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Non-autonomous Functional Differential Equations and Applications

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2.1 Flows over compact metric spaces

Let Ω be a compact metric space. A real *continuous flow* $(\Omega,\sigma,\mathbb{R})$ is defined by a continuous mapping $\sigma:\mathbb{R}\times\Omega\to\Omega,\ (t,\omega)\mapsto\sigma(t,\omega)$ satisfying

(i) $\sigma_0=\mathrm{Id}$.
(ii) $\sigma_{t+s}=\sigma_t\circ\sigma_s$ for each s, $t\in\mathbb{R}$,

where $\sigma_t(\omega)=\sigma(t,\omega)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$. The set $\{\sigma_t(\omega)\mid t\in\mathbb{R}\}$ is called the *orbit* or the *trajectory* of the point ω.

We say that a subset $\Omega_1\subset\Omega$ is σ*-invariant* if $\sigma_t(\Omega_1)=\Omega_1$ for every $t\in\mathbb{R}$. A mapping $f:\Omega\to\mathbb{R}$ is σ*-invariant* if it is constant along the trajectories, i.e., $f(\sigma_t(\omega))=f(\omega)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$. A subset $\Omega_1\subset\Omega$ is called *minimal* if it is compact, σ-invariant and it has no other nonempty compact σ-invariant subset but itself. Every compact and σ-invariant set contains a minimal subset; in particular it is easy to prove that a compact σ-invariant subset is minimal if and only if every trajectory is dense. We say that the continuous flow $(\Omega,\sigma,\mathbb{R})$ is *recurrent* or *minimal* if Ω is minimal.

If $\omega_0\in\Omega$ is a point such that the subset $\{\sigma_t(\omega_0)\mid t\geq t_0\}\subset\Omega$ is relatively compact for some $t_0>0$, then its *omega-limit set* can be defined by

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$$\bigcap_{s\geq t_0} \mathrm{cls}\{\sigma(t+s,\omega_0) \mid t \geq 0\},$$

which is a compact and invariant subset. Analogously, given $\omega_0 \in \Omega$ such that the set $\{\sigma_t(\omega_0) : t \leq -t_0\} \subset \Omega$ is relatively compact for some $t_0 > 0$, we can consider its *alpha-limit set*, defined by

$$\bigcap_{s\leq -t_0} \mathrm{cls}\{\sigma(t+s,\omega_0) \mid t \leq 0\}.$$

Both omega-limit set and alpha-limit set contain minimal subsets.

Let d be a metric on Ω. We say that the flow $(\Omega,\sigma,\mathbb{R})$ is *distal* when, for each pair ω_1, ω_2 of different elements of Ω, the orbits keep at a positive distance, that is, there is a $\delta > 0$ such that $\mathsf{d}(\sigma_t(\omega_1),\sigma_t(\omega_2)) > \delta$ for every $t \in \mathbb{R}$. Equivalently, $\inf\{\mathsf{d}(\sigma_t(\omega_1),\sigma_t(\omega_2)) \mid t \in \mathbb{R}\} = 0$ if and only if $\omega_1 = \omega_2$.

The flow $(\Omega,\sigma,\mathbb{R})$ is said to be *almost periodic* when for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if ω_1, $\omega_2 \in \Omega$ with $\mathsf{d}(\omega_1,\omega_2) < \delta$ then $\mathsf{d}(\sigma_t(\omega_1),\sigma_t(\omega_2)) < \varepsilon$ for every $t \in \mathbb{R}$; equivalently, the flow $(\Omega,\sigma,\mathbb{R})$ is almost periodic if the family $\{\sigma_t\}_{t\in\mathbb{R}}$ is equicontinuous. If $(\Omega,\sigma,\mathbb{R})$ is almost periodic, it is distal. The converse is not true; even if $(\Omega,\sigma,\mathbb{R})$ is minimal and distal, it does not need to be almost periodic.

We say that ω_1, $\omega_2 \in \Omega$ form a *proximal pair* if $\inf\{\mathsf{d}(\sigma_t(\omega_1),\sigma_t(\omega_2)) \mid t \in \mathbb{R}\} = 0$, otherwise the pair is said to be *distal*. It is said that the points ω_1 and ω_2 are a *positively* (resp. *negatively*) *proximal pair* if $\inf\{d(\sigma_t(\omega_1),\sigma_t(\omega_2)) \mid t \geq 0\} = 0$ (resp. $\inf\{d(\sigma_t(\omega_1),\sigma_t(\omega_2)) \mid t \leq 0\} = 0$). For the basic properties on almost periodic and distal flows we refer the reader to Ellis [14] and Sacker and Sell [63].

Given another continuous flow $(Y,\Psi,\mathbb{R})$, a *flow homomorphism* from $(Y,\Psi,\mathbb{R})$ into $(\Omega,\sigma,\mathbb{R})$ is a continuous mapping $\pi\colon Y \to \Omega$ such that, for every $y \in Y$ and $t \in \mathbb{R}$, $\pi(\Psi(t,y)) = \sigma(t,\pi(y))$. If π is also surjective, then it is called a *flow epimorphism*; in this case, Ω is a *factor* of Y, and Y is an *extension* of Ω. If π is a flow epimorphism and there exists $k \geq 1$ such that $\mathrm{card}(\pi^{-1}(\omega)) = k$ for all $\omega \in \Omega$, then it is said that the flow $(Y,\Psi,\mathbb{R})$ is a *k-cover* or a *k-copy* of $(\Omega,\sigma,\mathbb{R})$. If $k = 1$, then the flows are isomorphic; in particular, they have the same topological properties. In such a case, we will simply say that they are covers or copies. As for homomorphisms between distal flows, now we present a relevant result (see [72] and [63]).

Theorem 2.1. *Let $(\Omega,\sigma,\mathbb{R})$ be a minimal and distal flow, and consider a homomorphism between distal flows $\pi : (Y,\Psi,\mathbb{R}) \to (\Omega,\sigma,\mathbb{R})$. If there is an $\omega \in \Omega$ such that* $\mathrm{card}(\pi^{-1}(\omega)) = N$ *for some $N \in \mathbb{N}$, then*

(i) *Y is an N-copy of Ω;*
(ii) *$(Y,\Psi,\mathbb{R})$ is almost periodic if and only if $(\Omega,\sigma,\mathbb{R})$ is almost periodic.*

Let $\pi : (Y,\Psi,\mathbb{R}) \to (\Omega,\sigma,\mathbb{R})$ be a flow epimorphism, and suppose that $(Y,\Psi,\mathbb{R})$ is a minimal flow (then, so is $(\Omega,\sigma,\mathbb{R})$, because, given $\omega = \pi(y)$ and $\omega_0 = \pi(y_0)$, there exists $\{t_n\}_n \subset \mathbb{R}$ such that $\Psi_{t_n}(y_0) \to y$ as $n \to \infty$, and, due to the continuity of π and its being a homomorphism, we have that $\pi(\Psi_{t_n}(y_0)) = \sigma_{t_n}(\omega_0) \to \omega$ as $n \to \infty$). $(Y,\Psi,\mathbb{R})$ is said to be an *almost automorphic extension* of $(\Omega,\sigma,\mathbb{R})$ if there exists

$\omega \in \Omega$ such that $\mathrm{card}(\pi^{-1}(\omega)) = 1$. Furthermore, $(Y, \Psi, \mathbb{R})$ is said to be a *proximal extension* of $(\Omega, \sigma, \mathbb{R})$ if, whenever $\pi(y_1) = \pi(y_2)$ for some $y_1, y_2 \in Y$, then they are a proximal pair. An almost automorphic extension is always a proximal extension (see Veech [81]). From this last remark together with statement (i) of Theorem 2.1, it is deduced that, if $(Y, \Psi, \mathbb{R})$ is a minimal and almost periodic flow which is an almost automorphic extension of an almost periodic flow $(\Omega, \sigma, \mathbb{R})$, then it must be a copy of $(\Omega, \sigma, \mathbb{R})$.

A point $\omega_0 \in \Omega$ is said to be an *almost automorphic point* if, given any sequence $\{s_n\}_n \subset \mathbb{R}$, we can find a subsequence $\{t_n\}_n$ of it such that the limits $\lim_{n\to\infty} \sigma_{t_n}(\omega_0) = \omega_1$ and $\lim_{n\to\infty} \sigma_{-t_n}(\omega_1) = \omega_0$ exist. The flow $(\Omega, \sigma, \mathbb{R})$ is *almost automorphic* when there is an almost automorphic point which has a dense orbit. An almost automorphic flow is always minimal, that is, actually all the orbits are dense. Almost automorphic minimal flows were first introduced and studied by Veech [81, 82, 83]. The theorem known as *Veech almost automorphic structure theorem* says that a flow is almost automorphic if and only if it is an almost automorphic extension of an almost periodic (minimal) flow (see [81]).

If $(Y, \Psi, \mathbb{R})$ is an almost automorphic flow and $(\Omega, \sigma, \mathbb{R})$ is an almost periodic (and minimal) flow satisfying that there exists a flow epimorphism $p : (Y, \Psi, \mathbb{R}) \to (\Omega, \sigma, \mathbb{R})$ such that $\mathrm{card}(p^{-1}(\omega)) = 1$ for some $\omega \in \Omega$, then the subset of Y formed by all of the almost automorphic points in Y is given by $\{y \in Y \mid p^{-1}(p(y)) = \{y\}\}$, and it is a residual set (see Remark 2.6 in [72], part I).

We recall that a subset of a topological space E is said to be *residual* if its complementary is of first category in the sense of Baire, that is, its complementary is given by the union of countably many nowhere dense subsets of E.

A Borel measure on Ω will be a finite regular measure defined on the Borel sets. Let μ be a normalized Borel measure on Ω, μ is σ*-invariant* (or *invariant under* σ) if $\mu(\sigma_t(\Omega_1)) = \mu(\Omega_1)$ for every Borel subset $\Omega_1 \subset \Omega$ and every $t \in \mathbb{R}$. It is σ*-ergodic* (or *ergodic under* σ) if, in addition, $\mu(\Omega_1) = 0$ or $\mu(\Omega_1) = 1$ for every σ-invariant subset $\Omega_1 \subset \Omega$.

We denote by $\mathscr{M}_{\mathrm{inv}}(\Omega, \sigma)$ the set of positive and normalized σ-invariant measures on Ω. The Krylov-Bogoliubov theorem (see Nemytskii and Stepanoff [48]) asserts that $\mathscr{M}_{\mathrm{inv}}(\Omega, \sigma)$ is nonempty when Ω is a compact metric space. The extremal points of the convex and weakly compact set $\mathscr{M}_{\mathrm{inv}}(\Omega, \sigma)$ are the σ-ergodic measures, from which it is deduced that also the set of σ-ergodic measures is nonempty. The decomposition of the flow $(\Omega, \sigma, \mathbb{R})$ into ergodic components and the construction and representation theorems of σ-invariant measures from σ-ergodic measures are well known (see Phelps [62] and Mañé [41]).

We say that $(\Omega, \sigma, \mathbb{R})$ is *uniquely ergodic* (u.e.) if it has a unique normalized invariant measure which is then necessarily ergodic. If $(\Omega, \sigma, \mathbb{R})$ is u.e. it is not necessarily minimal; however, if $(\Omega, \sigma, \mathbb{R})$ is u.e. and $\mu(U) > 0$ for every non-empty open set U, then $(\Omega, \sigma, \mathbb{R})$ is minimal. An almost periodic and minimal flow $(\Omega, \sigma, \mathbb{R})$ is always u.e. but an almost automorphic minimal one can be non-uniquely ergodic and can admit positive topological entropy (see Markley and Paul [42]).

2.2 *Almost periodic and almost automorphic dynamics*

In order to find a link between non-autonomous differential equations with some recurrence in time and the theory of dynamical systems, we recall the basic definitions and results for the class of almost periodic and almost automorphic functions. We will give a brief explanation about the way this kind of equations give rise to skew-product flows or semiflows using the so-called hull as a base flow, which in turn will have some recurrence properties as well.

The concept of almost periodic function came up in the 1920s as an extension of the notion of periodicity. Authors like Bohr [7, 8], Favard [16], Besicovitch [5] and Bochner [6] studied exhaustively the properties of these functions. The book by Fink [18] is a detailed and well written reference on this topic.

Several equivalent definitions of almost periodic function may be found in the literature. Thus, in order to study harmonic functions, it is better to choose the characterization (as adopted by Corduneanu [13]) saying that a function is almost periodic whenever it can be approximated uniformly by a sequence of trigonometric polynomials on the whole real line, whereas, if our aim is to study differential equations, the preferred definition is the one introduced by Bohr, which is in the end the most frequently chosen one, (as seen in [5] and Amerio and Prouse [2]). A subset S of $\mathbb{R}$ is said to be *relatively dense* if there exists $l > 0$ such that every interval of length l intersects S. A complex function f, defined and continuous on $\mathbb{R}$, is *almost periodic* if, for all $\varepsilon > 0$, the set

$$T(f,\varepsilon) = \{s \in \mathbb{R} \mid |f(t+s) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense. The set $T(f,\varepsilon)$ is called *ε-translation set of f*. Almost periodic functions are bounded and uniformly continuous on $\mathbb{R}$. The set formed by all these functions is an algebra over $\mathbb{C}$, which is invariant by translations and closed under conjugation and uniform limits. Moreover, if f is almost periodic and $|f(t)| \geq m > 0$ for all $t \in \mathbb{R}$, then the function $1/f$ is almost periodic as well. Besides, if f is almost periodic and differentiable, then f' is almost periodic if and only if it is uniformly continuous on $\mathbb{R}$. As for integration, if a primitive of an almost periodic function is bounded, then it is also almost periodic.

The concept of almost periodicity can be extended to continuous functions taking values in a complete metric space (E,d) in a straightforward way: for each $\varepsilon > 0$, the set

$$T(f,\varepsilon) = \{s \in \mathbb{R} \mid \mathsf{d}(f(t+s), f(t)) < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

must be relatively dense in $\mathbb{R}$. The reference [2] contains a study about almost periodic functions taking values in a Banach space and their relation with the theory of functional equations.

Bochner introduced another equivalent definition in terms of sequences (adopted for instance in [18]): a continuous function f is almost periodic if, given any sequence $\{\alpha_n\}_n \subset \mathbb{R}$, we can find a subsequence $\{\alpha_{n_j}\}_j$ of the previous one such that $\lim_{j\to\infty} f(t+\alpha_{n_j})$ exists uniformly on $\mathbb{R}$.

Besides, Bochner pointed out that, in order to simplify the proofs involving almost periodic functions, a property satisfied by such functions with respect to a group G could be used (see [6]); when $G=\mathbb{R}$, this property can be stated as follows: given a complex function f, defined and continuous on $\mathbb{R}$, and given any sequence $\{\alpha_n\}_n$ of real numbers, we can find a subsequence $\{\alpha_{n_j}\}_j$ in such a manner that the following limits exist pointwise on $\mathbb{R}$:

$$\lim_{j\to\infty} f(t+\alpha_{n_j}) = g(t)\,, \qquad \lim_{j\to\infty} g(t-\alpha_{n_j}) = f(t)$$

for some function g. All the functions satisfying that property, whether they are almost periodic or not, are said to be *almost automorphic*. As we explained before, the fundamental properties of these functions with respect to groups, together with almost automorphic abstract minimal flows, were studied by Veech. In principle, the function g does not need to be continuous. If the function g is continuous for all sequences, then we say that f is *almost automorphic in the sense of Bohr*. From now on, we will assume that almost automorphic functions are almost automorphic in the sense of Bohr, so that almost automorphic functions are bounded and uniformly continuous on $\mathbb{R}$ (see [86]). Almost periodic functions are always almost automorphic, but the converse is not true; several examples can be found in the foregoing references.

One can define Fourier series both for almost periodic and almost automorphic functions valued in a Banach space but the one for an almost periodic function is unique and converges uniformly in terms of Bochner-Fejer summation, while the one for an almost automorphic function is in general non-unique and its Bochner-Fejer sum only converges pointwise ([82]). However, one can define the *frequency module* $\mathscr{M}(f)$ of an almost automorphic function f in the usual way as the smallest Abelian group containing a Fourier spectrum (the set of Fourier exponents associated with a Fourier series), and it has been shown that such a frequency module is uniquely defined ([72]). In the above sense both almost periodic and almost automorphic functions can be viewed as natural generalizations to the periodic ones in t he strongest and the weakest sense respectively.

In the early 1940s, Fréchet defined and studied the concept of asymptotic almost periodicity. A function f continuous on $\mathbb{R}^+=[0,\infty)$ is said to be *asymptotically almost periodic* if it can be represented as $f=f_1+f_2$, where f_1 is an almost periodic function, and f_2 vanishes pointwise as $t\to\infty$. In fact, that representation is unique.

The relation between almost periodic functions and almost periodic flows is quite simple (see [14], [48], and [18]). First, if $(\Omega,\sigma,\mathbb{R})$ is an almost periodic continuous real flow, then all the trajectories $t\in\mathbb{R}\mapsto\sigma(t,\omega)\in\Omega$ define almost periodic functions taking values in the compact metric space Ω. It is said that an element ω of a continuous real flow $(\Omega,\sigma,\mathbb{R})$ is an *almost periodic point* if, for any $\varepsilon>0$, the set

$$T(\omega,\varepsilon)=\{s\in\mathbb{R}\mid \mathsf{d}(\sigma(s,\omega),\omega)<\varepsilon\}$$

is relatively dense in $\mathbb{R}$; such points are sometimes referred to as points with a *recurrent* orbit (see [48]). This condition is equivalent to the fact that the closure of the trajectory of such point, $\mathrm{cls}\{\sigma(t,\omega)\mid t\in\mathbb{R}\}$, is a minimal subset for the flow. Notice that, if the flow is minimal, then all its points are almost periodic. As a consequence, the flow $(\Omega,\sigma,\mathbb{R})$ can be decomposed as the disjoint union of a family of minimal subsets if and only all its points are almost periodic. Clearly, if the trajectory of ω, $t\in\mathbb{R}\mapsto\sigma(t,\omega)\in\Omega$, is an almost periodic function, then ω is an almost periodic point; moreover, in this case, the closure of its orbit is an almost periodic minimal set which coincides with both the omega-limit and alpha-limit sets of ω. Specifically, almost periodic flows are decomposed as a disjoint union of almost periodic and minimal flows.

As for almost automorphic flows, we know that there is an almost automorphic point with a dense orbit. If a point $\omega\in\Omega$ is almost automorphic, then its trajectory, $t\in\mathbb{R}\mapsto\sigma(t,\omega)\in\Omega$, is an almost automorphic function taking values in Ω (as before, the definition can be extended to this case in a natural manner). However, now there is no need for all the points to be almost automorphic, though all the points in a residual subset of Ω are (as we remarked in Subsection 2.1). In fact, an almost automorphic minimal flow become almost periodic only if every point is almost automorphic (see [81, 85]).

Conversely, let us check how to obtain almost periodic and almost automorphic flows from functions with analogous properties.

Definition 2.2. A function $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ is said to be *admissible* if, for every compact subset $K\subset\mathbb{R}^n$, f is bounded and uniformly continuous on $\mathbb{R}\times K$. Besides, if f is of class C^r ($r\geq 1$) in $x\in\mathbb{R}^n$ and f and all its partial derivatives with respect to x up to order r are admissible, then we will say that f is either *C^r-admissible* or *admissible of class C^r*. A function $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^m)$ is *uniformly almost automorphic* (resp. *almost periodic*) if it is admissible and almost automorphic (resp. almost periodic) in $t\in\mathbb{R}$.

Given an admissible function $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^m)$, we consider the family of time translated functions $\{f_s\mid s\in\mathbb{R}\}$, where $f_s(t,x)=f(t+s,x)$ for all s, $t\in\mathbb{R}$, and all $x\in\mathbb{R}^n$. Hence, we can define the *hull* of f, which will be denoted by Ω or $H(f)$, as the closure within the space $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^m)$ of the set of time translated functions for the compact-open topology, that is, the topology of uniform convergence over compact subsets. Thanks to Arzelà-Ascoli's theorem, we can assure that the space $H(f)$ is compact and, furthermore, metrizable. Moreover, a continuous real flow is induced over the hull in a natural way, just by considering the mapping $\sigma:\mathbb{R}\times\Omega\to\Omega$, $(s,h)\mapsto h_s$, h translated a time s, that is, there is a flow over the hull defined by translation.

The next result assures that the initial function f admits a unique continuous extension to the hull and shows how the properties of recurrence of f are translated to the hull (see e.g. [72]).

Theorem 2.3. *Let $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^m)$ be an admissible function. The following statements hold:*

(i) *all the functions $h\in H(f)$ are admissible; in fact, if f is admissible of class C^r, so are all the functions $h\in H(f)$;*

(ii) *there exists a unique function* $F \in C(H(f) \times \mathbb{R}^n, \mathbb{R}^m)$ *which extends* f*, in the sense that* $F(f_t, x) = f(t,x)$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$*; besides, if* f *is* C^r*-admissible, then* F *is of class* C^r *in* x*;*

(iii) *the flow* $(H(f), \sigma)$ *is almost automorphic (resp. almost periodic) if* f *is uniformly almost automorphic (resp. almost periodic).*

It is convenient to point out that the function F is defined specifically by $F(h,x) = h(0,x)$, $(h,x) \in H(f) \times \mathbb{R}^n$. The construction of the flow on the hull is often used when dealing with differential equations, as we will see in the next subsection. In particular, systematic studies of almost automorphic dynamics in differential equations were made in the 90s in a series of works by Shen and Yi ([68]-[72]).

2.3 Some important ODE's examples

Let $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a C^r-admissible function such that the flow $(H(f), \sigma)$ is minimal, and consider its unique continuous extension to the hull $\Omega = H(f)$, $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, which, according to Theorem 2.3, is a function of class C^r in $x \in \mathbb{R}^m$. In particular, if the initial equation is given by a uniformly almost periodic or almost automorphic function, then we are in the foregoing context. This way, from a system of non-autonomous ordinary differential equations

$$x' = f(t,x),$$

we can obtain a family of differential equations with indexes in the hull

$$x'(t) = F(\omega{\cdot}t, x(t)), \quad \omega \in \Omega, \tag{2.1}$$

where the flow on Ω is denoted by $\omega{\cdot}t = \sigma(t,\omega)$. Notice that, fixing $\omega = f$, we get the original system, i.e. $x'(t) = f(t,x(t))$.

