We use the same notations $\langle\cdot,\cdot\rangle$ and $|\cdot|$ for the inner product and the corresponding norm in $\mathbb{R}^n$ and in $\mathbb{R}^{2n}$. It is well known that, formally, system (184) is the Euler-Lagrange equation associated to the (action) functional $\mathcal{A}$ defined on a suitable space of T-periodic functions by

$$\mathcal{A}(z) = \int_0^T \left[-\frac{1}{2}\langle Jz'(t), z(t)\rangle + H(t,z(t)) \right] dt,$$

or, in terms of the (p,q) variables and original data, after integrating by parts and using the periodicity, by

$$\mathcal{A}(p,q) = \int_0^T \left[-\langle p(t), q'(t)\rangle + \Phi^*(p(t)) - F(t,q(t)) - \langle h(t), q(t)\rangle \right] dt.$$

3.2.2 The Hamiltonian action functional

Define (see e.g. [86]) the space $H^{1/2}_{\#} = H^{1/2}_{\#}([0,T],\mathbb{R}^{2n})$ as the space of functions $z \in L^2(0,T;\mathbb{R}^{2n})$ with Fourier series $z(t) = \sum_{k\in\mathbb{Z}} e^{k\omega t J} z_k$ ($\omega = \frac{2\pi}{T}$), such that $z_k \in \mathbb{R}^{2n}$ ($k \in \mathbb{Z}$) and

$$\|z\|^2_{1/2} := \sum_{k\in\mathbb{Z}} (1+|k|)|z_k|^2 < +\infty.$$

With the corresponding inner product

$$(z|w) := \sum_{k\in\mathbb{Z}} (1+|k|)\langle z_k, w_k\rangle,$$

$H^{1/2}_{\#}$ is a Hilbert space such that $H^1_{\#}(0,T;\mathbb{R}^{2n}) \subset H^{1/2}_{\#} \subset L^s(0,T;\mathbb{R}^{2n})$ for any $s \geq 1$. We have also, by easy computations based on Fourier series and use of Cauchy-Schwarz inequality, for z smooth,

$$\left| \int_0^T [-\langle Jz'(t), w(t)\rangle]\, dt \right| \leq C\|z\|_{1/2}\|w\|_{1/2},$$

so that the bilinear form defined in the left-hand member can be extended to $H^{1/2}_{\#}$ as a continuous quadratic form $B(z,w)$, and the linear self-adjoint operator $A : H^{1/2}_{\#} \to H^{1/2}_{\#}$ defined through Riesz's representation theorem by the relation

$$(Lz|w) = B(z,w) \quad (z, w \in H^{1/2}_{\#}) \tag{185}$$

is continuous. In terms of Fourier series,

$$(Az|w) = 2\pi \sum_{k\in\mathbb{Z}} k\langle z_k, w_k\rangle$$

and hence

$$(Az|z) = 2\pi \sum_{k\in\mathbb{Z}} k|z_k|^2. \tag{186}$$

It is easily seen that the spectrum of A is made of the eigenvalues $\lambda_k = 2\pi\frac{k}{1+|k|}$ ($k \in \mathbb{Z}$), each of multiplicity $2n$, and of the elements -2π, 2π in the essential spectrum. Therefore, if we let, with $E(\lambda_k)$ the eigenspace associated to λ_k,

$$H^0 = \text{ker A} = E(\lambda_0) \simeq \mathbb{R}^{2n},$$

$$H^- = \overline{\{\cup_{k\leq -1} E(\lambda_k)\}}, \quad H^+ = \overline{\{\cup_{k\geq 1} E(\lambda_k)\}},$$

then $H^{1/2}_{\#} = H^- \oplus H^0 \oplus H^+$ (orthogonal sum with respect to $(\cdot|\cdot)$ and to $L^2(0,T;\mathbb{R}^{2n})$), and, using (186), we have, for $z^- \in H^-$, $z^+ \in H^+$,

$$
\begin{aligned}
(Az^-|z^-) &= 2\pi \sum_{k\leq -1} \frac{k}{1+|k|}(1+|k|)|z_k|^2 \\
&\leq -\pi \sum_{k\leq -1}(1+|k|)\|z_k\|^2 = -\pi|z^-|^2_{1/2}, \\
(Az^+|z^+) &= 2\pi \sum_{k\geq 1} \frac{k}{1+|k|}(1+|k|)|z_k|^2 \\
&\geq \pi \sum_{k\geq 1}(1+|k|)|z_k|^2 = \pi\|z^+\|^2_{1/2}. \qquad (187)
\end{aligned}
$$

Furthermore the subspaces H^- and H^+ are invariant for A.

