Twist mappings with non-periodic angles

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Preliminary draft

Contents

1 Symplectic maps in the plane and in the cylinder

We will work on the plane $\mathbb{R}^2$ with cartesian coordinates (θ, r). Sometimes we will also work on the cylinder $\mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. A generic point in the cylinder will be denoted by $(\overline{\theta}, r)$ with $\overline{\theta} = \theta + 2\pi\mathbb{Z}$. The covering map

$p : \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}$, $(\theta, r) \mapsto (\overline{\theta}, r)$ is useful to lift maps from the cylinder to the universal covering $\mathbb{R}^2$.

Let us start with the plane. We work with C^k embeddings, $k \geq 1$, defined on a strip $\Sigma = \mathbb{R} \times]a, b[$, $-\infty \leq a < b \leq +\infty$. More precisely, consider a C^k map

$$f : \Sigma \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (\theta, r) \mapsto (\theta_1, r_1)$$

satisfying

(i) f is one-to-one

(ii) $\det f'(\theta, r) \neq 0 \quad \forall (\theta, r) \in \Sigma$.

The class of these maps will be denoted by $\mathcal{E}^1(\Sigma)$. A map $f \in \mathcal{E}^1(\Sigma)$ is called *symplectic* if it preserves the differential form $\omega = d\theta \wedge dr$. This means that, in the set Σ,

$$d\theta_1 \wedge dr_1 = d\theta \wedge dr. \tag{1}$$

If we express the map f in coordinates

$$\theta_1 = F(\theta, r), \quad r_1 = G(\theta, r),$$

then the condition (1) can be reformulated as

$$\det f' = \frac{\partial F}{\partial \theta}\frac{\partial G}{\partial r} - \frac{\partial F}{\partial r}\frac{\partial G}{\partial \theta} = 1 \quad \text{on } \Sigma.$$

This is the classical definition of area-preserving map.

Exercise 1 *Prove that $f \in \mathcal{E}^1(\Sigma)$ is symplectic if and only if the two conditions below hold,*

(a) *f is orientation-preserving*

(b) *for each (Lebesgue) measurable set $\Omega \subset \Sigma$, the image $\Omega_1 = f(\Omega)$ is also measurable and $\mu(\Omega) = \mu(\Omega_1)$. Here μ is the Lebesgue measure in the plane.*

Given $f \in \mathcal{E}^2(\Sigma)$ we consider the 1-form

$$\alpha = r_1 d\theta_1 - r d\theta.$$

Then $d\alpha = -d\theta_1 \wedge dr_1 + d\theta \wedge dr$ and so α is closed if and only if f is symplectic. The strip Σ is contractible and therefore closed and exact forms coincide. In particular, if f is symplectic there must exist a function H with $\alpha = dH$. After taking differentials in this identity we conclude that the converse is also true. Summing up, a map $f \in \mathcal{E}^2(\Sigma)$ is symplectic if and only if

$$dH = r_1 d\theta_1 - r d\theta \quad \text{for some } H \in C^2(\Sigma). \tag{2}$$

This identity can be expressed as

$$H_\theta = GF_\theta - r, \quad H_r = GF_r. \tag{3}$$

The equivalence between closed and exact 1-forms is no longer true in the cylinder. Consider the strip immersed in the cylinder $\overline{\Sigma} = \mathbb{T}\times]a, b[$. Since Σ is its universal covering, all 1-forms on $\overline{\Sigma}$ can be expressed as

$$\beta = A(\theta, r)d\theta + B(\theta, r)dr$$

with $A, B \in C^1(\Sigma)$ and 2π-periodic in θ. When β is closed it is possible to find a function $H = H(\theta, r)$ with $dH = \beta$. The problem is that sometimes H is not periodic in θ and so it becomes a multi-valued function when regarded in the cylinder.

Exercise 2 *Prove that exact 1-forms in the cylinder can be characterized as closed 1-forms satisfying*

$$\int_0^{2\pi} A(\theta, r_*)d\theta = 0$$

for some $r_* \in]a, b[$.

This difference between the plane and the cylinder plays a role when one tries to extend the notion of symplectic map to $\mathbb{T} \times \mathbb{R}$. Let us start with a map

$$\overline{f} : \overline{\Sigma} \subset \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}, \quad (\overline{\theta}, r) \mapsto (\overline{\theta}_1, r_1)$$

satisfying the same conditions as in the case of the plane. The class of these maps is $\mathcal{E}^1(\overline{\Sigma})$. Every $f \in \mathcal{E}^1(\overline{\Sigma})$ has a lift $f = (F, G)$ in $\mathcal{E}^1(\Sigma)$. The coordinates satisfy

$$F(\theta + 2\pi, r) = F(\theta, r) + 2n\pi, \quad G(\theta + 2\pi, r) = G(\theta, r).$$

In principle n could be any integer but since our map is an orientation-preserving embedding it can only take the values $n = -1$ or $n = 1$. We say that $\overline{f}$ is symplectic when its lift is symplectic as a map of the plane. Notice that, up to an additive constant $2N\pi$, the lift is unique and so this definition is all right.