According to the standard theory of existence, uniqueness, and continuation of solutions for this kind of equations (see e.g Hale [23]), these families of systems give rise to a local flow of skew-product type

$$\tau\colon \mathcal{U} \subset \mathbb{R} \times \Omega \times \mathbb{R}^m \longrightarrow \Omega \times \mathbb{R}^m, \quad (t,\omega,x) \mapsto (\omega{\cdot}t, u(t,\omega,x)), \tag{2.2}$$

where $u(t,\omega,x)$ is the value of the solution of the system corresponding to ω with initial value $x(0) = x$ at time t, for t in the interval where the solution is defined. Thanks to the classical theorems of continuous dependence with respect to the initial values, u inherits the same regularity, C^r, with respect to x.

The use of this technique, that is, of including a non-autonomous system within a family of systems linked to one another by means of the flow on the hull, is focused to the application of the methods and results of the theory of skew-product flows to the new problem, where the solutions of the systems have been considered as a part of the trajectories of a dynamical system. It is noteworthy that, in the new family of systems generated from a given system, there are just their translated systems as well as their limits, so that the flow associated to this family is a good representation of the dynamics of the initial system and, in particular, the asymptotic behavior of its bounded solutions. Such a formulation was originated in Miller [44] and Sell [66].

The importance of the presence of almost automorphic dynamics in the study of almost periodic differential equations was motivated by the examples given by Levitan and Zhikov [40] and Johnson [33] of scalar almost periodic equations with almost automorphic but not almost periodic solutions. Ortega and Tarallo [61] describe a qualitative property, which is satisfied by the above examples, and it provides almost automorphic solutions of almost periodic linear systems. Similar phenomena occur in the Riccati equations obtained from the non-uniformly hyperbolic two-dimensional linear systems constructed by Millionščikov [45] and Vinograd [87]. We now review the importance of some of these examples in the study of the almost automorphic dynamics, as shown in [72]

Before, we recall that in the scalar and almost periodic case, $m=1$, every minimal set M is almost automorphic (see [72] for a more general version of this result, valid for scalar parabolic partial differential equations with the Neumann boundary condition).

Lemma 2.4. *Let $m=1$ and let M be a minimal set of* (2.2). *Then*

> (M,τ) *is an almost automorphic extension of the base flow (Ω,σ), and hence an almost automorphic minimal set if f is uniformly almost periodic.*

Proof. We define $x_i\colon \Omega\to\mathbb{R}$, $i=1,2$ by $x_1(\omega)=\inf\{x\in\mathbb{R}\mid(\omega,x)\in M\}$ and $x_2(\omega)=\sup\{x\in\mathbb{R}\mid(\omega,x)\in M\}$. It is easy to check that x_1 is lower semi-continuous and x_2 is upper semi-continuous. As a consequence (see Choquet [10]), there is a residual set $\Omega_0\subset\Omega$ of continuity points for x_1 and x_2. In fact, since the flow is scalar, and then monotone, it can be shown that

$$x_i(\omega{\cdot}t)=u(t,\omega,x_i(\omega))\,,\quad t\in\mathbb{R},\ \omega\in\Omega\,,\ i=1,2\,,$$

and that Ω_0 is an invariant set.

Let $\pi\colon\Omega\times\mathbb{R}\to\Omega$ be the natural projection and $\omega\in\Omega_0$. Next we check that $\mathrm{card}(M\cap\pi^{-1}(\omega))=1$. Since both $(\omega,x_1(\omega))$, $(\omega,x_2(\omega))\in M$, minimal, let $t_n\uparrow\infty$ such that $\lim_{n\to\infty}\tau(t_n,\omega,x_1(\omega))=(\omega,x_2(\omega))$, that is, $\lim_{n\to\infty}w{\cdot}t_n=\omega$ and $\lim_{n\to\infty}u(t_n,\omega,x_1(\omega))=x_2(\omega)$. Moreover, since $u(t_n,\omega,x_1(\omega))=x_1(\omega{\cdot}t_n)$ and ω is a continuity point of x_1, we conclude that $x_1(\omega)=x_2(w)$. Therefore, $\mathrm{card}(M\cap\pi^{-1}(\omega))=1$, and hence, M is an almost automorphic extension of the base flow (Ω,σ), as stated. □

2.3.1 Existence of a almost automorphic but non-almost periodic minimal set

Based on a previous example by Conley and Miller [12], Jonhson constructed in [33] a linear almost periodic scalar ordinary differential equation

$$x'+a(t)\,x=b(t) \tag{2.3}$$

satisfying the following properties:

(i) $a(t)$ y $b(t)$ are uniform limits of 2^n-periodic continuous functions $a_n(t)$ and $b_n(t)$ respectively;
(ii) $\lim_{t\to\infty}\int_0^t a(s)\,ds = \infty$;
(iii) if $x_0(t)$ is the solution of (2.3) with $x_0(0)=0$, then $|x_0(t)|\le 1$, and for $n\ge 4$, $x_0(2^n)=1/5$ if n is odd, and $x_0(2^n)=0$ if n is even.

As before, we take Ω the hull of the uniformly almost periodic function f given by $f(t,x)=-a(t)x+b(t)$, $(t,x)\in\mathbb{R}^2$, and the corresponding family of differential equations with indexes in the hull (2.1) and its induced skew-product flow (2.2).

Since $x_0(t)$ is a bounded solution with $x_0(0)=0$, the omega-limit set of the point $(f,0)\in\Omega\times\mathbb{R}$ contains a minimal set $M\subset\Omega\times\mathbb{R}$ for the flow, which is almost automorphic by Lemma 2.4. The uniqueness of the minimal set follows from (ii) because the existence of two different minimal sets would contradict the fact that the solutions of $x'+a(t)\,x=0$ tend to 0 at $+\infty$.

Johnson showed that (2.3) admits no almost periodic solutions and there is one of the equations in the hull with an almost automorphic but non-almost periodic solution $x(t,\widetilde{\omega},\widetilde{x})$. As a consequence, the unique minimal set M is almost automorphic but not almost periodic because the trajectory $\{\tau(t,\widetilde{\omega},\widetilde{x})=(\widetilde{\omega}{\cdot}t,x(t\widetilde{\omega},\widetilde{x}))\mid t\in\mathbb{R}\}$ is not almost periodic. In addition, Johnson showed in [34] that M is uniquely ergodic.

2.3.2 An omega-limit set which contains two minimal sets

A modification of the previous example provides an example of a skew-product scalar flow with an omega-limit set which contains two minimal sets. First notice that the family of equations in the hull corresponding to (2.3) could have be written in the form

$$x'+A(\omega{\cdot}t)\,x=B(\omega{\cdot}t)\,,\quad \omega\in\Omega\,, \tag{2.4}$$

for continuous functions A, $B\in C(\Omega,\mathbb{R})$. Let M be, as before, the unique minimal set for the induced skew-product flow. We take $y_0\in\mathbb{R}$ such that $(\omega,y_0)\notin M$ for all $\omega\in\Omega$. The change of variables $z=1/(x-y_0)$ takes (2.4) to

$$z'=A(\omega{\cdot}t)\,z+(B(\omega{\cdot}t)-A(\omega{\cdot}t)\,y_0)\,z^2\,,\quad \omega\in\Omega\,, \tag{2.5}$$

and we will denote by $\hat{\tau}$ the induced local skew-product flow for this family. Clearly, $M_1=\{(\omega,0)\mid\omega\in\Omega\}=\Omega\times\{0\}$ and $M_2=\{(\omega,1/(x-y_0)\mid(\omega,x)\in M\}$ are two different minimal sets for this flow.

We claim that there is an omega-limit set which contains M_1 and M_2. Since $\int_0^t a(s)\,ds$ is unbounded, also $\int_0^t A(\omega{\cdot}s)\,ds$ is unbounded for each $\omega\in\Omega$, and it can be shown (see [34] and [31]) that there is a point $\hat{\omega}\in\Omega$ (in fact a residual set) such that

$$\sup_{t\ge 0}\int_0^t A(\hat{\omega}{\cdot}s)\,ds=+\infty\,,\quad \inf_{t\ge 0}\int_0^t A(\hat{\omega}{\cdot}s)\,ds=-\infty\,, \tag{2.6}$$

and the equation (2.4) for $\hat\omega$ has a unique bounded solution $x(t,\hat\omega,\hat x)$. If we choose α such that $\big|x(t,\hat\omega,\hat x)+\alpha\exp(-\int_0^t A(\hat\omega{\cdot}s)\,ds)-y_0\big|\ge\varepsilon>0$ for each t, the function

$$z(t,\hat\omega,\hat z)=\frac{1}{x(t,\hat\omega,\hat x)+\alpha\exp(-\int_0^t A(\hat\omega{\cdot}s)\,ds)-y_0}$$

is a bounded solution of (2.5) for $\hat\omega$. Moreover from (2.6) there are sequences $\{t_n\}_n$ and $\{s_n\}_n\uparrow\infty$ such that $\lim_{n\to\infty}z(t_n,\hat\omega,\hat z)=0$ and $\lim_{n\to\infty}z(s_n,\hat\omega,\hat z)=z_0$ with $\lim_{n\to\infty}\hat\omega{\cdot}s_n=\omega_0$ and $(\omega_0,z_0)\in M_2$. Hence the omega-limit set of $(\hat\omega,\hat z)$, i.e., the closure of the set $\{\hat\tau(t,\hat\omega,\hat z)=(\hat\omega{\cdot}t,z(t,\hat\omega,\hat z))\mid t\in\mathbb{R}\}$ contains M_1 and M_2. Notice that M_2 is also almost automorphic but non-almost periodic.

2.3.3 Existence of non-uniquely ergodic minimal sets

The almost automorphic but non-almost periodic minimal set of subsection 2.3.1 is uniquely ergodic. An idea of constructing examples of non-uniquely ergodic almost automorphic minimal sets is suggested by Johnson in [32, 35] by studying the skew-product flow induced in the real projective bundle by a family of two-dimensional linear systems whose Sacker-Sell spectrum (see [64]) is a nondegenerate closed interval. We review the application of this technique to the non-uniformly hyperbolic family of systems obtained from the quasi-periodic Vinograd example [87]:

$$x'=\begin{pmatrix}0 & 1+a(\omega{\cdot}t)\\ 1-a(\omega{\cdot}t) & 0\end{pmatrix}x\,,\quad \omega=(\omega_1,\omega_2)\in\mathbb{T}^2\,, \tag{2.7}$$

where the flow on the base, which is a two torus, is given by a frequency vector $(1,\alpha)$ for an irrational number α, i.e., $\omega{\cdot}t=(\omega_1+t,\omega_2+\alpha t)$, and hence is minimal and almost periodic.

In polar coordinates (r,θ) ($\theta=\arg x$), (2.7) take the form $r'=r\sin(2\theta)$ and

$$\theta'=-a(\omega{\cdot}t)+\cos(2\theta)\,,\quad \omega\in\mathbb{T}^2\,, \tag{2.8}$$

and this family of scalar equations for the angular coordinate induces a skew-product flow on the projective bundle $\Sigma_p=\mathbb{T}^2\times\mathbb{P}^1$.

Moreover, the function $a(\omega)$ is constructed as the limit of a nondecreasing sequence of positive functions $a_n(\omega)$, satisfying that for each $n\ge1$ the system

$$x'=\begin{pmatrix}0 & 1+a_n(\omega{\cdot}t)\\ 1-a_n(\omega{\cdot}t) & 0\end{pmatrix}x\,,\quad \omega\in\mathbb{T}^2 \qquad (2.9)_{\omega,n}$$

- has two Lyapunov exponents $\beta_n,-\beta_n$ with $\beta_n>1/2$, and its Sacker-Sell spectrum is $\{-\beta_n,\beta_n\}$;
- the angular equation

$$\theta'=-a_n(\omega_0{\cdot}t)+\cos(2\theta)\,, \qquad (2.10)_n$$

for $\omega_0=(0,0)$ has two solutions $\theta_n^1(t)$ and $\theta_n^2(t)$ with

$$-\frac{\pi}{4} < \theta_n^1(t) < \theta_{n+1}^1(t) < \theta_{n+1}^2(t) < \theta_n^2(t) < \frac{\pi}{4}\,, \quad t \in \mathbb{R}\,,$$

and $0 < \inf_{t\in\mathbb{R}} |\theta_n^1(t) - \theta_n^2(t)| = \delta_n \to 0$ as $n \to \infty$;
- the sets $M_n^1 = \mathrm{cls}\{(\omega_0{\cdot}t, \theta_n^1(t) \mid t \in \mathbb{R}\}$ and $M_n^2 = \mathrm{cls}\{(\omega_0{\cdot}t, \theta_n^2(t) \mid t \in \mathbb{R}\}$ are disjoint almost period minimal sets for the skew-product flow induced on the projective bundle Σ_p, i.e., copies of the base: $M_n^1 = \{(\omega, h_n^1(\omega)) \mid \omega \in \mathbb{T}^2\}$ and $M_n^2 = \{(\omega, h_n^2(\omega)) \mid \omega \in \mathbb{T}^2\}$ with

$$-\frac{\pi}{4} < h_n^1(\omega) \le h_{n+1}^1(\omega) < h_{n+1}^2(\omega) \le h_n^2(\omega) < \frac{\pi}{4}\,.$$

As a consequence, in the limit, as studied in [35],

- the family (2.7) is non-uniformly hyperbolic: the positive Lyapunov exponent is $\beta \ge 1/2$ and the family of systems does not have an exponential dichotomy;
- the Sacker-Sell spectrum of (2.7) is a nondegenerate interval containing $\left[-\frac{1}{2}, \frac{1}{2}\right]$;
- the set $\Omega_1 = \{\omega \in \mathbb{T}^2 \mid h^1(\omega) = h^2(\omega)\}$ is a residual set of null measure, where $h^i(\omega) = \lim_{n\to\infty} h_n^i(\omega)$, $i = 1, 2$, and $J = \{(\omega, \varphi) \in \mathbb{T}^2 \times \mathbb{P}^1 \mid h^1(\omega) \le \varphi \le h^2(\omega)\}$ is a compact invariant set which contains a unique minimal set M satisfying our assertions: it is a non-uniquely ergodic minimal set which is almost automorphic but non-almost periodic. There are two different invariant measures concentrated on the sets $\{(\omega, x_i(\omega)) \mid \omega \in \mathbb{T}^2\} \subset M$, $i = 1, 2$, where, as in the proof of Lemma 2.4, x_1 and x_2 are defined by $x_1(\omega) = \inf\{x \in \mathbb{P}^1 \mid (\omega, x) \in M\}$, $x_2(\omega) = \sup\{x \in \mathbb{P}^1 \mid (\omega, x) \in M\}$, and they coincide a.e. with $h^1(\omega)$ and $h^2(\omega)$ respectively.

Similar assertions are obtained for the complex projective flow induced by the Riccati equations associated to the family (2.7)

$$z' = 1 - a(\omega{\cdot}t) - (1 + a(\omega{\cdot}t))\,z^2\,, \quad \omega \in \mathbb{T}^2\,.$$

We refer the reader to Novo *et al.* [52] for the study of how often this situation appears in a class of scalar convex or concave differential equations depending on a parameter, and to [31] for the relation of these minimal sets with the occurrence of strange non-chaotic attractors (SNA).

2.4 Ordered Banach spaces. Monotone skew-products semiflows

In the one-dimensional ODE case, the induced flow is obviously monotone, i.e. ordered initial states lead to ordered subsequent states. In general, this is not longer true, and in addition, we are interested in the study of functional differential equations in which the trajectories are not defined backwards and we obtain a semiflow.

Let E be a complete metric space and $\mathbb{R}^+ = \{t \in \mathbb{R} \,|\, t \ge 0\}$. A *semiflow* $(E, \Phi, \mathbb{R}^+)$ is determined by a continuous map $\Phi : \mathbb{R}^+ \times E \to E$, $(t, x) \mapsto \Phi(t, x)$ which satisfies

(i) $\Phi_0 = \mathrm{Id}$,
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \in \mathbb{R}^+$,

where $\Phi_t(x) = \Phi(t,x)$ for each $x \in E$ and $t \in \mathbb{R}^+$. The set $\{\Phi_t(x) \mid t \geq 0\}$ is the *semiorbit* of the point x. A subset E_1 of E is *positively invariant* (or just *Φ-invariant*) if $\Phi_t(E_1) \subset E_1$ for all $t \geq 0$. A semiflow $(E, \Phi, \mathbb{R}^+)$ admits a *flow extension* if there exists a continuous flow $(E, \widetilde{\Phi}, \mathbb{R})$ such that $\widetilde{\Phi}(t,x) = \Phi(t,x)$ for all $x \in E$ and $t \in \mathbb{R}^+$. A compact and positively invariant subset admits a flow extension if the semiflow restricted to it admits one.

Write $\mathbb{R}^- = \{t \in \mathbb{R} \,|\, t \leq 0\}$. A *backward orbit* of a point $x \in E$ in the semiflow $(E, \Phi, \mathbb{R}^+)$ is a continuous map $\psi : \mathbb{R}^- \to E$ such that $\psi(0) = x$ and for each $s \leq 0$ it holds that $\Phi(t, \psi(s)) = \psi(s+t)$ whenever $0 \leq t \leq -s$. If for $x \in E$ the semiorbit $\{\Phi(t,x) \mid t \geq 0\}$ is relatively compact, we can consider the *omega-limit set* of x,

$$\mathscr{O}(x) = \bigcap_{s \geq 0} \mathrm{closure}\{\Phi(t+s,x) \mid t \geq 0\}\,,$$

which is a nonempty compact connected and Φ-invariant set. Namely, it consists of the points $y \in E$ such that $y = \lim_{n\to\infty} \Phi(t_n,x)$ for some sequence $t_n \uparrow \infty$. It is well-known that every $y \in \mathscr{O}(x)$ admits a backward orbit inside this set. Actually, a compact positively invariant set M admits a flow extension if every point in M admits a unique backward orbit which remains inside the set M (see [72], part II).

A compact positively invariant set M for the semiflow $(E, \Phi, \mathbb{R}^+)$ is *minimal* if it does not contain any other nonempty compact positively invariant set than itself. If E is minimal, we say that the semiflow is minimal.

A semiflow is *of skew-product type* when it is defined on a vector bundle and has a triangular structure; more precisely, a semiflow $(\Omega \times X, \tau, \mathbb{R}^+)$ is a *skew-product* semiflow over the product space $\Omega \times X$, for a compact metric space (Ω, d) and a complete metric space (X, d), if the continuous map τ is as follows:

$$\tau\colon \mathbb{R}^+ \times \Omega \times X \longrightarrow \Omega \times X\,, \quad (t,\omega,x) \mapsto (\omega{\cdot}t, u(t,\omega,x))\,, \tag{2.11}$$

where $(\Omega, \sigma, \mathbb{R})$ is a real continuous flow $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t,\omega) \mapsto \omega{\cdot}t$, called the *base flow*. The skew-product semiflow (2.11) is *linear* if $u(t,\omega,x)$ is linear in x for each $(t,\omega) \in \mathbb{R}^+ \times \Omega$.

Now, we introduce some definitions concerning the stability of the trajectories. A forward orbit $\{\tau(t,\omega_0,x_0) \,|\, t \geq 0\}$ of the skew-product semiflow (2.11) is said to be *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$, called the *modulus of uniform stability*, such that, if $s \geq 0$ and $\mathsf{d}(u(s,\omega_0,x_0),x) \leq \delta(\varepsilon)$ for certain $x \in X$, then for each $t \geq 0$,

$$\mathsf{d}(u(t+s,\omega_0,x_0), u(t,\omega_0{\cdot}s,x)) = \mathsf{d}(u(t,\omega_0{\cdot}s,u(s,\omega_0,x_0)), u(t,\omega_0{\cdot}s,x)) \leq \varepsilon\,.$$

A forward orbit $\{\tau(t,\omega_0,x_0) \,|\, t \geq 0\}$ of the skew-product semiflow (2.11) is said to be *uniformly asymptotically stable* if it is uniformly stable and there is a $\delta_0 > 0$ with the following property: for each $\varepsilon > 0$ there is a $t_0(\varepsilon) > 0$ such that, if $s \geq 0$ and

$\mathsf{d}(u(s,\omega_0,x_0),x) \le \delta_0$, then

$$\mathsf{d}(u(t+s,\omega_0,x_0),u(t,\omega_0{\cdot}s,x)) \le \varepsilon \quad \text{for each } t \ge t_0(\varepsilon)\,.$$

Next we introduce the basic definitions and preliminary results of the theory of *monotone dynamical systems*, that is, dynamical systems on an ordered metric space X which have the property that ordered initial states lead to ordered subsequent states. We refer the reader to Smith [75], Amann [1] and Krasnoselskii *et al.* [37] for more details.

We say that X is a *strongly ordered* Banach space if there is a closed convex cone, that is, a nonempty closed subset $X_+ \subset X$ satisfying

$$\text{(i)}\quad X_+ + X_+ \subset X_+, \qquad \text{(ii)}\quad \mathbb{R}^+ X_+ \subset X_+, \qquad \text{(iii)}\quad X_+ \cap (-X_+) = \{0\}$$

with nonempty interior $\operatorname{Int} X_+ \neq \emptyset$. The *strong ordering* on X is defined as follows:

$$\begin{aligned} x \le y &\iff y - x \in X_+\,; \\ x < y &\iff y - x \in X_+ \text{ and } x_1 \neq x_2\,; \\ x \ll y &\iff y - x \in \operatorname{Int} X_+\,. \end{aligned}$$

The positive cone X_+ is said to be *normal* if the norm of the Banach space X is *semimonotone*, i.e., there is a positive constant $k > 0$ such that $0 \le x \le y$ implies $\|x\| \le k\,\|y\|$. A norm of X is called *monotone* if $0 \le x \le y$ implies $\|x\| \le \|y\|$.