Finally, using estimate (11), it is well known [86] that the assumptions (HΦ) and (HF) imply that $\mathcal{A}$ is of class C^1 on $H^{1/2}_{\#}$ and that any critical point $(\widehat{p},\widehat{q})$ of the functional

$$\mathcal{A}(p,q) = -\frac{1}{2}(A(p,q)|(p,q)) + \int_0^T [\Phi^*(p(t)) - F(t,q(t)) - \langle h(t), q(t)\rangle]\, dt$$

satisfies the Euler equation

$$(A(\widehat{p},\widehat{q})|(p,q)) + \int_0^T [\langle \nabla\Phi^*(\widehat{p}(t)), p(t)\rangle - \langle \nabla_q F(t,\widehat{q}(t)) - h(t), q(t)\rangle]\, dt = 0,$$

for all $(p,q) \in H^{1/2}_{\#}$. A classical reasoning shows then that $(\widehat{p},\widehat{q})$ is a (Carathéodory) solution of (184) (see e.g. [86]).

3.3 Multiplicity of periodic solutions

3.3.1 Szulkin's theorem

If X is a closed smooth manifold of dimension n, let $\Lambda^k(X)$ denote the vector space of all smooth differential k-forms on X. Taking for coboundary operator the exterior differential $d : \Lambda^k(X) \to \Lambda^{k+1}(X)$, one can define De Rham cohomology $H^*(X)$ through the vector spaces

$$Z^k(X) = \{\omega \in \Lambda^k(X) : d\omega = 0\}, \quad B^k(X) = dC^{k-1}(X),$$

by $H^k(X) = Z^k(X)/B^k(X)$. If $\omega_1 \in \Lambda^{k_1}(X)$, $\omega_2 \in \Lambda^{k_2}(X)$, then $\omega_1 \wedge \omega_2 \in \Lambda^{k_1+k_2}(X)$, and the easily checked fact that the exterior differential of the wedge product of two cocycles is a cocycle and the wedge product of a cocycle and a coboundary is a coboundary implies that the exterior product induces on $H^*(X)$ a product $\cup$, the cup product, operating as follows

$$\cup : H^{k_1}(X) \times H^{k_2}(X) \mapsto H^{k_1+k_2}(X).$$

Then the *cuplength* of X is the greatest number of elements of non-zero degree in $H^*(X)$ with non vanishing cup product, namely the largest integer m for which there exists $\alpha_j \in H^{k_j}(X)$, $1 \le j \le m$, such that $k_1, \ldots, k_m \ge 1$ and $\alpha_1 \cup \ldots \cup \alpha_m \ne 0$ in $H^{k_1+\ldots+k_m}(X)$. For the n-dimensional torus $\mathbb{T}^n$, cuplength$(\mathbb{T}^n) = n$.

Let E be a real Hilbert space with inner product $(\cdot|\cdot)$ and norm $\|\cdot\|$, and V^d a compact d-dimensional C^2-manifold without boundary. Let $L : E \to E$ be a bounded linear self-adjoint operator to which there corresponds an orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$ into invariant subspaces, with $E^0 = \ker L$, and a number $\varepsilon > 0$ such that

$$\langle Lx^+, x^+\rangle \ge \varepsilon \|x^+\|^2 \quad (x^+ \in E^+), \quad \langle Lx^-, x^-\rangle \le -\varepsilon \|x^-\|^2 \quad (x^- \in E^-).$$

The following result is due to Szulkin [98]

Lemma 23 *Let $\Psi \in C^1(E \times V^d, \mathbb{R})$ be given by $\Psi(x,v) = \frac{1}{2}(Lx|x) - \psi(x,v)$, where ψ' is compact. Suppose that $\psi'(E \times V^d)$ is a bounded set, E^0 is finite dimensional and, if dim $E^0 > 0$, $\psi(x^0, v) \to -\infty$ (or $\psi(x^0,v) \to +\infty$) as $\|x^0\| \to \infty$, $x^0 \in E^0$. Then Φ has at least cuplength $(V^d) + 1$ critical points.*

3.3.2 Existence of multiple periodic solutions

Lemma 23 applied to a suitable reformulation of $\mathcal{A}$ will give our multiplicity theorem.

Theorem 28 *If $\Phi : \overline{B}(a) \to \mathbb{R}$ satisfies Assumption (HΦ) and $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies Assumption (HF), then, for every $h \in L^s(0,T;\mathbb{R}^n)$ $(s > 1)$ verifying Assumption (Hh), problem (22) has at least* n+1 *geometrically distinct solutions.*