Exercise 3 *Extend exercise 1 to the cylinder using the measure transported from the plane via the covering map,*

$$\mu_{\mathbb{T}\times\mathbb{R}}(\overline{A}) = \mu_{\mathbb{R}^2}(p^{-1}(\overline{A}) \cap ([0, 2\pi] \times \mathbb{R})).$$

We say that $\overline{f} \in \mathcal{E}^2(\overline{\Sigma})$ is *exact symplectic* if there is a function $H \in C^2(\overline{\Sigma})$, and hence 2π-periodic in θ, such that

$$dH = r_1 d\theta_1 - r d\theta.$$

Just taking differentials in this identity we observe that exact symplectic maps are always symplectic. Since not all closed forms are exact in the cylinder, there are symplectic maps which are not exact.

Exercise 4 *Prove that $\overline{f} \in \mathcal{E}^2(\overline{\Sigma})$ is exact symplectic if and only if it is symplectic and*

$$\int_0^{2\pi} G(\theta, r_*) \frac{\partial F}{\partial \theta}(\theta, r_*) d\theta = 2\pi r_*$$

for some $r_* \in]a, b[$.

The notion of exact symplectic map can also be characterized in terms of measure theory. Given an arbitrary Jordan curve $\Gamma \subset \overline{\Sigma}$ which is C^1, regular and non-contractible, the image $\Gamma_1 = \overline{f}(\Gamma) \subset \mathbb{T} \times \mathbb{R}$ is another Jordan curve enjoying the same properties. Let us fix some $r_0 \in \mathbb{R}$ so that $\Gamma \cup \Gamma_1 \subset \{r > r_0\}$ and let A and A_1 denote the bounded components of $\{r > r_0\} \setminus \Gamma$ and $\{r > r_0\} \setminus \Gamma_1$. Then, if $\overline{f}$ is exact symplectic, $\mu(A) = \mu(A_1)$.

the regions between Γ_1 and Γ have equal positive and negative measure.

Exercise 5 *Prove that the previous property is a characterization of exact symplectic maps.*

To illustrate the above definitions we consider some simple maps in the cylinder and the corresponding lifts in the plane.

Example 1: $f(\theta, r) = (\theta + \omega, r)$ for fixed $\omega \in]0, 2\pi[$.

In the plane this map is a translation in the horizontal direction. It can be seen as the lift of a rotation

From $r_1 d\theta_1 - r d\theta = r d(\theta + \omega) - r d\theta = 0$ we deduce that the condition (2) holds with $H \equiv 0$. Hence, rotations are exact symplectic maps.

Example 2: $f(\theta, r) = (\theta, r + \lambda)$ for fixed $\lambda \in \mathbb{R} \setminus \{0\}$.

This map can be interpreted as a vertical translation.

Now, $r_1 d\theta_1 - r d\theta = (r + \lambda) d\theta - r d\theta = \lambda d\theta$. In the plane the condition (2) holds with $H(\theta, r) = \lambda\theta$. The differential form $\lambda d\theta$ is not exact in the cylinder and so $\overline{f}$ is symplectic but not exact.

Example 3: $f(\theta, r) = (\theta + r, r)$.

In the cylinder it can be interpreted as a rotation with changing angular velocity, increasing with r.

From $r_1 d\theta_1 - r d\theta = r d(\theta + r) - r d\theta = r dr = d(\frac{1}{2} r^2)$ we observe that (2) holds with $H(\theta, r) = \frac{1}{2} r^2$. Hence the map is exact symplectic.

We finish this section with another characterization of exact symplectic maps. It is less standard but it is useful to suggest how to introduce a related notion in the plane.

Exercise 6 *Assume that* $\overline{f} \in \mathcal{E}^2(\overline{\Sigma})$ *is symplectic and* $H \in C^2(\Sigma)$ *is such that* $dH = r_1 d\theta_1 - r d\theta$. *Prove that the three conditions below are equivalent:*

(i) $\overline{f}$ *is exact symplectic*

(ii) H *is* 2π*-periodic in* θ

(iii) H *is bounded on each strip* $\mathbb{R} \times [A, B]$ *with* $a < A < B < b$.