The skew-product semiflow $(\Omega \times X, \tau, \mathbb{R}^+)$ is *monotone* if

$$u(t,\omega,x) \le u(t,\omega,y) \quad \text{for each } \ t \ge 0,\ \omega \in \Omega \text{ and } x,y \in X \text{ with } x \le y\,,$$

it is *strongly monotone* if

$$u(t,\omega,x) \ll u(t,\omega,y) \quad \text{for each } \ t > 0,\ \omega \in \Omega \text{ and } x,y \in X \text{ with } x < y\,,$$

and it is *eventually strongly monotone* if there is a $t_0 > 0$ such that

$$u(t,\omega,x) \ll u(t,\omega,y) \quad \text{for each } \ t > t_0,\ \omega \in \Omega \text{ and } x,y \in X \text{ with } x < y\,.$$

3 Non-autonomous FDEs with Finite Delay

Throughout this section we will study the monotone skew-product semiflow induced by a finite-delay functional differential equation, as we explain in what follows. In this setting, the strongly ordered Banach space is the set $X = C([-r,0],\mathbb{R}^m)$, whose ordering relies in the usual one of $\mathbb{R}^m$,

$$\begin{aligned} v \le w \quad &\Longleftrightarrow \quad v_j \le w_j \quad \text{for } j = 1,\ldots,m\,,\\ v < w \quad &\Longleftrightarrow \quad v \le w \quad \text{and} \quad v_j < w_j \quad \text{for some } j \in \{1,\ldots,m\}\,,\\ v \ll w \quad &\Longleftrightarrow \quad v_j < w_j \quad \text{for } j = 1,\ldots,m\,, \end{aligned}$$

where v_j represents the j-th component of any point $v \in \mathbb{R}^m$.

The subset $X_+ = \{x \in X \,|\, x(s) \ge 0 \ \text{ for each } \ s \in [-r,0]\}$ is a normal positive cone in X. Since its interior is nonempty, this cone induces a strong order relation on X,

$$\begin{aligned} x \le y \quad &\Longleftrightarrow \quad x(s) \le y(s) \quad \text{for each } s \in [-r,0]\,,\\ x < y \quad &\Longleftrightarrow \quad x \le y \quad \text{and} \quad x \ne y\,,\\ x \ll y \quad &\Longleftrightarrow \quad x(s) \ll y(s) \quad \text{for each } s \in [-r,0]\,. \end{aligned}$$

The spaces $\mathbb{R}^m$ and X will be respectively endowed with the maximum norm $\|v\| = \max_{j=1,\ldots,m} |v_j|$ and with the supremum norm $\|x\|_\infty = \sup_{t\in[-r,0]} \|x(t)\|$, both of them monotone. As usual, given an interval $I \subset \mathbb{R}$, a point $t \in \mathbb{R}$ with $[t-r,t] \subset I$, and a continuous function $z : I \to \mathbb{R}^m$, z_t will denote the element of X given by $z_t(s) = z(t+s)$ for $s \in [-r,0]$.

Our starting point is the non-autonomous finite-delay FDE

$$z' = f(t,z_t)\,, \tag{3.1}$$

defined by a function $f : \mathbb{R} \times X \to \mathbb{R}^m$ satisfying the following conditions:

(C1) f is C^1-*admissible*; i.e. f is C^1 in the variable x, and the functions

$$\mathbb{R}\times X \to \mathbb{R}^m\,,\ (t,x) \mapsto f(t,x)\,, \qquad \mathbb{R}\times X \to L(X,\mathbb{R}^m)\,,\ (t,x) \mapsto f_x(t,x)$$

are admissible. As usual, $L(X,\mathbb{R}^m)$ represents the set of linear maps from X to $\mathbb{R}^m$. We recall that given a Banach space Y, a map $g : \mathbb{R}\times X \to Y$ is *admissible* if the family $\{g(t,\cdot) \,|\, t \in \mathbb{R}\}$ is equicontinuous at every $x_0 \in X$, and for each $x_0 \in X$, $\{g(t,x_0) \,|\, t \in \mathbb{R}\}$ is a relatively compact subset of Y.

(C2) f takes $\mathbb{R}\times B$ into a bounded set of $\mathbb{R}^m$ for any bounded subset B of X.

As in the ODE's case, let Ω be the *hull* of f, defined as the closure in the topology of uniform convergence on compact sets of the set of time-translated maps $\{f_t \,|\, t \in \mathbb{R}\}$, with $f_t(s,x) = f(t+s,x)$. Property (C1) and the separability of X guarantee that Ω is a compact metric space (see Hino *et al.* [24]). It is possible to define a real continuous flow on Ω, as $\sigma : \mathbb{R}\times\Omega \to \Omega$, $(t,\omega) \mapsto \omega{\cdot}t$ with $\omega{\cdot}t(s,x) = \omega(t+s,x)$. It is also known that each $\omega \in \Omega$ is also a C^1-admissible function and f has a unique extension to a continuous function $F : \Omega\times X \to \mathbb{R}^m$, $(\omega,x) \mapsto \omega(0,x)$. Thus we obtain the family of finite-delay equations

$$z' = F(\omega{\cdot}t, z_t)\,, \quad \omega \in \Omega\,. \tag{3.2}$$

Note that the element of this family corresponding to $\omega = f$ is our initial equation (3.1). A recurrence property is also assumed on f, namely

(C3) $(\Omega,\sigma,\mathbb{R})$ is a minimal flow.

This is satisfied, for instance, when f is a *uniformly almost periodic* or, more generally, a *uniformly almost automorphic function*; i.e. when it is admissible and almost periodic or almost automorphic in $t\in\mathbb{R}$ (see [72]).

Fix now an element $(\omega,x)\in\Omega\times X$. Condition (C1) ensures the existence of a unique maximal solution $z:[-r,\beta)\to\mathbb{R}^m$ of the initial value problem given by the equation (3.2) corresponding to ω and by the initial condition $z|_{[-r,0]}=x$, which in addition varies continuously with respect to the initial data (see e.g. Hale and Verduyn Lunel [23]). We represent this maximal solution by $z(t,\omega,x)$. In this context, maximality means that the solution cannot be continued to the right of β. Therefore, the family (3.2) induces a *local* skew-product semiflow

$$\tau\colon \mathbb{R}^+\times\Omega\times X\longrightarrow\Omega\times X\,,\quad (t,\omega,x)\mapsto(\omega{\cdot}t,u(t,\omega,x))\,, \tag{3.3}$$

where $u(t,\omega,x)$ is the element of X defined by $u(t,\omega,x)(s)=z(t+s,\omega,x)$ for each $s\in[-r,0]$.

On its turn, condition (C2) guarantees that if $z(t,\omega,x)$ is a bounded solution, then it is defined in $[-r,\infty)$; hence $u(t,\omega,x)$ exists for all $t\geq 0$ and the forward orbit $\{u(t,\omega,x)\,|\,t\geq 0\}$ is relatively compact in X.

Hypotheses (C1), (C2) and (C3), which allow us to define a skew-product semiflow with the mentioned properties, do not suffice to our purposes. In order to get this semiflow to be monotone, we also assume on f a *quasimonotone condition* of Kamke type (see [75]),

(C4) If $x\leq y$ and $x_j(0)=y_j(0)$ holds for some $j\in\{1,\dots,m\}$, then $f_j(t,x)\leq f_j(t,y)$ for each $t\in\mathbb{R}$.

It is easily seen that this condition is simultaneously satisfied for each function in Ω. Property (C4) has important consequences for the semiflow $(\Omega\times X,\tau,\mathbb{R}^+)$ given by (3.3): as checked e.g. in [75],

$$\begin{aligned} u(t,\omega,x)&\leq u(t,\omega,y)\quad\text{for } t\geq 0,\ \omega\in\Omega \text{ and } x,y\in X \text{ with } x\leq y\,,\\ u(t,\omega,x)&\ll u(t,\omega,y)\quad\text{for } t\geq 0,\ \omega\in\Omega \text{ and } x,y\in X \text{ with } x\ll y\,. \end{aligned} \tag{3.4}$$

In particular, $(\Omega\times X,\tau,\mathbb{R}^+)$ is a monotone semiflow.

3.1 Cooperative and irreducible systems of finite delay equations

In this section we consider the case in which the equation (3.1) takes the form

$$z'(t)=f(t,z(t),z(t-1))\,, \tag{3.5}$$

where $f:\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$, $(t,v,w)\mapsto f(t,v,w)$ is a C^1-admissible and uniformly almost periodic or uniformly almost automorphic function, and the family on the hull is

$$z'(t) = F(\omega{\cdot}t, z(t), z(t-1))\,, \quad \omega \in \Omega\,. \tag{3.6}$$

Notice that this case is included in the general previous formulation because (3.5) can be expressed as $z'(t) = g(t, z_t)$ with $g\colon \mathbb{R}\times X \to \mathbb{R}^m$, $(t,x) \mapsto f(t, x(0), x(-1)$ and the phase space is $X = C([-1,0],\mathbb{R}^m)$.

Now we provide sufficient conditions for the induced semiflow

$$\tau\colon \mathbb{R}^+ \times \Omega \times X \longrightarrow \Omega \times X\,, \quad (t,\omega,x) \mapsto (\omega{\cdot}t, u(t,\omega,x))\,, \tag{3.7}$$

to be eventually strongly monotone, where as before, $u(t,\omega,x)(s) = z(t+s,\omega,x)$ for $s \in [-1,0]$.

Definition 3.1. We say that (3.5) is a *cooperative system* with respect to $z(t)$ if

$$\frac{\partial f_i}{\partial v_j}(t,v,w) \geq 0 \quad \text{ for each } \; v,\, w \in \mathbb{R}^m,\; t \in \mathbb{R} \quad \text{and } \; i \neq j\,.$$

We say that the system (3.5) is *strongly irreducible* with respect to $z(t)$ if there is a $\delta_0 > 0$ such that if two nonempty subsets I, J form a partition of $N = \{1,2,\ldots,m\}$, then for any v, $w \in \mathbb{R}^m$, $t \in \mathbb{R}$, there is an $i \in I$ and $j \in J = N - I$ with

$$\left| \frac{\partial f_i}{\partial v_j}(t,v,w) \right| \geq \delta_0\,.$$

The delay system (3.5) is called *strongly monotone* with respect to $z(t-1)$ if there is a $\delta > 0$ such that

$$\frac{\partial f_i}{\partial w_j}(t,v,w) \geq \delta \quad \text{ for each } \; v,\, w \in \mathbb{R}^m,\; t \in \mathbb{R} \quad \text{and } \; i,\, j = 1,2,\ldots m\,.$$

It is easy to see that if (3.5) is a cooperative system, a strongly irreducible system or a strongly monotone system then so are (3.6) with the same constants $\delta_0 > 0$ and $\delta > 0$. The following result is proved in [72] and [51].

Proposition 3.2. *Let* (3.5) *be a cooperative and strongly irreducible system with respect to* $z(t)$ *and strongly monotone with respect to* $z(t-1)$*, then* (3.7) *is an eventually strongly monotone skew-product semiflow, i.e. for each* $\omega \in \Omega$

$$u(t,\omega,x) \ll u(t,\omega,y) \quad \textit{whenever} \quad x < y \quad \textit{and} \;\; t > 2,$$

and if we denote by $<_*$ *any of the relations* $<$, $\leq$ *or* $\ll$ *then*

$$u(t,\omega,x) <_* u(t,\omega,y) \quad \textit{whenever} \quad x <_* y \quad \textit{and} \;\; t > 0\,.$$

Analogously, in the ODE case $z' = f(t,z(t))$, a cooperative and strongly irreducible system induces a strongly monotone skew-product semiflow. In the rest of the section we will not consider the strongly monotone case, and we will impose additional hypotheses under which the omega-limit sets are 1-copies of the base flow.

When these stability assumptions do not hold, N-copies of the base could appear, as Takáç showed in [78]. To clarify this comment in the simplest case, $N = 2$, we recall the four-dimensional π-periodic system of ODEs given by

$$\begin{aligned} z_1' &= z_4 + 4\,z_1(\cos^2 t - z_1^2) \\ z_2' &= z_1 + 4\,z_2(\sin^2 t - z_2^2) \\ z_3' &= z_2 + 4\,z_3(\cos^2 t - z_3^2) \\ z_4' &= z_3 + 4\,z_4(\sin^2 t - z_4^2)\,. \end{aligned} \tag{3.8}$$

It is cooperative and strongly irreducible and, hence, the induced flow is strongly monotone. It has a global attractor, and $(\cos t, \sin t, -\sin t, \cos t)$ is a linearly stable 2π-periodic solution. Its omega-limit provides a 2-copy of the base flow, defined over the circumference $\mathbb{R}/\pi\mathbb{Z}$.

3.2 Semicontinuous equilibria and almost automorphic extensions

In this subsection, we consider the monotone skew-product semiflow (3.3) induced by equation (3.1) satisfying Hypotheses (C1)-(C4). However, the results apply to a general monotone skew-product semiflow over $\Omega \times X$ for a strongly ordered Banach space X and a minimal real continuous flow $(\Omega, \sigma, \mathbb{R})$ over a compact metric space Ω.

Definition 3.3. A measurable map $a : \Omega \to X$ such that $u(t, \omega, a(\omega))$ is defined for any $t \geq 0$ is

(a) an *equilibrium* if $a(\omega{\cdot}t) = u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geq 0$,
(b) a *super-equilibrium* if $a(\omega{\cdot}t) \geq u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geq 0$, and
(c) a *sub-equilibrium* if $a(\omega{\cdot}t) \leq u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geq 0$.

We will call *semi-equilibrium* to either a super or a sub-equilibrium.

The following dynamical interpretation of the concept of a super and a sub-equilibrium appeared in Novo *et al.* [49] in a topological framework.

Definition 3.4. A super-equilibrium (resp. sub-equilibrium) $a : \Omega \to X$ is *semicontinuous* if the following properties hold:

(1) $\Gamma_a = \text{closure}_X\{a(\omega) \mid \omega \in \Omega\}$ is a compact subset of X, and
(2) $C_a = \{(\omega,x) \mid x \leq a(\omega)\}$ (resp. $C_a = \{(\omega,x) \mid x \geq a(\omega)\}$) is a closed subset of $\Omega \times X$.

An equilibrium is *semicontinuous* in any of these cases.

A semicontinuous semi-equilibrium does always have a residual subset of continuity points, as it is derived from the next result, proved in [49].

Proposition 3.5. *Let $a\colon \Omega \to X$ be a map satisfying* (1) *and* (2) *in Definition* 3.4. *Then, it is continuous over a residual subset $\Omega_0 \subset \Omega$.*

A semicontinuous semi-equilibrium provides a minimal set which is an almost automorphic extension of the base if a relatively compact trajectory exists. Before proving this result, we introduce a topological tool that we call the *section map*, which will prove to be useful in the sequel. Given a compact and positively invariant set $K \subset \Omega \times X$, let us introduce the projection set of K into the fiber space

$$K_X = \{x \in X \mid \text{ there exists } \omega \in \Omega \text{ such that } (\omega,x) \in K\} \subseteq X\,.$$

From the compactness of K it is immediate to show that also K_X is a compact subset of X. Let $\mathscr{P}_c(K_X)$ denote the set of closed subsets of K_X, endowed with the Hausdorff metric ρ, that is, for any two sets A, $B \in \mathscr{P}_c(K_X)$,

$$\rho(A,B) = \sup\{\alpha(A,B),\ \alpha(B,A)\}\,,$$

where $\alpha(A,B) = \sup\{r(a,B) \mid a \in A\}$ and $r(a,B) = \inf\{\mathrm{d}(a,b) \mid b \in B\}$. Then, define the so-called *section map*

$$\Omega \longrightarrow \mathscr{P}_c(K_X)\,, \quad \omega \mapsto K_\omega = \{x \in X \mid (\omega,x) \in K\}\,. \tag{3.9}$$

Due to the minimality of Ω and the compactness of K, the set K_ω is nonempty for every $\omega \in \Omega$; besides, the map is trivially well-defined. The following result was proved in [54].

Lemma 3.6. *There exists a residual set $\Omega_0 \subseteq \Omega$ of continuity points for the section map* (3.9) *associated to a compact and positively invariant set $K \subset \Omega \times X$.*

A semicontinuous semi-equilibrium provides a minimal set which is an almost automorphic extension of the base if a relatively compact trajectory exists.

Proposition 3.7. *Let $a : \Omega \to X$ be a semicontinuous semi-equilibrium and assume that there is an $\omega_0 \in \Omega$ such that* $\mathrm{closure}_X\{u(t,\omega_0,a(\omega_0)) \mid t \ge 0\}$ *is a compact subset of X. Then:*

> *The omega-limit set $\mathscr{O}(\omega_0,a(\omega_0))$ contains a unique minimal set which is an almost automorphic extension of the base flow.*

Proof. We work in the case that a is a super-equilibrium, the proof being completely analogous in the case of a sub-equilibrium. Denote $K = \mathscr{O}(\omega_0,a(\omega_0))$ and let $(\omega,x) \in K$, i.e. for some $s_n \uparrow \infty$, $\omega_0{\cdot}s_n \to \omega$ and $u(s_n,\omega_0,a(\omega_0)) \to x$. Since $C_a = \{(\widetilde{\omega},\widetilde{x}) \mid \widetilde{x} \le a(\widetilde{\omega})\}$ is closed and $u(s_n,\omega_0,a(\omega_0)) \le a(\omega_0{\cdot}s_n)$, we deduce that $x \le a(\omega)$.

The proof of the result is done in Proposition 3.4 (part II) of [72] for a strongly monotone skew-product semiflow in a Banach space. A slight modification valid for our case is included here for completeness. From Lemma 3.6 there exists a residual set $\Omega_0 \subseteq \Omega$ of continuity points for the section map (3.9) associated to K. Let $\omega \in \Omega_0$ and take (ω,x), $(\omega,y) \in K$. Thus, $\omega_0{\cdot}s_n \to \omega$, $u(s_n,\omega_0,a(\omega_0)) \to x$ for some $s_n \uparrow$

∞. Besides, since $\lim_{n\to\infty} K_{\omega_0\cdot s_n} = K_\omega$ and $y \in K_\omega$ there are points $(\omega_0\cdot s_n, x_n) \in K$ such that $\lim_{n\to\infty} x_n = y$. In addition, since each point of K admits a backward orbit, $x_n = u(s_n, \omega_0, y_n)$ for some $(\omega_0, y_n) \in K$. Therefore, $y_n \le a(\omega_0)$ and the monotone character of the semiflow yields to

$$x_n = u(s_n, \omega_0, y_n) \le u(s_n, \omega_0, a(\omega_0))\,,$$

which as $n \to \infty$ provides $y \le x$. Analogously, we show that $x \le y$, that is, $y = x$ and $\mathrm{card}(K_\omega) = 1$ for each $\omega \in \Omega_0$. Notice that the same argument implies that there can only be one minimal set inside $\mathscr{O}(\omega_0, a(\omega_0))$. Notice that only (2) of Definition 3.4 has been used. □

Arnold and Chueshov [3, 4] show that, in the measurable case, the existence of a semi-equilibrium a with some additional compactness properties ensures the existence of an equilibrium for the semiflow. In our topological framework, also under the supplementary and somehow natural compactness conditions assumed on Proposition 3.7, the semicontinuous semi-equilibrium provides a semicontinuous equilibrium. We state the equivalent statements to this assumption.

Proposition 3.8. *Let $a : \Omega \to X$ be a semicontinuous semi-equilibrium. The following statements are equivalent:*

(i) $\Gamma = \mathrm{cls}_X\{u(t, \omega, a(\omega)) \mid t \ge 0,\ \omega \in \Omega\}$ *is a compact subset.*
(ii) *For each $\omega \in \Omega$, $\mathrm{cls}_X\{u(t, \omega, a(\omega)) \mid t \ge 0\}$ is a compact subset.*
(iii) *There is $\omega_0 \in \Omega$ such that* $\mathrm{cls}_X\{u(t, \omega_0, a(\omega_0)) \mid t \ge 0\}$ *is a compact subset.*

Theorem 3.9. *Let us assume the existence of a semicontinuous semi-equilibrium $a : \Omega \to X$ satisfying one of the equivalent statements of Proposition* 3.8. *Then,*

(i) *there exists a semicontinuous equilibrium $c : \Omega \to X$ with $c(\omega) \in \Gamma$ for any $\omega \in \Omega$.*
(ii) *Let ω_1 be a continuity point for c. Then, the restriction of the semiflow τ to*

$$K = \mathrm{cls}_{\Omega\times X}\{(\omega_1\cdot t, c(\omega_1\cdot t)) \mid t \ge 0\} \subset C_a$$

is an almost automorphic extension of the base flow $(\Omega, \sigma, \mathbb{R})$.
(iii) *K is the only minimal set contained in the omega-limit set $\mathscr{O}(\widehat{\omega}, a(\widehat{\omega}))$ for each point $\widehat{\omega} \in \Omega$.*

The result stated in Theorem 3.9 is optimum in the following sense: even the simultaneous existence of continuous super and sub-equilibria for an equation does not guarantee the occurrence of a minimal set given by a copy of the base, but just of an almost automorphic extension. As a first example, we mention the one constructed by Ortega and Tarallo in [60]. By taking the example $z' + a(t)\,z = b(t)$ due to Johnson [33] and explained in subsection 2.3.1 as starting point, they consider a first order scalar ordinary differential equation

$$z' + a(t)\,z = b(t) + D(z)\,,$$

where D is smooth, vanishes in a previously fixed interval $[m_1, m_2]$, and satisfies $zD(z) < 0$ outside the interval and $\liminf_{|z|\to\infty} D(z)/z < -\|a\|_\infty$.

It is easy to check that the last property guarantees the existence of a large enough positive constant k such that k and $-k$ are super and sub-equilibria for the equation (and, in addition, they are ordered); whereas, as shown in [60], the remaining conditions preclude the occurrence of almost periodic solutions and hence the existence of an almost periodic minimal set.

The same phenomenon occurs for the equation analyzed by Fink and Fredrickson in [19]: taking Opial's example [59] as starting point, they construct an equation which is almost periodic in time and for which all the solutions are ultimately uniformly bounded but no one is almost periodic. Again, this equation admits super and sub-equilibria, which allows us to conclude the existence of an almost automorphic and not almost periodic extension of the base flow.