Proof. Assumptions (HF) and (Hh) imply that, for any $(j_1, \ldots, j_n) \in \mathbb{Z}^n$,

$$\begin{aligned} & \mathcal{A}(p, q_1 + j_1\omega_1, \ldots, q_n + j_n\omega_n) \\ = \; & \frac{1}{2}(A(p,q)|(p,q)) + \int_0^T [\Phi^*(p(t)) - F(t, q_1(t) + j_1\omega_1, \ldots, q_n(t) + j_n\omega_n) \\ - \; & \langle h(t), q(t)\rangle - \sum_{k=1}^n h_k(t) j_k \omega_k] \, dt \\ = \; & \frac{1}{2}(A(p,q)|(p,q)) + \int_0^T [\Phi^*(p(t)) - F(t,q(t)) - \langle h(t), q(t)\rangle] \, dt = \mathcal{A}(p,q). \end{aligned}$$

To each critical point $(\widehat{p}, \widehat{q})$ of $\mathcal{A}$ on $H^{1/2}$, corresponds the orbit

$$(\widehat{p}, \widehat{q}_1 + j_1\omega_1, \ldots, \widehat{q}_n + j_n\omega_n) \quad ((j_1, \ldots, j_n \in \mathbb{Z}^n)$$

of critical points, which can be considered as a single critical point lying on the manifold $E \times V^n$, with V^n the n-torus $\mathbb{T}^n = \mathbb{R}^n/(\omega_1\mathbb{Z}, \ldots, \omega_n\mathbb{Z})$, and

$$E = \{(p,q) \in H^{1/2} : \overline{q} = 0\}.$$

Denoting by $L : E \to E$ the restriction to E of A given in (185), we have $E = H^- \oplus E^0 \oplus H^+$, where $E^0 \simeq \mathbb{R}^n = \{(p, 0) \in \mathbb{R}^{2n} : p \in \mathbb{R}^n\} = \ker L$. Hence, letting $\widetilde{q} = q - \overline{q}$, $\mathcal{A}$ has the equivalent form

$$\frac{1}{2}(L(p, \widetilde{q})|(p, \widetilde{q}) + \int_0^T [Phi^*(p(t)) - F(t, \overline{q} + \widetilde{q}(t)) - \langle h(t), \widetilde{q}(t)\rangle]\, dt,$$

namely

$$\Psi(x, v) = \frac{1}{2}\langle L(p, \widetilde{q}), (p, \widetilde{q})\rangle - \psi(p, \widetilde{q}; \overline{q})$$

requested by Szulkin's lemma with $x = (p, \widetilde{q})$, $v = \overline{q}$, considered as an element of V^n, and

$$\psi(p, \widetilde{q}; \overline{q}) := \int_0^T [F(t, \overline{q} + \widetilde{q}(t)) + \langle h(t), \widetilde{q}(t)\rangle - \Phi^*(p(t))]\, dt.$$

Therefore, for any $v, \widetilde{w}, \overline{w}$, we have

$$\begin{aligned} &(\psi'(p, \widetilde{q}; \overline{q})|(v, \widetilde{w}; \overline{w})) \\ = &\int_0^T [\langle \nabla_q F(t, \overline{q} + \widetilde{q}(t)), \overline{w} + \widetilde{w}\rangle + \langle h(t), \widetilde{w}(t)\rangle - \langle \phi^{-1}(p(t)), v(t)\rangle]\, dt. \end{aligned}$$

Because $\nabla_q F(t, \cdot)$ and ϕ^{-1} have a bounded range, ψ' has a bounded range, and ψ' is compact using the compact embedding of $H^{1/2}_{\#}$ in L^s for any $s \geq 1$. On the other hand, because of (11) and the fact that any $(p, \widetilde{q}) \in E^0$ has the form $(p^0, 0)$ with $p^0 \in \mathbb{R}^n$, we have, for $\|p^0\| \to \infty$,

$$\psi(p^0, \overline{q}) = \int_0^T [-F(t, \overline{q}) - \Phi^*(p^0)]\, dt = -T[\overline{F(\cdot, \overline{q})} + T\Phi^*(p^0)] \to -\infty.$$

All the assumptions of Lemma 23 are satisfied, and Ψ has at least cuplength $(\mathbb{T}^n) + 1 = n + 1$ critical points, i.e. $\mathcal{A}$ has at least $n + 1$ geometrically distinct critical points. ∎

Example 19 *For any $A_j \in \mathbb{R}$ and $h \in L^s$ $(s > 1)$ with mean value zero, the problem*

$$\left(\frac{q_j'}{\sqrt{1 - \|q'\|^2}}\right)' + A_j \sin q_j = h_j(t) \quad (j = 1, 2, \ldots, n),$$
$$q(0) = q(T), \quad q'(0) = q'(T)$$

has at least $n + 1$ geometrically distinct solutions.

In particular, *for any $A \in \mathbb{R}$ and $h \in L^s$ $(s > 1)$ with mean value zero, the forced relativistic pendulum problem*

$$\left(\frac{q'}{\sqrt{1 - q'^2}}\right)' + A \sin q = h(t), \quad q(0) = q(T), \quad q'(0) = q'(T)$$

has at least 2 geometrically distinct solutions.

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