Let us consider now a general map $f \in \mathcal{E}^2(\Sigma)$, possibly not 2π-periodic in θ. We say that f is *E-symplectic* if there exists a function $H \in C^2(\Sigma)$ satisfying

$$dH = r_1 d\theta_1 - r d\theta$$

and

$$\sup\{|H(\theta, r)| : \ \theta \in \mathbb{R}, \ A \le r \le B\} < \infty$$

for each A, B with $a < A < B < b$.

As an example consider the map

$$\theta_1 = \theta + r, \quad r_1 = r + a + b\sin(\theta + r) + c\sin\sqrt{2}(\theta + r)$$

with $a, b, c \in \mathbb{R}$. In this case $G(\theta, r)$ is not 2π-periodic but in the plane it satisfies (2) with

$$H(\theta, r) = \frac{1}{2}r^2 + a(\theta + r) - b\cos(\theta + r) - \frac{c}{\sqrt{2}}\cos\sqrt{2}(\theta + r).$$

Therefore f is E-symplectic when $a = 0$. Recall that θ is an unbounded variable.

2 The twist condition and the generating function

A map $f \in \mathcal{E}^1(\Sigma)$ has *twist* if

$$\frac{\partial F}{\partial r}(\theta, r) > 0 \quad \forall(\theta, r) \in \Sigma. \tag{4}$$

Geometrically this means that vertical segments are twisted to the right.

The "angle" θ_1 increases with r

From an analytic point of view the condition (4) is employed to solve the implicit function problem

$$\theta_1 = F(\theta, r). \tag{5}$$

In this way a function $r = R(\theta, \theta_1)$ is obtained. It is defined on the region

$$\Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \; F(\theta, a) < \theta_1 < F(\theta, b)\}$$

where

$$F(\theta, a) = \lim_{r \downarrow a} F(\theta, r), \quad F(\theta, b) = \lim_{r \uparrow b} F(\theta, r).$$

Notice that

$$-\infty \leq F(\theta, a) < F(\theta, b) \leq +\infty \;\text{ for each } \theta \in \mathbb{R}.$$

Exercise 7 *Prove that Ω is open and connected. Hint:* $\Omega = \bigcup \Omega_\epsilon$, $\Omega_\epsilon = \{(\theta, \theta_1) \in \mathbb{R}^2 : \; F(\theta, a + \epsilon) < \theta_1 < F(\theta, b - \epsilon)\}$.

The function R is in $C^1(\Omega)$ and, by implicit differentiation,

$$F_\theta \circ \mathcal{R} + (F_r \circ \mathcal{R}) R_\theta = 0, \quad (F_r \circ \mathcal{R}) R_{\theta_1} = 1, \tag{6}$$

where $\mathcal{R}(\theta, \theta_1) = (\theta, R(\theta, \theta_1))$.

The *generating function* of f is defined as

$$h(\theta, \theta_1) = -H(\theta, R(\theta, \theta_1)), \quad (\theta, \theta_1) \in \Omega,$$

where H is given by (2). Combining the identities (3), (6) and differentiating $h = -H \circ \mathcal{R}$ we obtain

$$\frac{\partial h}{\partial \theta}(\theta, \theta_1) = R(\theta, \theta_1), \;\; \frac{\partial h}{\partial \theta_1}(\theta, \theta_1) = -G(\theta, R(\theta, \theta_1)). \tag{7}$$

In a less formal language we can say that the map f given by $\theta_1 = F(\theta, r)$, $r_1 = G(\theta, r)$ is now expressed as

$$\frac{\partial h}{\partial \theta}(\theta, \theta_1) = r, \quad \frac{\partial h}{\partial \theta_1}(\theta, \theta_1) = -r_1.$$

This formula says that the map, originally defined in terms of two functions F and G, can be defined in terms of a single function, the generating function. This is reminiscent of the role played by the Hamiltonian function in the theory of Hamiltonian systems. The above formulas have also a consequence for the regularity of the generating function, because (7) implies that h is in $C^2(\Omega)$. Moreover,

$$\frac{\partial^2 h}{\partial \theta \partial \theta_1} > 0 \quad \text{in } \Omega.$$

This is a consequence of the twist condition together with (7) and (6), since

$$h_{\theta\theta_1} = R_{\theta_1} = 1/(F_r \circ \mathcal{R}) > 0.$$

Assume now that $\overline{f} \in \mathcal{E}^1(\overline{\Sigma})$ is a map in the cylinder whose coordinates satisfy

$$F(\theta + 2\pi, r) = F(\theta, r) + 2\pi, \quad G(\theta + 2\pi, r) = G(\theta, r).$$