In despite of this optimum character of the result, the method we have just described has a strong limitation: it does not detect recurrent solutions which do not determine almost automorphic extensions of the base flow. To clarify this comment, we recall the system of equations constructed by Takáç [78] explained in subsection 3.1. It is easy to check that (k,k,k,k) and $(-k,-k,-k,-k)$ define super and sub-equilibria for the skew-product flow induced by the system (3.8) if k is large enough. Hence we can assert the existence of solutions determining almost automorphic extensions of the base flow (in fact they are two distinct 1-copies of the base, determined by periodic solutions: they are the upper and lower boundaries of the global attractor). But this tool gives no evidence of the presence of the 2-copy of the base.

We finish this subsection with the definition and some properties of the so called *strong semi-equilibria* needed through the rest of the section.

Definition 3.10. A continuous sub-equilibrium (resp. super-equilibrium) is *strong* if there exists an $s_* > 0$ such that $a(\omega{\cdot}s_*) \ll u(s_*, \omega, a(\omega))$ (resp. $\gg$) for every $\omega \in \Omega$.

According to Proposition 4.2(i) in Novo et al. [49], it suffices for a continuous sub-equilibrium (resp. super-equilibrium) to be strong that there exist both an $s_* > 0$ and an $\omega_0 \in \Omega$ such that $a(\omega_0{\cdot}s_*) \ll u(s_*, \omega_0, a(\omega_0))$ (resp. $\gg$).

On the other hand, a continuous function $\widetilde{a}\colon \Omega \to \mathbb{R}^m$ is said to be $\mathscr{C}^1$ *along the* σ*-orbits* if for every $\omega \in \Omega$ the function

$$\mathbb{R} \to \mathbb{R}^m,\; s \mapsto \widetilde{a}'(\omega{\cdot}s) = (d/dt)\,\widetilde{a}(\omega{\cdot}(s+t))|_{t=0}$$

exists and is continuous. Then, $\widetilde{a}$ is said to be a *lower solution* for the family of systems (3.2) if it is $\mathscr{C}^1$ along the σ-orbits and the function $a\colon \Omega \to X$ given by $a(\omega)(s) = \widetilde{a}(\omega{\cdot}s)$ for $s \in [-r, 0]$ satisfies that $u(t, \omega, a(\omega))$ is defined for any $\omega \in \Omega$ and any $t \geq 0$ and that $\widetilde{a}'(\omega) \leq F(\omega, a(\omega))$ for every $\omega \in \Omega$. In this situation the map $a : \Omega \to X$ turns out to be a continuous sub-equilibrium: see Novo et al. [49] and Núñez et al. [58] for further details. For the sake of completeness, we include a detailed proof of the following result (see Lemma 2.11(i) in [58]).

Lemma 3.11. *Let $\widetilde{a}$ be a lower solution for the family of systems* (3.2)*. If at a certain $\omega_0 \in \Omega$ it holds $\widetilde{a}'(\omega_0) \ll F(\omega_0, a(\omega_0))$, then a is a strong sub-equilibrium.*

Proof. As mentioned before, if suffices to show the existence of an $s_* > 0$ such that $a(\omega_0{\cdot}s_*) \ll u(s_*, \omega_0, a(\omega_0))$. By the continuity of the maps involved and the fact that $\widetilde{a}'(\omega_0) \ll F(\omega_0, a(\omega_0))$, we find $\varepsilon > 0$ such that $\widetilde{a}'(\omega_0{\cdot}t) \ll F(\omega_0{\cdot}t, a(\omega_0{\cdot}t))$ for any $t \in [0, \varepsilon)$. This can be rewritten as $\widetilde{a}'_{\omega_0}(t) \ll F(\omega_0{\cdot}t, (\widetilde{a}_{\omega_0})_t)$ for any $t \in [0, \varepsilon)$, by denoting $\widetilde{a}_{\omega_0}(s) = \widetilde{a}(\omega_0{\cdot}s)$ for $s \in \mathbb{R}$. Then, a standard comparison argument for delay differential equations leads to $\widetilde{a}_{\omega_0}(t) \ll z(t, \omega_0, a(\omega_0))$ for any $t \in (0, \varepsilon)$. Now define the function $y(t) = z(t, \omega_0, a(\omega_0)) - \widetilde{a}_{\omega_0}(t)$, fix $t_0 \in (0, \varepsilon)$, and note that for every component $i = 1, \ldots, m$ it holds that $y_i(t_0) > 0$. As $\widetilde{a}$ is a lower solution, it follows that $y(t)$ satisfies the linear delay inequality $y'(t) \geq L(t)\, y_t$ for

$$L(t) = \int_0^1 D_x F(\omega_0{\cdot}t, \lambda\, u(t, \omega_0, a(\omega_0)) + (1-\lambda)\, a(\omega_0{\cdot}t))\, d\lambda\,.$$

Using again a comparison argument, we know that $y(t)$ remains above the solution of the linear delay system $z'(t) = L(t)\, z_t$ with initial data $z_{t_0} = y_{t_0}$, which we denote as $z(t, t_0, y_{t_0})$. At this point, as $z_i(t_0, t_0, y_{t_0}) = y_i(t_0) > 0$ for every component, we can apply Lemma 5.1.3 in Smith [75] to deduce that $y_i(t) \geq z_i(t, t_0, y_{t_0}) > 0$ for any $t \geq t_0$ and any $i = 1, \ldots, m$, that is, $y(t) \gg 0$ for any $t \geq t_0$. Therefore , we can take $s_* = t_0 + r > 0$ to complete the proof. □

3.3 Monotone and concave or sublinear cases

In this subsection, we consider the monotone skew-product semiflow (3.3) induced by equation (3.1) satisfying Hypotheses (C1)-(C4) and we will assume additional conditions providing the concave or sublinear character of the induced semiflow, important in applications, which will allow us to prove the existence of an attractor minimal set which is a copy of the base flow.

3.3.1 Monotone and concave semiflows

Theorem 3.9 shows how the existence of a semicontinuous super-equilibrium a allows us to construct a minimal set $K \subset \Omega \times X$ which is an almost automorphic extension of the base Ω. Now we will prove that, when this super-equilibrium is strong and under an additional concavity assumption for the semiflow, K turns out to be a copy of the base Ω attracting all the solutions $u(t, \omega, x)$ with initial condition above a; i.e. with $a(\omega) \leq x$. In particular, the flow on this set is almost periodic when the base flow is, which is the case when the initial function f is uniformly almost periodic.

(C5) The function $f(t,x)$ is *concave* in x; i.e. for each $t \in \mathbb{R}$ and $\lambda \in [0,1]$
$f(t, \lambda\, x + (1-\lambda)\, y) \geq \lambda\, f(t,x) + (1-\lambda)\, f(t,y)$ whenever $x \leq y$.

This property is satisfied simultaneously for all the functions of the hull Ω. Conditions (C4), (C5) and standard arguments of comparison of solutions (see [53]) ensure the *concave* character of the monotone semiflow $(\Omega\times X,\tau,\mathbb{R}^+)$. In other words, u is an *order concave map* in x:

$$u(t,\omega,\lambda\,x+(1-\lambda)y)\geq\lambda\,u(t,\omega,x)+(1-\lambda)\,u(t,\omega,y) \tag{3.10}$$

for $x\leq y$ and for each $t\geq 0$, $\lambda\in[0,1]$ and $\omega\in\Omega$.

The definition of *convex* semiflow is obtained by substituting the inequality $\geq$ by $\leq$ in (3.10). Finally, it is easy to check that if we take the set $X_-=\{-x\,|\,x\in X_+\}$ as positive cone, inverting in this way the order relation, a monotone and convex skew-product semiflow becomes monotone and concave. So that the results obtained for concave semiflows are easily adapted to the case of convexity.

Theorem 3.12. *Assume conditions* (C1)-(C5) *for the semiflow* (3.3)*. Let $a:\Omega\to X$ be a strong semicontinuous sub-equilibrium satisfying one of the equivalent statements of Proposition* 3.8*, and let c and K be the equilibrium and the almost automorphic extension of the base flow provided by Theorem* 3.9*. Then,*

(i) *K defines a copy of the base flow; more precisely, $c:\Omega\to X$ is continuous and $K=\{(\omega,c(\omega))\,|\,\omega\in\Omega\}$.*
(ii) *All the semiorbits corresponding to initial data (ω,x) with $a(\omega)\leq x$ are globally defined and approach asymptotically K, i.e.*

$$\lim_{t\to\infty}\|u(t,\omega,x)-c(\omega{\cdot}t)\|=0\,. \tag{3.11}$$

Next, we can weaken the assumption of the sub-equilibrium to be strong by strength the assumption on concavity:

(C6) there is an $\omega_1\in\Omega$ and a $t_1>0$ such that for each $\lambda\in(0,1)$

$$u(t_1,\omega_1,\lambda\,x+(1-\lambda)y)\gg\lambda\,u(t_1,\omega_1,x)+(1-\lambda)\,u(t_1,\omega_1,y)$$

whenever $x\gg y$.

One way, but not the only one, of obtaining this is to assume that $f(t,x)$ is *strongly concave* in x for t in an interval I of length $r'>r$, i.e., for each $\lambda\in(0,1)$ and $t\in I$

$$f(t,\lambda\,x+(1-\lambda)\,y)\gg\lambda\,f(t,x)+(1-\lambda)\,f(t,y)\qquad\text{whenever } x\ll y.$$

Theorem 3.13. *Assume conditions* (C1)-(C6) *for the semiflow* (3.3)*. Let $a:\Omega\to X$ be a semicontinuous sub-equilibrium and let $(\omega_0,x_0)\in\Omega\times X$ with $a(\omega_0)\ll x_0$ be such that its semitrajectory $\{(\omega_0{\cdot}t,u(t,\omega_0,x_0))\mid t\geq 0\}$ is bounded and there is a $y_0\gg 0$ with $u(t,\omega_0,x_0)-a(\omega_0{\cdot}t)\geq y_0$ for each $t\geq 0$. Then,*

(i) *$K=\mathcal{O}(\omega_0,x_0)$ defines a copy of the base flow, i,e. $K=\{(\omega,e(\omega))\,|\,\omega\in\Omega\}$.*
(ii) *All the semiorbits corresponding to initial data (ω,x) with $a(\omega)\ll x$ are globally defined and approach asymptotically K, i.e.*

$$\lim_{t\to\infty}\|u(t,\omega,x)-e(\omega{\cdot}t)\|=0\,.$$

3.3.2 Monotone and sublinear semiflows

When dealing with sublinear systems, the natural space for solutions is the positive cone. For that reason in this subsection we restrict the study to systems given by functions $f\colon \mathbb{R}\times X_+ \to \mathbb{R}^m$. The purpose is to analyze the conditions ensuring the existence of a unique and asymptotically stable copy of the base when the concavity hypotheses are replaced by some sublinearity properties. Apart from (C1)-(C4) we will assume

(C7) the function $f\colon \mathbb{R}\times X^+ \to \mathbb{R}^m$ is *sublinear* in x; that is, for each $t\in\mathbb{R}$

$$f(t,\lambda\, x) \ge \lambda\, f(t,x) \quad \text{whenever } x\in X^+ \text{ and } \lambda\in[0,1].$$

Again, this property is satisfied simultaneously for all the functions of the hull Ω, and together with condition (C4) provides the *sublinear* character of the monotone semiflow $(\Omega\times X,\tau,\mathbb{R}^+)$, that is, for each $t\ge 0$, $\lambda\in[0,1]$ and $\omega\in\Omega$

$$u(t,\omega,\lambda\, x) \ge \lambda\, u(t,\omega,x) \quad \text{whenever } \; x\in X_+\,. \tag{3.12}$$

In addition, we will assume any condition on the family implying the following strong sublinearity property for the semiflow

(C8) there is an $\omega_1\in\Omega$ and a $t_1>0$ such that for each $\lambda\in(0,1)$

$$u(t_1,\omega_1,\lambda\, x) \gg \lambda\, u(t_1,\omega_1,x) \quad \text{whenever } x\gg 0\,. \tag{3.13}$$

One way is to assume that $f\colon \mathbb{R}\times X^+ \to \mathbb{R}^m$ is *strongly sublinear* in x when t belongs to an interval of length r, that is, for each $\lambda\in(0,1)$ and $t\in[t_0,t_0+r]$

$$f(t,\lambda\, x) \gg \lambda\, f(t,x) \quad \text{whenever } x\gg 0.$$

We include a proof of the main result of this subsection (see Núñez *et al.* [57]).

Theorem 3.14. *Let us assume conditions* (C1)-(C4) and (C7)-(C8) *for the semiflow* (3.3)*. Let* $(\omega_0,x_0)\in\Omega\times X$ *with* $0\ll x_0$ *be such that its semitrajectory* $\{(\omega_0{\cdot}t,u(t,\omega_0,x_0))\mid t\ge 0\}$ *is bounded and there is a* $y_0\gg 0$ *with* $u(t,\omega_0,x_0)\ge y_0$ *for each* $t\ge 0$*. Then,*

(i) $\mathcal{O}(\omega_0,x_0)$ *is a copy of the base flow, i,e.* $\mathcal{O}(\omega_0,x_0)=\{(\omega,e(\omega))\mid \omega\in\Omega\}$.
(ii) *All the semiorbits corresponding to initial data* $(\omega,x)\in\Omega\times X_+$ *with* $0\ll x$ *are globally defined and approach asymptotically* $\mathcal{O}(\omega_0,x_0)$*, i.e.*

$$\lim_{t\to\infty}\|u(t,\omega,x)-e(\omega{\cdot}t)\|=0\,.$$

Proof. First notice that the omega limit set $\mathcal{O}(\omega_0,x_0)$ is strongly positive, that is, $y\gg 0$ for each $(\omega,y)\in\mathcal{O}(\omega_0,x_0)$ and we can fix e_1 and $e_2\in X$ such that $0\ll e_1\le y\le e_2$ for each $(\omega,y)\in\mathcal{O}(\omega_0,x_0)$. Next, we check that all the semiorbits with initial data $x\ge 0$ are bounded and hence relatively compact. We take $(\omega,y)\in\mathcal{O}(\omega_0,x_0)$ and $\mu>1$ such that $x\le\mu\, y$. Monotonicity and sublinearity ensure that $0\le u(t,\omega,x)\le\mu\, u(t,\omega,y)$ for $t\ge 0$, and the boundedness follows from the semimonotonicity of the norm. If, in addition, $x \gg 0$ we check that the semiorbit is uniformly stable. We choose $\lambda \in (0,1)$ such that $\lambda\, y \le x \le (1/\lambda)\, y$. Again (3.4), (3.12) and $e_1 \le u(t,\omega,y) \le e_2$ lead us to

$$0 \ll \lambda\, e_1 \le \lambda\, u(t,\omega,y) \le u(t,\omega,x) \le (1/\lambda)\, u(t,\omega,y) \le (1/\lambda)\, e_2$$

for each $t \ge 0$. Let us now fix $\alpha \in (0,1)$. It is easy to deduce the existence of $\delta = \delta(\alpha) > 0$ such that, if $z \in X_+$ satisfies $\|u(s,\omega,x) - z\| < \delta$ for certain $s \ge 0$, then $\alpha\, u(s,\omega,x) \le z \le (1/\alpha)\, u(s,\omega,x)$, and hence, again the monotonicity and sublinearity properties, $\alpha\, u(s+t,\omega,x) \le u(t,\omega{\cdot}s,z) \le (1/\alpha)\, u(s+t,\omega,x)$ for each $t \ge 0$. Therefore,

$$\begin{aligned}(1-1/\alpha)\,\lambda e_1 \le (1-1/\alpha)\, u(s+t,\omega,x) &\le u(s+t,\omega,x) - u(t,\omega{\cdot}s,z)\\ &\le (1-\alpha)\, u(s+t,\omega,x) \le (1-\alpha)\,(1/\lambda)\, e_2\,,\end{aligned}$$

and the uniform stability follows easily.

Let M be a strongly positive minimal set. It is clear that there is at least one because each minimal set $M \subset \mathscr{O}(\omega_0,x_0)$ satisfies $y \ge e_1 \gg 0$ for each $(\omega,y) \in M$. Then, let $\omega_1 \in \Omega$ be the point of condition (C8) and $(\omega_1,x) \in \Omega \times X_+$ with $0 \ll x \ll y$ for any $(\omega_1,y) \in M$. We fix $(\omega_1,y) \in M$ and define

$$\lambda(t) = \sup\{\lambda \in [0,1] \mid u(t,\omega_1,x) \ge \lambda\, u(t,\omega_1,y)\}\,.$$

The cocycle property $u(t+s,\omega_1,x) = u(s,\omega_1{\cdot}t,u(t,\omega_1,x))$, the monotonicity and the sublinearity imply that $u(t+s,\omega_1,x) \ge \lambda(t)\, u(t+s,\omega_1,y)$ if $s > 0$, which means that $\lambda(t+s) \ge \lambda(t)$, and $\lambda\colon [0,\infty) \to [0,1]$ is an increasing function. Let $\lambda^* = \lim_{t\to\infty} \lambda(t)$. We claim that $\lambda^* = 1$. Assume on the contrary that $\lambda^* < 1$. Then, for an adequate sequence $\{t_n\} \uparrow \infty$

$$\begin{aligned}&\lim_{n\to\infty} (\omega{\cdot}t_n, u(t_n,\omega_1,x)) = (\omega^*,x^*) \in \Omega \times X_+\,,\\ &\lim_{n\to\infty} (\omega{\cdot}t_n, u(t_n,\omega_1,y)) = (\omega^*,y^*) \in M\,,\end{aligned}$$

and together with $u(t_n,\omega_1,x) \ge \lambda(t_n)\, u(t_n,\omega_1,y)$ we deduce that $x^* \ge \lambda^*\, y^*$. Moreover, from $\lambda^* < 1$ the strong sublinearity (3.13) leads us to

$$u(t_1,\omega^*,x^*) \ge u(t_1,\omega^*,\lambda^*\, y^*) \gg \lambda^* u(t_1,\omega^*,y^*)\,.$$

The above limits imply that there is n_0 with $u(t_n+t_1,\omega_1,x) \gg \lambda^* u(t_n+t_1,\omega_1,y)$ for each $n \ge n_0$, and hence $\lambda(t_n+t_1) > \lambda^*$, contradicting the definition of λ^*, and showing that $\lambda^* = 1$, as claimed.

In addition, from $u(t,\omega_1,y) \ge u(t,\omega_1,x) \ge \lambda(t)\, u(t,\omega_1,y)$ and $u(t,\omega_1,y) \le e_2$ we have $0 \le u(t,\omega_1,y) - u(t,\omega_1,x) \le (1-\lambda(t))\, e_2$, and we conclude that

$$\lim_{t\to\infty} \|u(t,\omega_1,x) - u(t,\omega_1,y)\| = 0\,,$$

$\mathscr{O}(\omega_1,x) = M$. Since this is true for each $x \gg 0$ sufficiently small, M is the only strongly positive minimal set and $M \subset \mathscr{O}(\omega_0,x_0)$.

The same proof of Proposition 3.7, taking now into account that $x \le y$ for each $(\omega_1,y) \in M$, shows that M is an almost automorphic extension of the base flow, and we take $\hat\omega \in \Omega$ such that $M_{\hat\omega}$ reduces to a point $(\hat\omega,\hat y)$. We claim that M is a copy of the base. Assume, on the contrary, that there are $(\widetilde\omega,\widetilde y)$, $(\widetilde\omega,\widetilde z) \in M$ with $\|\widetilde y-\widetilde z\| > \varepsilon$, and let $\{s_n\} \downarrow -\infty$ be such that $\widetilde\omega{\cdot}s_n \to \omega_1$ as $n \to \infty$ and, hence for an adequate subsequence, let us take the whole sequence,

$$\lim_{n\to\infty}(\widetilde\omega{\cdot}s_n,u(s_n,\widetilde\omega,\widetilde y)) = \lim_{n\to\infty}(\widetilde\omega{\cdot}s_n,u(s_n,\widetilde\omega,\widetilde z)) = (\hat\omega,\hat y)\,. \tag{3.14}$$

As shown above, all the semiorbits are uniformly stable and, it is immediate from the proof that all the semiorbits in M have the same modulus of uniform stability $\delta(\varepsilon)$. From (3.14) we take n_0 such that $\|u(s_{n_0},\widetilde\omega,\widetilde y)-u(s_{n_0},\widetilde\omega,\widetilde z)\| \le \delta(\varepsilon)$, and the uniform stability provides $\|\widetilde y-\widetilde z\| \le \varepsilon$, a contradiction.

We denote $M = \{(w,e(w)) \mid \omega \in \Omega\}$ for a continuous map $e\colon \Omega \to X$ and, in order to finish the proof we check that $M = \mathscr{O}(\omega,x)$ for each $\omega \in \Omega$ and $x \gg 0$, in particular, $M = \mathscr{O}(\omega_0,x_0)$. We take again $\delta(\varepsilon)$ the modulus of uniform stability of the trajectories in M. Since $(\omega,e(\omega)) \in M \subset \mathscr{O}(\omega,x)$,

$$\lim_{n\to\infty}(\omega{\cdot}t_n,u(t_n,\omega,x)) = (\omega,e(\omega)) = \lim_{n\to\infty}(\omega{\cdot}t_n,e(\omega{\cdot}t_n))\,,$$

and there is an n_0 such that $\|u(t_{n_0},\omega,x)-e(\omega{\cdot}t_{n_0})\| \le \delta(\varepsilon)$. Hence, the uniform stability yields to $\|u(t+t_{n_0},\omega,x)-e(\omega{\cdot}(t_{n_0}+t))\| \le \varepsilon$ for each $t>0$, and the proof is finished. $\square$

3.4 A non-autonomous cyclic feedback system

Next we show some application of the previous results, related to the mathematical model of biochemical feedback in protein synthesis represented by the non-autonomous system of finite-delay functional differential equations

$$\begin{aligned} z_1'(t) &= g(t,z_m(t-r_m)) - \alpha_1(t)\,z_1(t)\,,\\ z_j'(t) &= z_{j-1}(t-r_{j-1}) - \alpha_j(t)\,z_j(t)\,, \quad \text{for } j=2,\dots,m\,. \end{aligned} \tag{3.15}$$

Here, $\alpha_j(t)$ are positive almost periodic functions for $j=1,\dots,m$, $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^m$ is a C^1-admissible uniformly almost periodic function, $r_j \ge 0$ for $j=1,\dots,m$ and $\max\{r_j \mid j=1,\dots,m\}>0$.

The system (3.15) expresses a model for a biochemical control circuit in which each of the z_j represents the concentration of an enzyme; hence $z_j \ge 0$ for $j=1,\dots,m$. The autonomous ordinary case was firstly introduced by Selgrade [65]. Different extensions to the periodic and the autonomous functional cases are explored in Smith [75], Krause and Ranft [38], Smith and Thieme [77], and references therein. Chueshov [11] analyzes the random case and Novo *et al.* [51] the ordinary non-autonomous case. The present case of finite-delay was considered, for the concave case, in Novo *et al.* [49].

3.4.1 The concave case

The next result provides conditions which ensure the existence of a global attractor for (3.15) in the interior of the positive cone X_+. In contrast to previous works, the approach we present here just requires the semiflow to be monotone and concave. Similar conclusions apply when we change almost periodicity for almost automorphy or, in general, recurrence.

Define $r = \max\{r_j \mid j = 1,\dots,m\} > 0$ and consider, as in the previous subsections, the strongly ordered Banach space $X = C([-r,0],\mathbb{R}^m)$. As usual, given any element $x_0 \in X$, $z(t,x_0)$ represents the solution of equation (3.15) satisfying $z(t,x_0) = x_0(t)$ for $t \in [-r,0]$.

Theorem 3.15. *Let us assume that*

- $g(t,y) > 0$ *for each* $t \in \mathbb{R}$ *and* $y > 0$, *and* $g_y(t,y) \geq 0$ *for each* $t \in \mathbb{R}$ *and* $y \geq 0$.
- $g(t,y)$ *is concave in* y.

Moreover, there are positive constants β_j, α_j *with*

$$0 < \alpha_j \leq \alpha_j(t) < \beta_j \quad \text{for each } j = 1,\dots,m \text{ and } t \in \mathbb{R}\,, \tag{3.16}$$

and a real function $g_0 \in C(\mathbb{R})$ *such that*

- $g_0(y) > 0$ *for each* $y > 0$;
- $\prod_{j=1}^m \beta_j < \limsup_{y\to 0^+} g_0(y)/y \leq +\infty$;
- $g_0(y) \leq g(t,y) \leq \alpha y + \delta$ *for each* $t \in \mathbb{R}$, $y \geq 0$ *and some positive constants* α, $\delta > 0$ *with* $0 < \alpha < \prod_{j=1}^m \alpha_j$.

Then, there is a unique almost periodic solution $z^*(t) \gg 0$ *of* (3.15) *such that*

$$\lim_{t\to\infty} \|z(t,x_0) - z^*(t)\| = 0 \quad \text{for each } x_0 \in C([-r,0],\mathbb{R}^m) \text{ with } x_0 \gg 0\,.$$

Proof. We can write (3.15) as $z' = f(t,z_t)$ where $f : \mathbb{R}\times X \to \mathbb{R}^m$ is the C^1-admissible function defined for each $t \in \mathbb{R}$ and $x \in X$ by

$$\begin{aligned} f_1(t,x) &= g(t,x_m(-r_m)) - \alpha_1(t)\,x_1(0)\,,\\ f_j(t,x) &= x_{j-1}(-r_{j-1}) - \alpha_j(t)\,x_j(0) \quad \text{for } j = 2,\dots,m\,. \end{aligned} \tag{3.17}$$

As explained in the introduction of Section 3, system (3.15) is included in the family of systems $z'(t) = F(\omega{\cdot}t, z_t)$ for $\omega \in \Omega$, where Ω is the hull of f. The almost periodicity of the coefficients ensures that $(\Omega,\sigma,\mathbb{R})$ is minimal and almost periodic. Consequently, hypotheses (C1), (C2) and (C3) are satisfied.

It is also easy to check that condition (C4) and (C5) hold. Moreover, from $\prod_{j=1}^{m} \beta_j < \limsup_{y\to 0^+} g_0(y)/y \le +\infty$ we can find a sequence $\varepsilon_n > 0$ tending to 0 such that, for each $n \in \mathbb{N}$,

$$g_0(\varepsilon_n) - \varepsilon_n \prod_{j=1}^{m} \beta_j > 0 \tag{3.18}$$

Now we consider the constant function $a_n = (a_{n,1},\ldots,a_{n,d}) \gg 0$, with $a_{n,d} = \varepsilon_n$ and $a_{n,l} = \varepsilon_n \prod_{j=l+1}^{m} \beta_j$ for $l = 1,\ldots,m-1$. From inequality $g_0(y) \le g(t,y)$ and relations (3.16) and (3.18),

$$f_1(t,a_n) = g(t,a_{n,d}) - \alpha_1(t)\,a_{n,1} > g_0(\varepsilon_n) - g_0(\varepsilon_n) - \varepsilon_n \prod_{j=1}^{m} \beta_j > 0$$

$$f_l(t,a_n) = a_{n,l-1} - \alpha_l(t)\,a_{n,l} = \varepsilon_n \prod_{j=l}^{m} \beta_j - \alpha_l(t)\,\varepsilon_n \prod_{j=l+1}^{m} \beta_j > 0\,, \quad l = 2,\ldots,m\,.$$

which implies that $f(t,a_n) \gg 0$ for each $n \in \mathbb{N}$. Hence $F(\omega,a_n) \ge 0$ for each $n \in \mathbb{N}$ and $\omega \in \Omega$, and Lemma 3.11 allows us to assure that a_n is a strong sub-equilibrium for each $n \in \mathbb{N}$.

We define $\delta_1 = \delta/(\prod_{j=1}^{m} \alpha_j - \alpha) > 0$ and take $b = (b_1,\ldots,b_m) \gg 0$, with $b_m = \delta_1$ and $b_l = \delta_1 \prod_{j=l+1}^{m} \alpha_j$ for $l = 1,\ldots,m-1$. As before, from $g(t,y) \le \alpha y + \delta$ and (3.16) we check that $f(t,b) \ll 0$ and $F(\omega,b) \le 0$, and hence b is a strong super-equilibrium. In addition, it is not hard to check that $a_n \ll b$ for each $n \in \mathbb{N}$. Consequently, since b is a super-equilibrium and the flow is monotone, we deduce that

$$a_n \le u(t,\omega,a_n) \ll u(t,\omega,b) \le b$$

for $t \ge 0$ and $n \in \mathbb{N}$. In particular, the equivalent statements of Proposition 3.8 are satisfied for each one of the sub-equilibria a_n, as deduced from Arzelà-Ascoli theorem.

Finally, Theorem 3.12 applied to each a_n yields to the existence of a copy of the base K_n which is a global attractor in the set $A_n = \{(\omega,x) \in \Omega \times X \mid x \ge a_n\}$. Consequently, $K_n = \{(\omega,e(\omega)) \mid \omega \in \Omega\}$ is the same for each $n \in \mathbb{N}$ and, since a_n tends to 0, it is a global attractor in $A = \{(\omega,x) \in \Omega \times X \mid x \gg 0\}$. The almost periodic function $z^*(t) = e(\omega_0{\cdot}t)$ with $\omega_0 = f$ satisfies the statement.

3.4.2 The sublinear case

Now we increase the range of applications by changing concavity by sublinearity but we strength the monotonicity assumptions to provide new conditions which ensure the existence of a global attractor for (3.15) in the interior of the positive cone X_+. Again, similar conclusions apply when we change almost periodicity for almost automorphy or, in general, recurrence.

Theorem 3.16. *Let us assume that*

- $g(t,y) > 0$ *for each* $t \in \mathbb{R}$ *and* $y > 0$, *and* $g_y(t,y) > 0$ *for each* $t \in \mathbb{R}$ *and* $y > 0$.
- $g(t,y)$ *is sublinear in* y *and* $y\,g_y(t,y) < g(t,y)$ *for each* $t \in \mathbb{R}$ *and* $y > 0$.

Moreover, there are positive constants β_j, α_j *with*

$$0 < \alpha_j \le \alpha_j(t) < \beta_j \quad \text{for each } \; j = 1,\dots,d \; \text{ and } t \in \mathbb{R}\,,$$

and a real function $g_0 \in C(\mathbb{R})$ *such that*

- $g_0(y) > 0$ *for each* $y > 0$*;*
- $\prod_{j=1}^{n} \beta_j < \limsup_{y\to 0^+} g_0(y)/y \le +\infty$*;*
- $g_0(y) \le g(t,y) \le \alpha\,y + \delta$ *for each* $t \in \mathbb{R}$, $y \ge 0$ *and some positive constants* α, $\delta > 0$ *with* $0 < \alpha < \prod_{j=1}^{m} \alpha_j$.

Then, there is a unique almost periodic solution $z^*(t) \gg 0$ *of* (3.15) *such that*

$$\lim_{t\to\infty} \|z(t,x_0) - z^*(t)\| = 0 \quad \text{for each } x_0 \in C([-r,0],\mathbb{R}^m) \; \text{ with } \; x_0 \gg 0\,.$$

Proof. In this case, for simplicity in the notation, we will consider all the delays to be equal to r. If the delays are different, the proof is completely similar changing the phase space to $X = \prod_{j=1}^{m} C([-r_j,0],\mathbb{R})$.

As in the proof of Theorem 3.15, it is easy to prove that conditions (C1)-(C4) and (C7) hold, the semitrajectory $\{(\omega{\cdot}t, u(t,\omega,b)) \mid t \ge 0\}$ is bounded and $u(t,\omega,b) \ge a_1 \gg 0$ for each $t \ge 0$. Therefore, if we check condition (C8), the result follows from Theorem 3.14.

We take $\omega_0 = f$, defined by (3.17), and notice that $u(t,\omega_0,x)(s) = z(t+s,x)$ for $s \in [-r,0]$. We check that there is a $t_0 > 0$ such that

$$u(t,\omega_0,x) \ll u(t,\omega_0,y) \quad \text{ if } \; 0 \ll x < y \; \text{ and } t > t_0\,. \tag{3.19}$$

For this it is enough to check that if, as usual, we denote by $u_x(t,\omega,x)\colon X \to X$ the linear differential operator with respect to the third variable, we have

$$u_x(t,\omega_0,x)\,v \gg 0 \quad \text{ if } \; v > 0\,,\; x \gg 0 \; \text{ and } \; t > t_0\,.$$

Notice that $(u_x(t,\omega_0,x)\,v)(s) = h(t+s)$ where $h(t)$ satisfies the variational problem

$$\begin{aligned} h_1'(t) &= g_y(t, z_m(t-r,x)\,h_m(t-r) - \alpha_1(t)\,h_1(t)\,,\\ h_j'(t) &= h_{j-1}(t-r) - \alpha_j(t)\,h_j(t)\,, \quad \text{ for } j = 2,\dots,m\,. \end{aligned} \tag{3.20}$$

with $h(s) = v(s)$ for $s \in [-r,0]$. Therefore the assertion holds if there is $\widetilde{t} > r$ such that $h(t) \gg 0$ for each $t \ge \widetilde{t}$. Moreover, from Lemma 1.3 of Chapter 5 in [75], if $h_i(t_1) > 0$ for some $t_1 \ge 0$, then $h_i(t) > 0$ for each $t > t_1$. Then, we contradict the assertion if we assume that there is one $j \in \{1,\dots,m\}$ such that $h_j(t) = 0$ for each $t \ge 0$. If $j \ge 2$, from (3.20) we would deduce that $h_{j-1}(t-r) = 0$ for each $t \ge 0$. In a recursive way we will obtain the same result for $j-2,\dots,1$. Next, from (3.20) we obtain

$$g_y(t, z_m(t-r,x)\,h_m(t-r) = 0 \quad \text{ for each } t \ge 0\,.$$

Since $x \gg 0$ and the semiflow is monotone $u(t,\omega_0,x) = z(t+\cdot,x) \gg 0$ for each $t \geq 0$ and hence, $z_m(t-r,x) > 0$ for $t \geq 0$, and $g_y(t,z_m(t-r,x) > 0$, which lead to $h_m(t-r)$ for each $t \geq 0$. The result for the rest of the indices from $d-1$ to $j+1$ follows analogously, and we conclude that $h(t-r) = 0$ for each $t \geq 0$, which contradicts that $v > 0$.

Next, we fix $\lambda \in (0,1)$ and we consider $y(t) = \lambda z(t,x)$. Then $y'(t) = \lambda f(t,z_t)$, with f defined by (3.17). From $y\,g_y(t,y) < g(t,y)$ we deduce that g is strongly sublinear in y, that is, $g(t,\lambda y) > \lambda\, g(t,y)$ for $y > 0$, and consequently, $f(t,\lambda z_t) > \lambda f(t,z_t)$. Therefore $y' < f(t,y_t)$ and comparison theorems for this kind of ordinary differential equations [75] lead to $y(t) < z(t,\lambda x)$ for each $t > 0$, or equivalently $\lambda\, u(t,\omega_0,x) < u(t,\omega_0,\lambda x)$ for each $t > r$. In addition, from relations (3.19) and (3.12), if we take $t^* > 2\max(t_0,r)$ we deduce that

$$\begin{aligned} u(t^*+t^*,\omega_0{\cdot}(-t^*),\lambda x) &= u(t^*,\omega_0,u(t^*,\omega_0,\lambda x)) \gg u(t^*,\omega_0,\lambda u(t^*,\omega_0,x)) \\ &\geq \lambda u(t^*,\omega_0,u(t^*,\omega_0,x)) = \lambda\, u(t^*+t^*,\omega_0{\cdot}(-t^*),x)\,, \end{aligned}$$

and (C8) holds for $t_1 = 2t^*$ and $\omega_1 = \omega_0(-t^*)$, which finishes the proof. □

3.5 Cellular neural networks

We finish this section by explaining some applications to neural networks. The models we consider are among the so-called delayed cellular neural networks, which in particular include the Hopfield-type models (see the early works by Hopfield [25, 26] and Marcus-Westervelt [43] where the delay is introduced). The autonomous case, with finite or distributed delays, has been intensively investigated (see Golpasamy-He [20], Van den Driessche-Zou [79], Van den Driessche et al. [80] and Zhao [90], among many others). The main interest has focused on determining sufficient conditions to guarantee the existence and uniqueness of a globally asymptotically stable equilibrium point for the system.

Nevertheless, the non-autonomous periodic or almost periodic case with constant or time-varying delays has just recently been considered in some papers (see for instance Liang-Cao [39], Mohamad [46], Fan-Ye [15], Jiang et al. [30], Chen-Cao [9] and Huang et al. [27]). Also recently, Novo et al. [51, 53] have considered quite general non-autonomous finite-delay monotone and concave models of Hopfield neural networks, giving conditions for the existence of attracting solutions with the same recurrence properties as the model coefficients. Besides, in Novo et al. [55] the existence of a global exponentially attracting solution of finite-delay cellular neural networks is deduced from the uniform asymptotical stability of the null solution of an associated non-autonomous linear system.

Let us consider a non-autonomous system of finite-delay FDEs

$$
\begin{aligned}
z_i'(t) = -\widetilde{a}_i(t)\,z_i(t) + \sum_{j=1}^{m} \widetilde{b}_{ij}(t)\,f_j(z_j(t)) \\
+ \sum_{j=1}^{m} \widetilde{c}_{ij}(t) \int_{-r}^{0} g_j(z_j(t+s))\,d\mu_{ij}(s) + \widetilde{I}_i(t)\,, \quad t \geq 0\,,\; i = 1,\ldots,m\,,
\end{aligned}
\tag{3.21}
$$

which describes the dynamics of a network of m neurons (or amplifiers) with delayed interconnections. The variable $z_i(t)$ represents the state of the i-th neuron in the network at time t. The coupling coefficients can be arranged in two interconnection matrices $[\widetilde{b}_{ij}(t)]$ and $[\widetilde{c}_{ij}(t)]$, whose entries, as well as the coefficients $\widetilde{a}_i$ and the external input functions $\widetilde{I}_i$ are bounded, uniformly continuous and present some recurrent behavior in time, such as almost periodicity. The functions f_j and g_j are the so-called real signal propagation functions or activation functions. More precisely, we make the following assumptions on system (3.21).

(A1) The coefficient functions $\widetilde{a}_i(t)$, $\widetilde{b}_{ij}(t)$, $\widetilde{c}_{ij}(t)$ and $\widetilde{I}_i(t)$ are all bounded and uniformly continuous on $\mathbb{R}$, and recurrent, that is, the *hull* is minimal.
(A2) $\widetilde{a}_i(t) \geq a_0 > 0$ for every $t \in \mathbb{R}$ and $i \in \{1,\ldots,m\}$.
(A3) $\widetilde{b}_{ij}(t), \widetilde{c}_{ij}(t) \geq 0$ for every $t \in \mathbb{R}$ and $i, j \in \{1,\ldots,m\}$.
(A4) The activation functions $f_j, g_j : \mathbb{R} \to \mathbb{R}$ satisfy:

   (i) they are $\mathscr{C}^1$-functions on $\mathbb{R}$;
   (ii) $\lim_{|s|\to\infty} |f_j(s)|/|s| = \lim_{|s|\to\infty} |g_j(s)|/|s| = 0$ for each $j = 1,\ldots,m$;
   (iii) $0 \leq f_j'(s)$ and $0 \leq g_j'(s)$ for each $j = 1,\ldots,m$ and any $s \in \mathbb{R}$.

(A5) Each μ_{ij} is a normalized positive regular Borel measure on $(-r,0]$, i.e., $\mu_{ij}((-r,0]) = 1$.
(A6) f_j, g_j are convex for $s < 0$ and concave for $s > 0$, for each $j \in \{1,\ldots,m\}$.
(A7) For each $j \in \{1,\ldots,m\}$, f_j, g_j are strictly convex for $s < 0$ and strictly concave for $s > 0$, and there is an interval I of length r such that
$$\sum_{j=1}^{m} \widetilde{b}_{ij}(t) + \sum_{j=1}^{m} \widetilde{c}_{ij}(t) > 0 \;\text{ for each } i \in \{1,\ldots,m\} \text{ and } t \in I.$$

Under condition (A1) we can build the so-called *hull* of the system, by considering the set of functions $\{(\widetilde{a}_i(t+s), \widetilde{b}_{ij}(t+s), \widetilde{c}_{ij}(t+s), \widetilde{I}_i(t+s)),\ (t \in \mathbb{R}) \mid s \in \mathbb{R}\}$ formed by the time-translations of the coefficient functions determining system (3.21), together with their limit points for the compact-open topology. In this way, the hull turns out to be a compact metric space of functions, denoted by Ω, where a continuous flow $\sigma(t,\omega) = \omega{\cdot}t$ can be defined just by translation. Then we can consider the family of systems over the hull, which can be written for short as

$$z'(t) = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\; \omega \in \Omega\,, \tag{3.22}$$

where $z_t : (-r,0] \to \mathbb{R}^m$ is defined by $z_t(s) = z(t+s)$ for $s \leq 0$ and $t \geq 0$ (whenever it makes sense) and the function F is given componentwise by

$$
\begin{aligned}
F_i(\omega,x) = -a_i(\omega)\,x_i(0) + \sum_{j=1}^{m} b_{ij}(\omega)\,f_j(x_j(0)) \\
+ \sum_{j=1}^{m} c_{ij}(\omega)\int_{-r}^{0} g_j(x_j(s))\,d\mu_{ij}(s) + I_i(\omega)\,, \quad i=1,\ldots,m
\end{aligned}
\tag{3.23}
$$

for $\omega \in \Omega$ and a continuous map $x : (-\infty,0] \to \mathbb{R}^m$, with $(a_i, b_{ij}, c_{ij}, I_i)$ defined on Ω by evaluation at 0, i.e., $(a_i, b_{ij}, c_{ij}, I_i)(\omega) = \omega(0)$, so that $(a_i, b_{ij}, c_{ij}, I_i)(\omega{\cdot}t) = \omega(t)$ for $t \in \mathbb{R}$. Therefore, taking $\omega = (\widetilde{a}_i, \widetilde{b}_{ij}, \widetilde{c}_{ij}, \widetilde{I}_i)$ we recover in (3.22) the initial system (3.21). Note that by the construction, the functions a_i are positive on Ω, and the functions b_{ij} and c_{ij} are nonnegative on Ω. The recurrence hypothesis in (A1) means that the translation flow on Ω is minimal, that is, orbits are dense; this is a usual requirement in the non-autonomous setting and it happens for example if the coefficient functions are all almost periodic or almost automorphic. Consequently, hypothesis (C3) is satisfied. As in the introduction of the chapter, we consider the skew-product semiflow (3.3) induced by (3.22).

The condition stated in hypothesis (A2) is commonly imposed in the bibliography, representing the passive decay rates. Hypothesis (A3) together with (A4)-(iii) and (A5) will imply the quasimonotone condition (C4). With regard to hypothesis (A5), note also that the measures μ_{ij} are not necessarily absolutely continuous with respect to the Lebesgue measure. The concavity condition (C5) is deduced from Hypotheses (A5) and (A6), and the strongest one (C6) from (A5), (A6) and (A7). We now show the boundedness of solutions for the family of systems (3.22).

Proposition 3.17. *Assume that the initial system* (3.21) *satisfies hypotheses* (A1)-(A5). *Then, all solutions of the family of systems* (3.22) *are bounded and consequently the induced semiflow* (3.3) *is globally defined.*

Proof. Let $k \in \mathbb{R}$, and denote by $\mathbf{k}$ the constant map $(-r,0] \to \mathbb{R}^m$, $s \mapsto (k,\ldots,k)$ of $X = C([-r,0],\mathbb{R}^m)$. As the induced semiflow (3.3) is monotone, we know that if $-\mathbf{k} \le x \le \mathbf{k}$ for some $k > 0$ and $x \in X$, then $u(t,\omega,-\mathbf{k}) \le u(t,\omega,x) \le u(t,\omega,\mathbf{k})$ for any $t \ge 0$ for which all the terms exist. Therefore, it suffices to make sure that the solutions $u(t,\omega,-\mathbf{k})$ and $u(t,\omega,\mathbf{k})$ remain bounded for sufficiently large $k > 0$. For this, it is enough to check that $-\mathbf{k}$ defines a lower solution and $\mathbf{k}$ defines an upper solution (see Remark 2.5 in [56]), that is, $F(\omega,-\mathbf{k}) \ge 0$ and $F(\omega,\mathbf{k}) \le 0$ for $k > 0$ big enough. In order to show this assertion, just note first that by hypothesis (A2), $0 < a_0 \le a_i(\omega)$ for any $\omega \in \Omega$, and that all the functions a_i, b_{ij}, c_{ij} and I_i are bounded on the compact set Ω. Note also that by hypothesis (A4)-(ii) given any $\varepsilon > 0$ there is k_0 such that for any $k \ge k_0$, $-\varepsilon k \le f_j(k), g_j(k) \le \varepsilon k$ for any $j = 1,\ldots,m$. The proof is complete. □

We now provide a global attractivity result in X_+ for the neural networks under consideration, and a similar result in X_- using convexity instead of concavity in the proofs. We define the $m \times m$ diagonal matrices $F = \operatorname{diag}(f_1'(0),\ldots,f_m'(0))$, $G = \operatorname{diag}(g_1'(0),\ldots,g_m'(0))$ and $A(\omega) = \operatorname{diag}(a_1(\omega),\ldots,a_m(\omega))$, $\omega \in \Omega$, and the matrices $B(\omega) = [b_{ij}(\omega)]$, $C(\omega) = [c_{ij}(\omega)]$, $\omega \in \Omega$, and we denote

$$\widetilde{I}^- = \inf_{\substack{i=1,\ldots,m\\ t\in\mathbb{R}}} \widetilde{I}_i(t) \quad\text{and}\quad \widetilde{I}^+ = \sup_{\substack{i=1,\ldots,m\\ t\in\mathbb{R}}} \widetilde{I}_i(t)\,.$$

Theorem 3.18. *Assume that* $f_j(0) = g_j(0) = 0$*,* $j = 1,\ldots,m$*, and* $\widetilde{I}^- \geq 0$ *in* (3.21).

(i) *Under assumptions* (A1)-(A7)*, if there exists a vector* $v \gg 0$ *such that*

$$\big(A(\omega) - B(\omega)\,F - C(\omega)\,G\big)\,v \ll 0\,, \quad \omega\in\Omega\,, \tag{3.24}$$

then there is a unique solution $z^*(t) \gg 0$ *of* (3.21) *such that*

$$\lim_{t\to\infty} \|z(t,x_0) - z^*(t)\| = 0 \quad \textit{for each } x_0\in X \textit{ with } x_0 \gg 0\,.$$

Furthermore, the solution $z^*(t)$ *has the same recurrence property as that of the coefficients, meaning for instance that if the coefficients of* (3.21) *are all almost periodic, then this solution is almost periodic too.*

(ii) *Under assumptions* (A1)-(A6)*, if there exists a vector* $v \gg 0$ *such that*

$$\big(A(\omega) - B(\omega)\,F - C(\omega)\,G\big)\,v \gg 0\,, \quad \omega\in\Omega\,, \tag{3.25}$$

then any solution $z(t,x_0)$ *of* (3.21) *with* $x_0 \geq 0$ *satisfies*

$$\lim_{t\to\infty} \|z(t,x_0) - z(t,0)\| = 0\,.$$

In particular, if there are not external input functions, all the solutions with initial data in X_+ *tend to* 0*.*

Proof. (i) The proof relies on an application of Theorem 3.13. For that, since $\widetilde{I}^- \geq 0$ and $f_j(0) = g_j(0) = 0$ for $j = 1,\ldots,m$, we can simply take $a \equiv 0$ as a semicontinuous sub-equilibrium. In particular this implies that the set $\Omega\times X_+$ is invariant for the dynamics.

Next we claim that δv, as a constant function from Ω to $\mathbb{R}^m$, defines a lower solution for the family of systems (3.22) for sufficiently small $\delta > 0$, or equivalently, δv, as a constant function from Ω to X is a sub-equilibrium. It suffices to check that $F(\omega,\delta v) \geq 0$ where here δv is to be understood as an element in X. By Taylor's approximation theorem for the $\mathscr{C}^1$ activation functions, we can write $f_j(s) = f_j'(0)\,s + \varepsilon_j^1(s)\,s$ and $g_j(s) = g_j'(0)\,s + \varepsilon_j^2(s)\,s$ for certain functions ε_j^1, ε_j^2 with $\lim_{s\to 0}\varepsilon_j^i(s) = 0$ for any $j = 1,\ldots,m$ and $i = 1,2$. Then, writing

$$\begin{aligned} F_i(\omega,\delta v) = \delta\Bigg(&-a_i(\omega)\,v_i + \sum_{j=1}^m b_{ij}(\omega)\,f_j'(0)\,v_j + \sum_{j=1}^m c_{ij}(\omega)\,g_j'(0)\,v_j \\ &+ \sum_{j=1}^m b_{ij}(\omega)\,\varepsilon_j^1(\delta v_j)\,v_j + \sum_{j=1}^m c_{ij}(\omega)\,\varepsilon_j^2(\delta v_j)\,v_j\Bigg) + I_i(\omega)\,, \end{aligned}$$

and taking (3.24) into account, it is easy to check that $F_i(\omega,\delta v) \geq 0$, $i = 1,\ldots,m$, provided that $\delta > 0$ is taken small enough.

Therefore, if we fix $\omega_0 \in \Omega$ and $x_0 = \delta v \in X_+$, from Proposition 3.17 its semi-trajectory $\{(\omega_0{\cdot}t, u(t,\omega_0,x_0)) \mid t \geq 0\}$ is bounded, and since x_0 is a subequilibrium, $u(t,\omega_0,x_0) \geq x_0$ for each $t \geq 0$. Then, Theorem 3.13 applies and asserts that $K = \mathcal{O}(\omega_0,x_0)$ is a copy of the base Ω, that is, $K = \{(\omega, e(\omega)) \mid \omega \in \Omega\}$ for a continuous and τ-invariant map $e : \Omega \to X$, and it is an attractor: all the semiorbits corresponding to initial data (ω,x) with $x \gg 0$ are globally defined and approach K in $\Omega \times X$ as $t \to \infty$. From here all the statements in (i) are easily deduced.

(ii) First of all, note that if $\omega \in \Omega$ and $x_0 \in X_+$, because of monotonicity and concavity we can write $0 \leq u(t,\omega,x_0) - u(t,\omega,0) \leq u_x(t,\omega,0)\,x_0$, where $u_x(t,\omega,0)\,x_0$ is the solution in X of the linearized equations along the orbit of $(\omega,0)$ with initial value x_0; the linearized equations being

$$
\begin{aligned}
y_i'(t) &= -a_i(\omega{\cdot}t)\,y_i(t) + \sum_{j=1}^{m} b_{ij}(\omega{\cdot}t)\,f_j'(z_j(t,\omega,0))\,y_j(t) \\
&\quad + \sum_{j=1}^{m} c_{ij}(\omega{\cdot}t)\int_{-\infty}^{0} g_j'(z_j(t+s,\omega,0))\,y_j(t+s)\,d\mu_{ij}(s)\,, \quad i = 1,\dots,m.
\end{aligned}
$$

Now, consider the following family of monotone and linear systems for $\omega \in \Omega$,

$$
\begin{aligned}
y_i'(t) &= -a_i(\omega{\cdot}t)\,y_i(t) + \sum_{j=1}^{m} b_{ij}(\omega{\cdot}t)\,f_j'(0)\,y_j(t) \\
&\quad + \sum_{j=1}^{m} c_{ij}(\omega{\cdot}t)\,g_j'(0)\int_{-\infty}^{0} y_j(t+s)\,d\mu_{ij}(s)\,, \quad i = 1,\dots,m,
\end{aligned}
\tag{3.26}
$$

whose solution for each $\omega \in \Omega$ and each initial condition $x_0 \in X_+$ we denote by $\Phi(t,\omega)\,x_0 \in X_+$. Under the concavity assumptions, $f_j'(z_j(t,\omega,0)) \leq f_j'(0)$ and $g_j'(z_j(t+s,\omega,0)) \leq g_j'(0)$ for any $t \geq 0$ and $s \leq 0$. Therefore, a standard argument of comparison of solutions leads to the upper bound

$$
0 \leq u(t,\omega,x_0) - u(t,\omega,0) \leq u_x(t,\omega,0)\,x_0 \leq \Phi(t,\omega)\,x_0\,,
$$

so that we just need to prove the asymptotic convergence of $\Phi(t,\omega)\,x_0$ to 0. For that, note that condition (3.25) means that $-k\,v$ defines a strong lower solution for the family of monotone and linear (in particular concave) systems (3.26) for each $k > 0$, or equivalently, a strong sub-equilibrium. Thus, given $x_0 \in X_+$ there exists $k > 0$ so that $-k\,v \leq -x_0$ and from Theorem 3.12 we deduce that the semiorbit with initial data $(\omega,-x_0)$ approach asymptotically the unique minimal set K which, in this linear case, coincide with $K = \{(\omega,0) \mid \omega \in \Omega\}$. Hence, $\Phi(t,\omega)\,x_0 \to 0$ as $t \to \infty$, as claimed, and the proof is finished. □

Remark 3.19. Different conditions on the initial system of neural networks (3.21) imply conditions (3.24) or (3.25) in Theorem 3.18. For example, with the notation $\widetilde{b}_{ij}^- = \inf_{t\in\mathbb{R}} \widetilde{b}_{ij}(t)$, $\widetilde{c}_{ij}^- = \inf_{t\in\mathbb{R}} \widetilde{c}_{ij}(t)$, and $\widetilde{a}_i^+ = \sup_{t\in\mathbb{R}} \widetilde{a}_i(t)$, it is sufficient for condition (3.24) to hold that for $i = 1,\dots,m$,

$$\widetilde{a}_i^+ < \sum_{j=1}^m \widetilde{b}_{ij}^- f_j'(0) + \sum_{j=1}^m \widetilde{c}_{ij}^- g_j'(0)\,.$$

Remark 3.20. Conditions (3.24) and (3.25) are well known to be a priori incompatible with one another (see Fiedler [17]). In fact, condition (3.25) is equivalent to saying that the matrix with nonpositive nondiagonal entries $A(\omega)-B(\omega)\,F-C(\omega)\,G$ is a nonsingular M-matrix uniformly in $\omega\in\Omega$.

The proof of the following result is analogous to the proof of Theorem 3.18, just making the obvious modifications because of convexity instead of concavity.

Theorem 3.21. *Assume that $f_j(0)=g_j(0)=0$, $j=1,\dots,m$, and $\widetilde{I}^+\le 0$ in* (3.21).

(i) *Under assumptions* (A1)-(A7)*, if there exists a vector $v\gg 0$ such that condition* (3.24) *holds, then there exists a unique solution $z_*(t)\ll 0$ such that*

$$\lim_{t\to\infty}\|z(t,x_0)-z_*(t)\|=0 \quad \textit{for each } x_0\in X \textit{ with } x_0\ll 0\,.$$

Furthermore, the solution $z_*(t)$ has the same recurrence property as that of the coefficients, meaning for instance that if the coefficients of (3.21) *are all almost periodic, then this solution is almost periodic too.*

(ii) *Under assumptions* (A1)-(A6)*, if there exists a vector $v\gg 0$ such that condition* (3.25) *holds, then any solution $z(t,x_0)$ of* (3.21) *with $x_0\le 0$ satisfies*

$$\lim_{t\to\infty}\|z(t,x_0)-z(t,0)\|=0\,.$$

In particular, if there are not external input functions, all the solutions with initial data in X_- tend to 0*.*

4 Non-autonomous FDEs with Infinite Delay

This section is devoted to the study of dynamical properties of a monotone skew-product semiflow determined by a family of functional differential equations with infinite delay. Many essential results in the theory of monotone dynamical systems deduced in the last decades require strong monotonicity. This condition never holds when we consider infinite delay differential equations with the usual order. Although the definition of an alternative order is possible in some particular cases (see for instance Wu [88]), this explains the reason why monotone methods have not been systematically applied to this kind of problems. We extend to this context recent results with significative dynamical meaning which only require the monotonicity of the semiflow.

Before considering this particular skew-product semiflow, we describe the structure of the compact invariant sets obtained in [54], which becomes essential to understand the global picture of the dynamics, for an abstract skew-product semiflow

$(\Omega \times X, \tau, \mathbb{R}^+)$ where $(\Omega, \sigma, \mathbb{R})$ stands for a minimal flow on a compact metric space and X is a complete metric space.

4.1 Stability and extensibility results for omega-limit sets

As explained above, in this subsection we give some new results in the area of topological dynamics for a continuous skew-product semiflow $(\Omega \times X, \tau, \mathbb{R}^+)$

$$\tau\colon \mathbb{R}^+ \times \Omega \times X \longrightarrow \Omega \times X\,, \quad (t, \omega, x) \mapsto (\omega{\cdot}t, u(t, \omega, x))\,, \tag{4.1}$$

over a minimal base flow $(\Omega, \sigma, \mathbb{R})$ and a complete metric space (X, d). In particular, we extend classical stability and extensibility results to the case of a non-distal base flow, which allow us to generalize in a straightforward way known results for monotone semiflows induced by non-autonomous differential equations when the flow in the base is only minimal.

To begin with, we state the definitions of uniform stability and uniform asymptotic stability for a compact τ-invariant set $K \subset \Omega \times X$.

Definition 4.1. Let C be a positively invariant and closed set in $\Omega \times X$. A compact positively invariant set $K \subseteq C$ is *uniformly stable* (with respect to C) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$, called the *modulus of uniform stability*, such that, if $(\omega, x) \in K$, $(\omega, y) \in C$ are such that $\mathsf{d}(x, y) < \delta(\varepsilon)$, then $\mathsf{d}(u(t, \omega, x), u(t, \omega, y)) \leq \varepsilon$ for all $t \geq 0$. K is *uniformly asymptotically stable* if it is uniformly stable and besides, there exists a $\delta_0 > 0$ such that, if $(\omega, x) \in K$, $(\omega, y) \in C$ satisfy $\mathsf{d}(x, y) < \delta_0$, then, uniformly in $(\omega, x) \in K$, $\lim_{t\to\infty} \mathsf{d}(u(t, \omega, x), u(t, \omega, y)) = 0$.

Very often one deals with either $C = \Omega \times X$ or $C = K$. If no mention to C is made, we assume that it is the whole space, whereas if the restricted semiflow $(K, \tau, \mathbb{R}^+)$ is said to be uniformly stable, we mean that $C = K$. Besides, as it is to be expected, if $C = \Omega \times X$, all the trajectories in a uniformly (asymptotically) stable set are uniformly (asymptotically) stable.

Conversely, if a trajectory has some stability properties, its omega-limit set inherits them: it is not difficult to prove that, if the semiorbit of certain (ω, x) is relatively compact and uniformly (asymptotically) stable, then the omega-limit set of (ω, x) is a uniformly (asymptotically) stable set with the same modulus of uniform stability as that of the semiorbit (see Sell [67]).

The next result, proved in [54], relates the property of uniform stability to that of fiber distallity, provided that there exists a flow extension. The important topological tool, known as the section map of a positively invariant set $K \subset \Omega \times X$, was already recalled in subsection 3.2.

Theorem 4.2. *Let $K \subset \Omega \times X$ be a compact τ-invariant set admitting a flow extension. If $(K, \tau, \mathbb{R})$ is uniformly stable as $t \to \infty$, then it is a fiber distal flow which is also uniformly stable as $t \to -\infty$. Furthermore, the section map for K, $\omega \in \Omega \mapsto K_\omega = \{x \in X \mid (\omega, x) \in K\} \in \mathscr{P}_c(K_X)$, is continuous at every $\omega \in \Omega$.*

The next step is to prove the same result without assuming that K has a flow extension but considering the existence of backward extensions of semiorbits (see also [54]).

Theorem 4.3. *Let $K \subset \Omega \times X$ be a compact positively invariant set such that every point of K admits a backward orbit. If the semiflow $(K, \tau, \mathbb{R}^+)$ is uniformly stable, then it admits a flow extension which is fiber distal and uniformly stable as $t \to -\infty$. Besides, the section map for K, $\omega \in \Omega \mapsto K_\omega \in \mathscr{P}_c(K_X)$, is continuous at every $\omega \in \Omega$.*

We can now easily state the theorem on the structure of uniformly asymptotically stable sets admitting backward semiorbits. We prove that these sets K are *N-covers* of the base flow, that is, maintaining the notation introduced for the section map (3.9), $\mathrm{card}(K_\omega) = N$ for every $\omega \in \Omega$. Without distallity on the base flow, we combine Theorem 4.3 with previous ideas by Sacker-Sell [63].

Theorem 4.4. *Consider a compact positively invariant set $K \subset \Omega \times X$ for the skew-product semiflow* (4.1) *and assume that every semiorbit in K admits a backward extension. If $(K, \tau, \mathbb{R}^+)$ is uniformly asymptotically stable, then it is an N-cover of the base flow $(\Omega, \sigma, \mathbb{R})$.*

Proof. By Theorem 4.3 we know that K admits a flow extension which is fiber distal. Let us fix any $\omega \in \Omega$ and let us check that $\mathrm{card}(K_\omega)$ must be finite. Suppose for contradiction that it is infinite. Then, we can take a sequence of pairwise distinct elements $\{x_n\} \subset K_\omega$ such that $\lim_{n\to\infty} x_n = x_0 \in K_\omega$. Let $\delta_0 > 0$ be the positive radius of attraction for K given in Definition 4.1. Choosing n sufficiently large, we have that $0 < \mathrm{d}(x_n, x_0) < \delta_0$, so that $\lim_{t\to\infty} \mathrm{d}(u(t, \omega, x_n), u(t, \omega, x_0)) = 0$, in contradiction with the fiber distallity of K. Therefore, there is a finite N such that $\mathrm{card}(K_\omega) = N$.

Finally, it suffices to apply a classical result by Sacker-Sell (see Theorem 3 in [63]) or just the continuity of the section map proved in Theorem 4.3 to conclude that it must be $\mathrm{card}(K_\omega) = N$ for all $\omega \in \Omega$, as we claimed. □

As a consequence, we extend old results by Miller [45] and Sacker-Sell [63] on the structure of omega-limit sets with an almost periodic minimal base flow, to the case of a non-distal base flow.

Proposition 4.5. *Let $\{\tau(t, \widetilde{\omega}, \widetilde{x}) \,|\, t \geq 0\}$ be a forward orbit of the skew-product semiflow* (4.1) *which is relatively compact and let $\widetilde{K}$ denote the omega-limit set of $(\widetilde{\omega}, \widetilde{x})$. The following statements hold:*

(i) *If $\widetilde{K}$ contains a minimal set K which is uniformly stable, then $\widetilde{K} = K$ and it admits a fiber distal flow extension.*
(ii) *If the semiorbit is uniformly stable, then the omega-limit set $\widetilde{K}$ is a uniformly stable minimal set which admits a fiber distal flow extension.*
(iii) *If the semiorbit is uniformly asymptotically stable, then the omega-limit set $\widetilde{K}$ is a uniformly asymptotically stable minimal set which is an N-cover of the base flow.*

Proof. (i) We just need to show that $\widetilde{K} \subseteq K$. So, take an element $(\omega,x) \in \widetilde{K}$ and let us prove that $(\omega,x) \in K$. As K is in particular closed, it suffices to see that for any fixed $\varepsilon > 0$ there exists $(\omega,x^*) \in K$ such that $\mathsf{d}(x,x^*) \leq \varepsilon$. Let $\delta(\varepsilon) > 0$ be the modulus of uniform stability for K.

First of all, there exists $s_n \uparrow \infty$ such that $\lim_{n\to\infty}(\widetilde{\omega}{\cdot}s_n, u(s_n,\widetilde{\omega},\widetilde{x})) = (\omega,x)$. Now, take a pair $(\omega,x_0) \in K \subseteq \widetilde{K}$. Then, there exists a sequence $t_n \uparrow \infty$ such that

$$(\omega,x_0) = \lim_{t\to\infty}(\widetilde{\omega}{\cdot}t_n, u(t_n,\widetilde{\omega},\widetilde{x}))\,.$$

As it is well-known, in omega-limit sets and in minimal sets there always exist backward continuations of semiorbits. Then, we can apply Theorem 4.3 to K so that the section map (3.9) turns out to be continuous at any point. As $\widetilde{\omega}{\cdot}t_n \to \omega$, we deduce that $K_{\widetilde{\omega}\cdot t_n} \to K_\omega$ in the Hausdorff metric. Then, for $x_0 \in K_\omega$ there exists a sequence $x_n \in K_{\widetilde{\omega}\cdot t_n}$, $n \geq 1$, such that $x_n \to x_0$ as $n \to \infty$. Altogether, there exists $n_0 \in \mathbb{N}$ such that $\mathsf{d}(u(t_{n_0},\widetilde{\omega},\widetilde{x}),x_{n_0}) < \delta(\varepsilon)$. By the uniform stability,

$$\mathsf{d}(u(t+t_{n_0},\widetilde{\omega},\widetilde{x}),u(t,\widetilde{\omega}{\cdot}t_{n_0},x_{n_0})) \leq \varepsilon \quad \text{for all } t \geq 0\,.$$

In particular, if n_1 is such that $s_n - t_{n_0} \geq 0$ for $n \geq n_1$, we obtain that

$$\mathsf{d}(u(s_n,\widetilde{\omega},\widetilde{x}),u(s_n - t_{n_0},\widetilde{\omega}{\cdot}t_{n_0},x_{n_0})) \leq \varepsilon \quad \text{for all } n \geq n_1\,. \tag{4.2}$$

Now, it remains to notice that, as $(\widetilde{\omega}{\cdot}t_{n_0},x_{n_0}) \in K$, also $\tau(s_n - t_{n_0},\widetilde{\omega}{\cdot}t_{n_0},x_{n_0}) = (\widetilde{\omega}{\cdot}s_n, u(s_n - t_{n_0},\widetilde{\omega}{\cdot}t_{n_0},x_{n_0})) \in K$ for all $n \geq n_1$. Therefore, there is a convergent subsequence towards a pair $(\omega,x^*) \in K$, and taking limits in (4.2), we deduce that $\mathsf{d}(x,x^*) \leq \varepsilon$, as we wanted.

(ii) We already remarked that in this case $\widetilde{K}$ is uniformly stable. The fact that it is minimal is a straightforward consequence of (i). For the fiber distal flow extension one just needs to apply Theorem 4.3.

(iii) It follows from previous comments as well as from Theorem 4.4.

Remark 4.6. Notice that the stability and extensibility results obtained in this subsection allow us to extend many of the results of Shen-Yi [72] and Jiang-Zhao [36], proved with distallity on the base, to the case of just a minimal base flow.

4.2 Monotone FDEs with infinite delay

Let $(\Omega,\sigma,\mathbb{R})$ be a minimal flow over a compact metric space (Ω,d) and denote $\sigma(t,\omega) = \omega{\cdot}t$ for each $\omega \in \Omega$ and $t \in \mathbb{R}$. In $\mathbb{R}^m$ we take the maximum norm $\|v\| = \max_{j=1,\ldots,m}|v_j|$ and the usual partial order relation, already introduced in Section 3. We consider the Fréchet space $X = C((-\infty,0],\mathbb{R}^m)$ endowed with the compact-open topology, i.e. the topology of uniform convergence over compact subsets, which is a metric space for the distance

$$\mathsf{d}(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|x-y\|_n}{1+\|x-y\|_n}\,,\quad x,y\in X\,,$$

where $\|x\|_n=\sup_{s\in[-n,0]}\|x(s)\|$. The subset

$$X_+=\{x\in X\mid x(s)\geq 0 \text{ for each } s\in(-\infty,0]\}$$

is a normal positive cone in X and has an empty interior. As usual, a partial order relation in X is induced, given by

$$\begin{aligned} x\leq y &\iff x(s)\leq y(s) \quad \text{for each } s\in(-\infty,0]\,,\\ x<y &\iff x\leq y \quad \text{and} \quad x\neq y\,. \end{aligned}$$

Let $BU\subset X$ be the Banach space

$$BU=\{x\in X\mid x \text{ is bounded and uniformly continuous}\}$$

with the supremum norm $\|x\|_\infty=\sup_{s\in(-\infty,0]}\|x(s)\|$. Given $r>0$ we will denote

$$B_r=\{x\in BU\mid \|x\|_\infty\leq r\}\,.$$

As usual, given $I=(-\infty,a]\subset\mathbb{R}$, $t\in I$, and a continuous function $z:I\to\mathbb{R}^m$, z_t will denote the element of X defined by $z_t(s)=z(t+s)$ for $s\in(-\infty,0]$.

We consider the family of non-autonomous infinite delay FDEs

$$z'(t)=F(\omega{\cdot}t,z_t)\,,\quad t\geq 0\,,\ \omega\in\Omega\,, \qquad (4.3)_\omega$$

defined by a function $F\colon\,\Omega\times BU\to\mathbb{R}^m$, $(\omega,x)\mapsto F(\omega,x)$ satisfying the following conditions:

(H1) The functions $F_x:\Omega\times BU\to L(BU,\mathbb{R}^m)$, $(\omega,x)\mapsto F_x(\omega,x)$ and F are continuous on $\Omega\times BU$.
(H2) For each $r>0$, $F(\Omega\times B_r)$ is a bounded subset of $\mathbb{R}^m$.
(H3) For each $r>0$, $F\colon\,\Omega\times B_r\to\mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r, i.e. if $\omega_n\to\omega$ and $x_n\xrightarrow{\mathsf{d}}x$ as $n\to\infty$ with $x\in B_r$, then $\lim_{n\to\infty}F(\omega_n,x_n)=F(\omega,x)$.
(H4) If $x,y\in BU$ with $x\leq y$ and $x_j(0)=y_j(0)$ holds for some $j\in\{1,\ldots,m\}$, then $F_j(\omega,x)\leq F_j(\omega,y)$ for each $\omega\in\Omega$.

From hypothesis (H1), the standard theory of infinite delay functional differential equations (see Hino *et al.* [24]) assures that for each $x\in BU$ and each $\omega\in\Omega$ the system $(4.3)_\omega$ locally admits a unique solution $z(t,\omega,x)$ with initial value x, i.e. $z(s,\omega,x)=x(s)$ for each $s\in(-\infty,0]$. Therefore, the family $(4.3)_\omega$ induces a local skew-product semiflow

$$\tau\colon\,\mathbb{R}^+\times\Omega\times BU\longrightarrow\Omega\times BU\,,\quad (t,\omega,x)\mapsto(\omega{\cdot}t,u(t,\omega,x))\,, \qquad (4.4)$$

where $u(t,\omega,x)\in BU$ and $u(t,\omega,x)(s)=z(t+s,\omega,x)$ for $s\in(-\infty,0]$.

As shown in [54], from hypotheses (H1) and (H2), each bounded solution $z(t,\omega_0,x_0)$, that is, $r=\sup_{t\in\mathbb{R}}\|z(t,\omega_0,x_0)\|<\infty$, provides a relatively compact trajectory, i.e. the closure $\mathrm{cls}_X\{u(t,\omega_0,x_0)\mid t\ge 0\}$ is a compact subset of BU for the compact-open topology. In addition, from hypotheses (H1), (H2) and (H3) and the next result, we can deduce the continuity of the semiflow restricted to some compact subsets $K\subset\Omega\times BU$ when the compact-open topology is considered in BU.

Proposition 4.7. *Let $\{(\omega_n,x_n)\}\subset\Omega\times B_R$ for some $R>0$ be such that $\omega_n\to\omega$ and $x_n\stackrel{\mathsf{d}}{\to}x$ with $(\omega,x)\in\Omega\times B_R$. If there is $t>0$ such that $u(t,\omega,x)$ is defined, then there is n_0 such that $u(t,\omega_n,x_n)$ is defined for each $n\ge n_0$ and $u(t,\omega_n,x_n)\stackrel{\mathsf{d}}{\to}u(t,\omega,x)$.*

Proof. If $s\le -t$, $u(t,\omega_n,x_n)(s)-u(t,\omega,x)(s)=x_n(t+s)-x(t+s)$, and $x_n\stackrel{\mathsf{d}}{\to}x$. Thus, it suffices to show that $u(t,\omega_n,x_n)(s)\to u(t,\omega,x)(s)$ uniformly for $s\in[-t,0]$ or, equivalently, $z(s,\omega_n,x_n)\to z(s,\omega,x)$ uniformly for $s\in[0,t]$.

First, we claim that the result holds if $z(t,\omega_n,x_n)$ is defined for each $n\ge 1$ and $\sup\{\|z(s,\omega_n,x_n)\|\mid s\in[0,t],\ n\ge 1\}\le r_0$ for some $r_0>0$. In fact, the set $\mathscr{F}=\{z(\cdot,\omega_n,x_n)|_{[0,t]}\mid n\ge 1\}\subset(C([0,t],\mathbb{R}^m),\|\cdot\|_\infty)$ is uniformly bounded, and it is uniformly equicontinuous, because of the mean value theorem and (H2). Then, by Arzelà-Ascoli theorem, $\mathscr{F}$ is relatively compact. We just need to prove that $z(\cdot,\omega,x)|_{[0,t]}$ is its only limit point. So, assume for simplicity that $z(s,\omega_n,x_n)\to v(s)$ uniformly on $[0,t]$. We extend the function v with continuity to all $(-\infty,t]$ by defining $v(s)=x(s)$ for any $s\le 0$. Then, it trivially holds that $u(s,\omega_n,x_n)\stackrel{\mathsf{d}}{\to}v_s$ and $v_s\in B_{r_0}$ for every $s\in[0,t]$. Now, for each $n\ge 1$, integrating in the equation it satisfies, we have that for any $s\in[0,t]$,

$$z(s,\omega_n,x_n)=x_n(0)+\int_0^s F(\omega_n{\cdot}r,u(r,\omega_n,x_n))\,dr\,.$$

Because of (H2) we can apply Lebesgue convergence theorem, and because of the continuity of the flow on Ω and (H3), when we take limits we obtain that

$$v(s)=x(0)+\int_0^s F(\omega{\cdot}r,v_r)\,dr\quad\text{ for every }\ s\in[0,t]\,.$$

As we have uniqueness of solutions for the initial value problem, it must be $v(s)=z(s,\omega,x)$ for every $s\in[0,t]$, as we wanted to see.

Next, we take $r_1\ge R+1$ such that $\sup\{\|z(s,\omega,x)\|\mid s\in[0,t]\}<r_1-1$ and we define a C^∞ function $\varphi\colon\mathbb{R}^m\to\mathbb{R}$ such that $\varphi(y)=1$ if $\|y\|\le r_1-1$ and $\varphi(y)=0$ if $\|y\|\ge r_1$. We consider the new family

$$y'(s)=F(\omega{\cdot}s,y_s)\varphi(y(s))\,,\quad \omega\in\Omega\,. \tag{4.5}$$

It is clear that $y(s)=z(s,\omega,x)$ satisfies the equation for $s\in[0,t]$. Moreover, denoting as usual, by $y(s,\omega_n,x_n)$ the solution of (4.5) for ω_n with initial data x_n, $y(t,\omega_n,x_n)$ is defined for $n\ge 1$ and $\sup\{\|y(s,\omega_n,x_n)\|\mid s\in[0,t],\ n\ge 1\}\le r_1$. Then, we can apply the first part of the proof to deduce that $y(s,\omega_n,x_n)\to y(s)=z(s,\omega,x)$ uniformly in $[0,t]$. Finally from $\|z(s,\omega,x)\|<r_1-1$, there is n_0 such that $\|y(s,\omega_n,x_n)\|<r_1-1$

for each $n \geq n_0$ and $s \in [0,t]$. Therefore, $z(s,\omega_n,x_n) = y(s,\omega_n,x_n)$ for $s \in [0,t]$, and the proof is finished. □

As mentioned above, a bounded solution $z(t,\omega_0,x_0)$ provides a relatively compact trajectory, we can define the omega-limit set as

$$\mathscr{O}(\omega_0,x_0) = \{(\omega,x) \in \Omega \times BU \mid \exists t_n \uparrow \infty \text{ with } \omega_0{\cdot}t_n \to \omega\,,\; u(t_n,\omega_0,x_0) \overset{\mathsf{d}}{\to} x\}\,,$$

and it is a positively invariant compact set admitting a flow extension (see [54]).

From hypothesis (H4) the monotone character of the semiflow is deduced, that is, for each $\omega \in \Omega$ and $x,y \in BU$ such that $x \leq y$ it holds that $u(t,\omega,x) \leq u(t,\omega,y)$ whenever they are defined. The proof is completely analogous to the one given in Theorem 2.6 of Wu [88] or Theorem 5.1.1 of Smith [75].

The techniques and conclusions derived in the in the previous subsection allow us to prove results concerning the existence of minimal sets which are almost automorphic extensions of the flow on the base. These minimal sets are copies of the base flow assuming additional hypotheses of stability. More precisely, in [54] we extend previous results of Novo *et al.* [49], explained and stated in subsection 3.2, deducing the presence of almost automorphic dynamics from the existence of a semicontinuous semiequilibrium which satisfies additional compactness conditions. In the present situation it is natural to assume that the range of a semi-equilibrium is the set BU. If the base $(\Omega,\sigma,\mathbb{R})$ is almost periodic these methods ensure the existence of almost automorphic minimal sets, which in many cases become exact copies of the base and hence are almost periodic. We refer the reader to [54] but we include the statements of the results for completion.

Proposition 4.8. *Let $a : \Omega \to BU$ be a semicontinuous semi-equilibrium and assume that there is an $\omega_0 \in \Omega$ such that* $\text{closure}_X\{u(t,\omega_0,a(\omega_0))\mid t\geq 0\}$ *is a compact subset of X for the compact-open topology. Then:*

(i) *The omega-limit set $\mathscr{O}(\omega_0,a(\omega_0))$ contains a unique minimal set which is an almost automorphic extension of the base flow.*
(ii) *If the orbit $\{\tau(t,\omega_0,a(\omega_0))\mid t\geq 0\}$ is uniformly stable, then $\mathscr{O}(\omega_0,a(\omega_0))$ is a copy of the base.*

Proposition 4.9. *Let $a : \Omega \to BU$ be a semicontinuous semi-equilibrium such that $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty < \infty$ and $\Gamma_a \subset BU$. The following statements are equivalent:*

(i) $\Gamma = \text{closure}_X\{u(t,\omega,a(\omega))\mid t\geq 0,\ \omega\in\Omega\}$ *is a compact subset of BU for the compact-open topology.*
(ii) *For each $\omega\in\Omega$, the* $\text{closure}_X\{u(t,\omega,a(\omega))\mid t\geq 0\}$ *is a compact subset of BU for the compact-open topology.*
(iii) *There is an $\omega_0\in\Omega$ such that the* $\text{closure}_X\{u(t,\omega_0,a(\omega_0))\mid t\geq 0\}$ *is a compact subset of BU for the compact-open topology.*

Theorem 4.10. *Let us assume the existence of a semicontinuous semi-equilibrium $a : \Omega \to BU$ satisfying $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty < \infty$, $\Gamma_a \subset BU$ and one of the equivalent statements of Proposition* 4.9. *Then,*

(i) *there exists a semicontinuous equilibrium* $c : \Omega \to BU$ *with* $c(\omega) \in \Gamma$ *for any* $\omega \in \Omega$.

(ii) *Let* ω_1 *be a continuity point for c. Then, the restriction of the semiflow* τ *to the minimal set*

$$K^* = \text{closure}_{\,\Omega\times X}\{(\omega_1{\cdot}t, c(\omega_1{\cdot}t)) \mid t \ge 0\} \subset C_a \tag{4.6}$$

is an almost automorphic extension of the base flow $(\Omega, \sigma, \mathbb{R})$.

(iii) K^* *is the only minimal set contained in the omega-limit set* $\mathscr{O}(\widehat{\omega}, a(\widehat{\omega}))$ *for each point* $\widehat{\omega} \in \Omega$.

(iv) *If there is a point* $\widetilde{\omega} \in \Omega$ *such that the trajectory* $\{\tau(t, \widetilde{\omega}, a(\widetilde{\omega})) \mid t \ge 0\}$ *is uniformly stable, then for each* $\widehat{\omega} \in \Omega$,

$$\mathscr{O}(\widehat{\omega}, a(\widehat{\omega})) = K^* = \{(\omega, c(\omega)) \mid \omega \in \Omega\},$$

i.e. it is a copy of the base determined by the equilibrium c of (i)*, which is a continuous map.*

Next we obtain an infinite delay version of significative results proved by Jiang-Zhao [36], again without the assumption of distal flow on the base. They established the 1-covering property of omega limit sets for monotone and uniformly stable skew-product semiflows with the componentwise separating property of bounded and ordered full orbits, where the ordered space is a product Banach space.

A componentwise separation property has been frequently considered for ordinary and finite delayed cooperative differential equations (see for instance Smith [75] and Shen-Zhao [73]). We show that this is also a natural condition for cooperative retarded differential equations with infinite delay, in fact, can be deduced from Hypotheses (H1)-(H4).

Proposition 4.11. *Under Hypotheses* (H1)-(H4)*, if* $x, y \in BU$ *with* $x \le y$ *and* $x_i(0) < y_i(0)$ *holds for some* $i \in \{1, \dots, m\}$*, then* $z_i(t, \omega, x) < z_i(t, \omega, y)$ *for each* $\omega \in \Omega$ *and whenever they are defined.*

Proof. We fix $\omega \in \Omega$ and $x, y \in BU$ satisfying the assumptions in the statement. We take $\widetilde{x} = y + (x - y)\, g$ with $g : (-\infty, 0] \to \mathbb{R}$ continuous, $0 \le g \le 1$ and $g(t) = 0$ if $t \le -1$. Notice that $y - \widetilde{x} = (y - x)\, g$ has compact support and $x \le \widetilde{x} \le y$.

Let $J = [0, T]$ be an interval of definition of $z(t, \omega, y)$ and $z(t, \omega, x)$, and hence, also of $z(t, \omega, \widetilde{x})$. We denote $h(t) = z(t, \omega, y) - z(t, \omega, \widetilde{x})$ for $t \in J = [0, T]$. Then

$$\begin{aligned} h'(t) &= F(\omega{\cdot}t, u(t, \omega, y)) - F(\omega{\cdot}t, u(t, \omega, \widetilde{x})) \\ &= \int_0^1 F_x(\omega{\cdot}t, r\,u(t, \omega, y) + (1 - r)\, u(t, \omega, \widetilde{x}))\, h_t\, dr = L(t)\, h_t \end{aligned}$$

where $L(t) : BU \to \mathbb{R}^m$ is linear and continuous for the norm.

From Riesz representation theorem we obtain that for each φ of compact support, i.e., i.e. $\varphi_i \in C_c(-\infty, 0]$, $i = 1, \dots m$

$$L(t)\varphi=\int_{-\infty}^{0}[d\mu(t)(s)]\,\varphi(s)\,,$$

where $\mu(t)=[\mu_{ij}(t)]$ is a matrix of real regular Borel measures $\mu_{ij}(t)$ with finite total variation $|\mu_{ij}(t)|(-\infty,0]<\infty$, for all i, $j\in\{1,\dots,m\}$.

Next, we can express $L(t)\varphi$ as

$$L(t)\varphi=D(t)\,\varphi(0)+\int_{-\infty}^{0}[d\nu(t)(s)]\,\varphi(s)=D(t)\,\varphi(0)+\widetilde{L}(t)\varphi\,,$$

where $D(t)=\operatorname{diag}(a_1(t),\dots,a_m(t))$ with $a_i(t)=\mu_{ii}(t)(\{0\})$, $\nu(t)=[\nu_{ij}(t)]$ with $\nu_{ij}(t)=\mu_{i,j}(t)$ if $i\neq j$, and $\nu_{ii}(A)=\mu_{ii}(A-\{0\})$ for each Borel set $A\subset(-\infty,0]$.

Moreover, from (H4) we deduce that whenever $\varphi\geq 0$ and $\varphi_i(0)=0$ then $L_i(t)\varphi\geq 0$, and from (H3) we know that $L\colon J\to L(BU,\mathbb{R}^m)$ is continuous. Hence, as in Lemma 5.1.2 of [75], it is shown that $\widetilde{L}(t)\varphi\geq 0$ whenever $\varphi\geq 0$, and both D and $\widetilde{L}$ vary continuously with t.

Fix $t>0$ and notice that $h_t=u(t,\omega,y)-u(t,\omega,\widetilde{x})\geq 0$ has compact support because $h_t(s)=y(s)-\widetilde{x}(s)$ for each $s\leq -1-t$. Therefore,

$$h_i'(t)=L_i(t)\,h_t=a_i(t)\,h_t(0)+\widetilde{L}_i(t)\,h_t\geq a_i(t)\,h_i(t)\,,$$

which implies, since $h_i(0)=y_i(0)-\widetilde{x}_i(0)>0$, that $h_i(t)=z_i(t,\omega,y)-z_i(t,\omega,\widetilde{x})>0$ for each $t\in J$. Finally, from $x\leq\widetilde{x}$ and the monotonicity we deduce that $z(t,\omega,x)\leq z(t,\omega,\widetilde{x})$ and hence, $z_i(t,\omega,x)<z_i(t,\omega,y)$, as claimed.

We establish the 1-covering property of omega limit sets when, in addition to hypotheses (F1-F4), the uniform stability is assumed:

(H5) There is an $r>0$ such that all the trajectories with initial data in B_r are uniformly stable in $B_{r'}$ for each $r'>r$, and relatively compact for the product metric topology.

Theorem 4.12. *Assume that Hypotheses* (H1-H5) hold and let $(\omega_0,x_0)\in\Omega\times B_r$ be such that $K=\mathscr{O}(\omega_0,x_0)\subset\Omega\times B_r$. Then $K=\mathscr{O}(\omega_0,x_0)=\{(\omega,c(\omega))\mid\omega\in\Omega\}$ is a copy of the base and

$$\lim_{t\to\infty}\mathsf{d}(u(t,\omega_0,x_0),c(\omega_0{\cdot}t))=0\,,$$

where $c:\Omega\to BU$ is a continuous equilibrium.

Proof. For each $\omega\in\Omega$ we define the map $a(\omega)$ on $(-\infty,0]$ by

$$a(\omega)(s)=\inf\{x(s)\mid(\omega,x)\in K\}\quad\text{for each }\ s\leq 0\,. \tag{4.7}$$

Then, we claim that $a\colon\Omega\to BU$, $\omega\mapsto a(\omega)$ is well-defined, it is a continuous super-equilibrium with $\Gamma_a=\mathrm{closure}_X\{a(\omega)\mid\omega\in\Omega\}\subset BU$, $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty<\infty$, and it satisfies the equivalent statements of Proposition 4.9.

It is not hard to check that for any $(\widetilde{\omega},\widetilde{x})\in K$, $\widetilde{x}$ is Lipschitzian with Lipschitz constant $L=\sup\{\|F(\omega,x)\|\mid(\omega,x)\in\Omega\times B_r\}$. From this one can prove that each

$a(\omega)$ is also Lipschitzian with the same constant L and so, $a(\omega) \in B_r$ for any $\omega \in \Omega$ (see Proposition 5.6 in [51] for more details). Then, it holds that Γ_a is a compact subset of X, and actually $\Gamma_a \subset BU$.

Let us check that a defines a super-equilibrium. Notice that, as $a(\omega) \in B_r$, it follows from hypothesis (H5) that $u(t,\omega,a(\omega))$ exists for any $\omega \in \Omega$ and $t \geq 0$. Now, fix $\omega \in \Omega$ and $t \geq 0$ and consider any $(\omega{\cdot}t, y) \in K$. As we have a flow on K, $\tau(-t,\omega{\cdot}t,y) = (\omega, u(-t,\omega{\cdot}t,y)) \in K$ and therefore, $a(\omega) \leq u(-t,\omega{\cdot}t,y)$. Applying monotonicity, $u(t,\omega,a(\omega)) \leq y$. As this happens for any $(\omega{\cdot}t,y) \in K$, we get that $u(t,\omega,a(\omega)) \leq a(\omega{\cdot}t)$. Besides, as done in Proposition 5.6 in [51], we have that, if $\omega_n \to \omega$ and $a(\omega_n) \xrightarrow{\mathsf{d}} x$, then $a(\omega) \leq x$.

Now let us prove that a is continuous on Ω. From hypothesis (H5) and Proposition 4.5 we know that K is uniformly stable, and then Theorem 4.2 asserts that the section map (3.9) for K, $\omega \in \Omega \mapsto K_\omega$, is continuous at every $\omega \in \Omega$. Fix $\omega \in \Omega$ and $\omega_n \to \omega$ such that $a(\omega_n) \xrightarrow{\mathsf{d}} x$. As we have just noted, $a(\omega) \leq x$. On the other hand, as $K_{\omega_n} \to K_\omega$ in the Hausdorff metric, for any $y \in K_\omega$ there exist $x_n \in K_{\omega_n}$, $n \geq 1$, such that $x_n \xrightarrow{\mathsf{d}} y$. Then, $(\omega_n, x_n) \in K$ implies that $a(\omega_n) \leq x_n$ and taking limits, $x \leq y$. As again this happens for any $y \in K_\omega$, we conclude that $x \leq a(\omega)$. In all, $a(\omega) = x$, as wanted.

Hence, from Theorem 4.10 we deduce that there is a continuous equilibrium $c : \Omega \to BU$ such that for each $\widehat{\omega} \in \Omega$,

$$\mathscr{O}(\widehat{\omega}, a(\widehat{\omega})) = K^* = \{(\omega, c(\omega)) \mid \omega \in \Omega\}\,. \tag{4.8}$$

The definition of a yields to $a(\omega) \leq x$ for each $(\omega,x) \in K$ and hence $c(\omega) \leq x$ by the construction of c. As in Jiang-Zhao [36] we prove that there is a subset $J \subset \{1,\ldots,m\}$ such that

$$\begin{aligned} c_i(\omega) &= x_i \quad \text{for each } (\omega,x) \in K \text{ and } i \notin J\,, \\ c_i(\omega) &< x_i \quad \text{for each } (\omega,x) \in K \text{ and } i \in J\,. \end{aligned} \tag{4.9}$$

It is enough to check that if $c_i(\widetilde{\omega})(0) = \widetilde{x}_i(0)$ for some $i \in \{1,\ldots,m\}$ and $(\widetilde{\omega},\widetilde{x}) \in K$, then $c_i(\omega) = x_i$ for any $(\omega,x) \in K$. We first notice that $c_i(\widetilde{\omega}) = \widetilde{x}_i$. Otherwise, there would be $s \in (-\infty,0]$ with $c_i(\widetilde{\omega})(s) < \widetilde{x}_i(s)$. Then, since $u_i(s,\widetilde{\omega},\widetilde{x})(0) = \widetilde{x}_i(s)$ because K admits a flow extension, $u(t,\widetilde{\omega},c(\widetilde{\omega})) = c(\widetilde{\omega}{\cdot}t)$ for each $t \in \mathbb{R}$ because c is an equilibrium, and Proposition 4.11, we would deduce that $c_i(\widetilde{\omega})(0) < \widetilde{x}_i(0)$, a contradiction. Next, as K is minimal from (H5) and Proposition 4.5, we take $(\omega,x) \in K$ and a sequence $s_n \downarrow -\infty$ such that $\widetilde{\omega}{\cdot}s_n \to \omega$ and $u(s_n,\widetilde{\omega},\widetilde{x}) \xrightarrow{\mathsf{d}} x$. Then,

$$\begin{aligned} x_i(0) &= \lim_{n\to\infty} u_i(s_n,\widetilde{\omega},\widetilde{x})(0) = \lim_{n\to\infty} \widetilde{x}_i(s_n) \\ &= \lim_{n\to\infty} c_i(\widetilde{\omega})(s_n) = \lim_{n\to\infty} c_i(\widetilde{\omega}{\cdot}s_n)(0) = c_i(\omega)(0)\,, \end{aligned}$$

and as before this implies that $c_i(\omega) = x_i$, as wanted.

Let $(\omega,x) \in K$ and define $x_\alpha = (1-\alpha)\,a(\omega) + \alpha\, x \in B_r \subset BU$ for $\alpha \in [0,1]$, and

$$L=\{\alpha\in[0,1]\mid \mathscr{O}(\omega,x_\alpha)=K^*\}\,.$$

If we prove that $L=[0,1]$, then $K=K^*$, $J=\emptyset$ and the proof is finished. From the monotone character of the semiflow and since $\mathscr{O}(\omega,a(\omega))=K^*$, it is immediate to check that if $0<\alpha\in L$ then $[0,\alpha]\subset L$.

Next we show that L is closed, that is, if $[0,\alpha)\subset L$ then $\alpha\in L$. From Hypothesis (H5), $\{\tau(t,\omega,x_\alpha)\mid t\geq 0\}$ is uniformly stable; let $\delta(\varepsilon)>0$ be the modulus of uniform stability for $\varepsilon>0$. Thus, we take $\beta\in[0,\alpha)$ with $\mathsf{d}(x_\alpha,x_\beta)<\delta(\varepsilon)$ and we obtain $\mathsf{d}(u(t,\omega,x_\alpha),u(t,\omega,x_\beta))<\varepsilon$ for each $t\geq 0$. Moreover, $\mathscr{O}(\omega,x_\beta)=K^*$ and hence, there is a t_0 such that $\mathsf{d}(u(t,\omega,x_\beta),c(\omega{\cdot}t))<\varepsilon$ for each $t\geq t_0$. Then, we deduce that $\mathsf{d}(u(t,\omega,x_\alpha),c(\omega{\cdot}t))<2\,\varepsilon$ for each $t\geq t_0$ and $\mathscr{O}(\omega,x_\alpha)=K^*$, as claimed.

Finally, we prove that the case $L=[0,\alpha]$ with $0\leq\alpha<1$ is impossible. For each $i\in J$ we consider the continuous map

$$K\longrightarrow(0,\infty)\,,\quad (\widetilde{\omega},\widetilde{x})\mapsto\widetilde{x}_i(0)-c_i(\widetilde{\omega})(0)\,.$$

Hence, there is an $\varepsilon>0$ such that $\widetilde{x}_i(0)-c_i(\widetilde{\omega})(0)\geq\varepsilon>0$ for each $i\in J$ and $(\widetilde{\omega},\widetilde{x})\in K$. Moreover, since $(\widetilde{\omega}{\cdot}s,u(s,\widetilde{\omega},\widetilde{x}))\in K$, $u_i(s,\widetilde{\omega},\widetilde{x})(0)=\widetilde{x}_i(s)$ for each $s\leq 0$ because K admits a flow extension, and $c_i(\widetilde{\omega})(s)=c_i(\widetilde{\omega}{\cdot}s)(0)$, we deduce that $\widetilde{x}_i(s)-c_i(\widetilde{\omega})(s)\geq\varepsilon>0$ for each $s\in(-\infty,0]$ and $(\widetilde{\omega},\widetilde{x})\in K$.

As before, let $\delta(\varepsilon/4)>0$ be the modulus of uniform stability for the trajectory $\{\tau(t,\omega,x_\alpha)\mid t\geq 0\}$ and take $\alpha<\gamma\leq 1$ with $\mathsf{d}(x_\alpha,x_\gamma)<\delta(\varepsilon/4)$. For each $t\geq 0$ we have $\|u(t,\omega,x_\alpha)(0)-u(t,\omega,x_\gamma)(0)\|<\varepsilon/4$ and, as above, from $\mathscr{O}(\omega,x_\alpha)=K^*$ we deduce that there is a $t_0\geq 0$ such that $\|u(t,\omega,x_\alpha)(0)-c(\omega{\cdot}t)(0)\|<\varepsilon/4$ for each $t\geq t_0$. Consequently, for each $t\geq t_0$

$$\|u(t,\omega,x_\gamma)(0)-c(\omega{\cdot}t)(0)\|<\frac{\varepsilon}{2}\,. \tag{4.10}$$

Let $(\widetilde{\omega},\widetilde{x})\in\mathscr{O}(\omega,x_\gamma)$, i.e. $(\widetilde{\omega},\widetilde{x})=\lim_{n\to\infty}(\omega{\cdot}t_n,u(t_n,\omega,x_\gamma))$ for some $t_n\uparrow\infty$. The monotonicity and $c(\omega)\leq x_\gamma$ imply that $c(\omega{\cdot}t_n)\leq u(t_n,\omega,x_\gamma)$, which yields to $c(\widetilde{\omega})\leq\widetilde{x}$. Moreover, from $c(\omega)\leq x_\gamma\leq x$ we have $c(\omega{\cdot}t_n)\leq u(t_n,\omega,x_\gamma)\leq u(t_n,\omega,x)$ and hence from (4.9) we deduce that $c_i(\omega{\cdot}t_n)=u_i(t_n,\omega,x_\gamma)$ for each $i\notin J$. This yields to $c_i(\widetilde{\omega})=\widetilde{x}_i$ for $i\notin J$. Given any $(\widetilde{\omega},z)\in K$, from (4.9) we know that $c_i(\widetilde{\omega})=z_i$ for each $i\notin J$ and, as shown above,

$$z_i(s)-c_i(\widetilde{\omega})(s)\geq\varepsilon\quad\text{ for each } s\in(-\infty,0]\ \text{ and } i\in J\,. \tag{4.11}$$

From (4.10) there is an n_0 such that $0\leq u_i(t_n,\omega,x_\gamma)(0)-c_i(\omega{\cdot}t_n)(0)<\varepsilon/2$ for each $n\geq n_0$, and consequently, $0\leq\widetilde{x}_i(0)-c_i(\widetilde{\omega})(0)\leq\varepsilon/2$. As before, since this is true for each $(\widetilde{\omega},\widetilde{x})\in\mathscr{O}(\omega,x_\gamma)$ admitting a flow extension, we deduce that $0\leq\widetilde{x}_i(s)-c_i(\widetilde{\omega})(s)\leq\varepsilon/2$ for each $s\in(-\infty,0]$ and $i\in J$, which combined with (4.11) and $c_i(\widetilde{\omega})=\widetilde{x}_i=z_i$ for $i\notin J$ show that $c(\widetilde{\omega})\leq\widetilde{x}\leq z$. Since this holds for each $(\widetilde{\omega},z)\in K$, the definition of a provides $c(\widetilde{\omega})\leq\widetilde{x}\leq a(\widetilde{\omega})$. From (4.8) we know that $\mathscr{O}(\widetilde{\omega},a(\widetilde{\omega}))=K^*$ and therefore $\mathscr{O}(\widetilde{\omega},\widetilde{x})=K^*\subseteq\mathscr{O}(\omega,x_\gamma)$. Once more from (H5) and Proposition 4.5 we conclude that $\mathscr{O}(\widetilde{\omega},\widetilde{x}) = \mathscr{O}(\omega,x_\gamma) = K^*$, a contradiction. Therefore, $L = [0,1]$, i.e. $J = \emptyset$ and $\mathscr{O}(\omega_0,x_0) = K^*$, as stated.

4.3 Compartmental systems

Compartmental systems have been widely used as a mathematical model for the study of the dynamical behavior of many processes in biological and physical sciences which depend on local mass balance conditions (see Jacquez and Simon [28, 29] for a review of compartmental systems with or without delay, Györi [22], Györi and Eller [21] and Wu and Freedman [89]).

In this subsection, we apply the previous result, that is, the 1-covering property of omega limit sets, to show that the solutions of a compartmental system given by a monotone FDE with infinite delay are asymptotically of the same type as the transport functions.

Firstly, we introduce the model with which we are going to deal as well as some notation. Let us suppose that we have a system formed by m compartments $C_1,\dots,C_m$, denote by C_0 the environment surrounding the system, and by $z_i(t)$ the amount of material within compartment C_i at time t for each $i \in \{1,\dots,m\}$. Material flows from compartment C_j into compartment C_i through a pipe P_{ij} having a transit time distribution given by a positive regular Borel measure μ_{ij} with finite total variation $\mu_{ij}(-\infty,0] = 1$, for each $i,\, j \in \{1,\dots,m\}$. Let $\widetilde{g}_{ij} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$ be the so-called *transport function* determining the volume of material flowing from C_j to C_i given in terms of the time t and the value of z_j in t for $i \in \{0,\dots,m\}$, $j \in \{1,\dots,m\}$. For each $i \in \{1,\dots,m\}$, we will assume that there exists an incoming flow of material $\widetilde{I}_i$ from the environment into the compartment C_i which only depends on time.

Thus, taking into account that the change of the amount of material of any compartment C_i, $1 \le i \le m$, equals to the difference between the amount of total influx into and total outflux out of C_i, we obtain a model governed by the the following system of infinite delay differential equations:

$$z_i'(t) = -\widetilde{g}_{0i}(t,z_i(t)) - \sum_{j=1}^{n} \widetilde{g}_{ji}(t,z_i(t)) + \sum_{j=1}^{n} \int_{-\infty}^{0} \widetilde{g}_{ij}(t+s,z_j(t+s))\,d\mu_{ij}(s) + \widetilde{I}_i(t), \tag{4.12}$$

$i \in \{1,\dots,m\}$. For simplicity, we denote $\widetilde{g}_{i0} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$, $(t,v) \mapsto \widetilde{I}_i(t)$ for each $i \in \{1,\dots,m\}$ and let $\widetilde{g} = (\widetilde{g}_{ij})_{i,j} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{m(m+2)}$. We will assume that

(C1) $\widetilde{g}$ is C^1-*admissible*, i.e. $\widetilde{g}$ is C^1 in its second variable and $\widetilde{g}$, $\frac{\partial}{\partial v}\widetilde{g}$ are uniformly continuous and bounded on $\mathbb{R}\times\{v_0\}$ for all $v_0 \in \mathbb{R}$;
(C2) all the component of $\widetilde{g}$ are monotone in the second variable, and $\widetilde{g}_{ij}(t,0) = 0$ for each $t \in \mathbb{R}$, $i \in \{0,\dots,m\}$ and $j \in \{1,\dots,m\}$;
(C3) $\widetilde{g}$ is a recurrent function, i.e. its *hull* is minimal;
(C4) $\mu_{ij}(-\infty,0] = 1$ and $\int_{-\infty}^{0} |s|\,d\mu_{ij}(s) < \infty$;

As usual, we include the non-autonomous system (4.12) into a family of non-autonomous FDEs with infinite delay of the form $(4.3)_\omega$ as follows.

Let Ω be the *hull* of $\widetilde{g}$, namely, the closure of the set of mappings $\{\widetilde{g}_t \mid t \in \mathbb{R}\}$, with $\widetilde{g}_t(s,v) = \widetilde{g}(t+s,v)$, $(s,v) \in \mathbb{R}^2$, with the topology of uniform convergence on compact sets, which from (C1) is a compact metric space. Let $(\Omega,\sigma,\mathbb{R})$ be the continuous flow defined on Ω by translation, $\sigma : \mathbb{R}\times\Omega \to \Omega$, $(t,\omega) \mapsto \omega{\cdot}t$, with $\omega{\cdot}t(s,v) = \omega(t+s,v)$. By hypothesis (C3), the flow $(\Omega,\sigma,\mathbb{R})$ is minimal. In addition, if $\widetilde{g}$ is almost periodic (resp. almost automorphic) the flow will be almost periodic (resp. almost automorphic). Notice that these two cases are included in our formulation.

Let $g : \Omega\times\mathbb{R} \to \mathbb{R}^{m(m+2)}$, $(\omega,v) \mapsto \omega(0,v)$, continuous on $\Omega\times\mathbb{R}$ and denote $g = (g_{ij})_{i,j}$. It is easy to check that, for all $\omega = (\omega_{ij})_{i,j} \in \Omega$ and all $i \in \{1,\dots,m\}$, ω_{i0} is a function dependent only on t; thus, we can define $I_i = \omega_{i0}$, $i \in \{1,\dots,m\}$. Let $F : \Omega\times BU \to \mathbb{R}^m$ be the map defined by

$$F_i(\omega,x) = -g_{0i}(\omega,x_i(0)) - \sum_{j=1}^m g_{ji}(\omega,x_i(0)) + \sum_{j=1}^m \int_{-\infty}^0 g_{ij}(\omega{\cdot}s,x_j(s))\,d\mu_{ij}(s) + I_i(\omega),$$

for $(\omega,x) \in \Omega\times BU$ and $i \in \{1,\dots,m\}$. Hence, the family

$$z'(t) = F(\omega{\cdot}t,z_t)\,,\quad t\geq 0\,,\ \omega\in\Omega\,, \qquad (4.13)_\omega$$

includes system (4.12) when $\omega = \widetilde{g}$.

It is easy to check that this family satisfies hypotheses (F1)-(F4). Next we will study some cases in which hypothesis (F5) is satisfied. In order to do this, we define $M\colon \Omega\times BU \to \mathbb{R}$, the *total mass* of the system $(4.13)_\omega$ as

$$M(\omega,x) = \sum_{i=1}^m x_i(0) + \sum_{i=1}^m\sum_{j=1}^m \int_{-\infty}^0 \left(\int_s^0 g_{ji}(\omega{\cdot}r,x_i(r))\,dr\right) d\mu_{ji}(s)\,, \qquad (4.14)$$

for all $\omega \in \Omega$ and $x \in BU$, which is well defined from condition (C4). The next result shows the continuity properties of M and its variation along the flow.

Proposition 4.13. *The total mass M is a continuous function on all the sets of the form $\Omega\times B_r$ with $r>0$ for the product metric topology. Moreover, for each $t\geq 0$*

$$M(\omega{\cdot}t,z_t(\omega,x)) = M(\omega,x) + \sum_{i=1}^m \int_0^t [I_i(\omega{\cdot}s) - g_{0i}(\omega{\cdot}s,z_i(s,\omega,x))]\,ds\,. \qquad (4.15)$$

Theorem 4.14. *Under Assumptions* (C1)-(C4)*, if there exists $\omega_0 \in \Omega$ such that* $(4.13)_{\omega_0}$ *has a bounded solution, then all solutions of* $(4.13)_\omega$ *are bounded as well, hypothesis* (H5) *holds, and all omega-limit sets are copies of the base.*

Proof. As explained before, from hypothesis (H4) the monotone character of the semiflow is deduced, that is, for each $\omega\in\Omega$ and $x,y\in BU$ such that $x\leq y$ it holds that $u(t,\omega,x)\leq u(t,\omega,y)$ whenever they are defined. Therefore, $z_i(t,\omega,x)\leq z_i(t,\omega,y)$ for each $i=1,\dots,m$. In addition, the monotonicity of transport functions yields $g_{ij}(\omega,z_j(t,\omega,x)) \le g_{ij}(\omega,z_j(t,\omega,y))$ for each $\omega \in \Omega$. From all these inequalities, (4.14) and (4.15) we deduce that

$$\begin{aligned} 0 \le z_i(t,\omega,y) - z_i(t,\omega,x) &\le M(\omega{\cdot}t, z_t(\omega,y)) - M(\omega{\cdot}t, z_t(\omega,x)) \\ &\le M(\omega,y) - M(\omega,x)\,, \end{aligned}$$

for each $i = 1,\dots,m$ and whenever $z(t,\omega,x)$ and $z(t,\omega,y)$ are defined. Hence, from the continuity of M, given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|z(t,\omega,y) - z(t,\omega,x)\| < \varepsilon$ provided that x, $y \in B_r$, $\mathsf{d}(x,y) < \delta$ and $x \le y$. The case in which x and y are not ordered follows easily from this one. The boundedness of all solutions is an easy consequence of this fact and the continuity of the semiflow.

Let $(\omega,x) \in \Omega \times BU$ and $r' > 0$ such that $z_t(\omega,x) \in B_{r'}$ for all $t \ge 0$. Then, as above, we deduce that given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|z(t+s,\omega,x) - z(t,\omega{\cdot}s,y)\| = \|z(t,\omega{\cdot}s,z_s(\omega,x)) - z(t,\omega{\cdot}s,y)\| < \varepsilon$$

for all $t \ge 0$ whenever $y \in B_{r'}$ and $\mathsf{d}(z_s(\omega,x),y) < \delta$, which shows the uniform stability of the trajectories in $B_{r'}$ for each $r' > 0$, hypothesis (H5) holds for all $r > 0$, and Theorem 4.12 applies for all initial data, which finishes the proof. □

Concerning the solutions of the original compartmental system, we obtain the following result providing a non trivial generalization of the autonomous case, in which the asymptotically constancy of the solutions was shown (see Wu and Freedman [89]). Although the theorem is stated in the almost periodic case, similar conclusions are obtained changing almost periodicity for periodicity, almost automorphy or recurrence, that is, all solutions are asymptotically of the same type as the transport functions.

Theorem 4.15. *Under Assumptions* (C1)-(C4) *and in the almost periodic case, if there is a bounded solution of* (4.12)*, then there is at least an almost periodic solution and all the solutions are asymptotically almost periodic. For closed systems, i.e.,* $\widetilde{I}_i \equiv 0$ *and* $\widetilde{g}_{0i} \equiv 0$ *for each* $i = 1,\dots,m$*, there are infinitely many almost periodic solutions and the rest of them are asymptotically almost periodic.*

Proof. The first statement is an easy consequence of the previous theorem. We take $\omega_0 = \widetilde{g}$. The omega-limit of each solution $z(t,\omega_0,x_0)$ is a copy of the base $\mathscr{O}(\omega_0,x_0) = \{(\omega,x(\omega)) \mid \omega \in \Omega\}$ and hence, $z(t,\omega_0,x(\omega_0)) = x(\omega_0{\cdot}t)(0)$ is an almost periodic solution of (4.12) and

$$\lim_{t\to\infty} \|z(t,\omega_0,x_0) - z(t,\omega_0,x(\omega_0))\| = 0\,.$$

The statement for closed systems follows in addition from (4.15), which implies that the mass is constant along the trajectories. Hence, there are infinitely many minimal subsets because from the definition of the mass, given $c > 0$ there is an $(\omega_0,x_0) \in \Omega \times BU^+$ such that $M(\omega_0,x_0) = c$ and hence $M(\omega,x) = c$ for each $(\omega,x) \in \mathscr{O}(\omega_0,x_0)$. □

Acknowledgements The authors were partly supported by Junta de Castilla y León under project VA060A09, and Ministerio de Ciencia e Innovación under project MTM2008-00700/MTM.

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