When the lift is symplectic and has twist, the domain Ω and the function R are invariant under the translation

$$T(\theta, \theta_1) = (\theta + 2\pi, \theta_1 + 2\pi).$$

This means that $T(\Omega) = \Omega$ and $R \circ T = R$. The second identity is a consequence of the uniqueness of solution for the implicit function problem (5) and the generalized periodicity of F. The generating function is not always invariant under T. Indeed the identity $h \circ T = h$ is equivalent to

$$H(\theta + 2\pi, R(\theta, \theta_1)) = H(\theta, R(\theta, \theta_1)),$$

as can be deduced from the definition of h and the periodicity of R. For fixed θ the function $\theta_1 \mapsto R(\theta, \theta_1)$ maps the interval $]F(\theta, a), F(\theta, b)[$ onto $]a, b[$. Hence the above identity, valid for all θ and θ_1, is equivalent to

$$H(\theta + 2\pi, r) = H(\theta, r) \;\; \forall (\theta, r) \in \mathbb{R}\times]a, b[.$$

This is just the periodicity of H with respect to θ and, from the definition of h, we deduce that

$$h(\theta + 2\pi, \theta_1 + 2\pi) = h(\theta, \theta_1), \quad (\theta, \theta_1) \in \Omega \tag{8}$$

holds whenever $\overline{f}$ is exact symplectic.

To illustrate the previous notions let us go back to the example at the end of the previous section. We consider the symplectic map in $\Sigma = \mathbb{R}^2$,

$$\theta_1 = \theta + r, \quad r_1 = r + a + b \sin(\theta + r) + c \sin \sqrt{2}(\theta + r).$$

Since $\frac{\partial \theta_1}{\partial r} = 1$ the map has twist. Moreover $\Omega = \mathbb{R}^2$ and $R(\theta, \theta_1) = \theta_1 - \theta$. The generating function is

$$h(\theta, \theta_1) = -\frac{1}{2}(\theta_1 - \theta)^2 - a\theta_1 + b \cos \theta_1 + \frac{c}{\sqrt{2}} \cos \sqrt{2}\theta_1.$$

When $c = 0$ this map can be defined on the cylinder and h satisfies the periodicity condition (8) when $c = 0$ and $a = 0$.

Exercise 8 *Assume that* $h \in C^2(\mathbb{R}^2)$*,* $h = h(\theta, \theta_1)$*, is a function satisfying* $\frac{\partial^2 h}{\partial\theta\partial\theta_1} > 0$ *and, for some numbers* a*,* b*,*

$$\sup_{\theta_1 \in \mathbb{R}} \frac{\partial h}{\partial \theta}(\theta, \theta_1) \leq a < b \leq \inf_{\theta_1 \in \mathbb{R}} \frac{\partial h}{\partial \theta}(\theta, \theta_1)$$

for each $\theta \in \mathbb{R}$*. Then there exists a twist symplectic map* $f \in \mathcal{E}^1(\Sigma)$*,* $\Sigma = \mathbb{R} \times]a, b[$*, such that* h *is its generating function. Here it is understood that* h *is restricted to an appropriate domain.*

3 The variational principle

We will construct a functional such that its critical points are in correspondence with the orbits generated by symplectic twist maps. First we present a concrete example, arising in solid state physics.

3.1 The Frenkel-Kontorowa model

Let us imagine an infinite chain of atoms placed on a line, the positions of the atoms being described by bi-infinite sequences $(\theta_n)_{n \in \mathbb{Z}}$. It is assumed that every atom n is attracted by its neighbors $n - 1$ and $n + 1$, according to Hooke's law (with constant C). In addition there is a force derived from a potential $V = V(\theta)$ acting on the real line.

To find the equilibrium positions of the chain it is enough to impose that the sum of forces acting on each atom vanishes. That is,

$$C(\theta_{n-1} - \theta_n) + C(\theta_{n+1} - \theta_n) - V'(\theta_n) = 0.$$

We arrive at the second order difference equation

$$\theta_{n+1} + \theta_{n-1} - 2\theta_n = \frac{1}{C} V'(\theta_n), \ \ n \in \mathbb{Z}, \tag{9}$$

which can be seen as a discrete counterpart of the Newtonian equation $\ddot{\theta} = \frac{1}{C} V'(\theta)$.

Alternatively we can look for critical points of the potential energy

$$\Phi((\theta_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} [\frac{1}{2} C(\theta_{n+1} - \theta_n)^2 + V(\theta_n))].$$

It is straightforward to check that the conditions $\frac{\partial \Phi}{\partial \theta_n} = 0$ lead to the equation (9). Of course this computation is purely formal since typically the series defining Φ is divergent. One way to proceed rigorously is to consider finite strings $(\theta_n)_{|n|\leq N}$ and to assume that the end points are fixed and known, say $\theta_{-N} = A_{-N}$ and $\theta_N = A_N$. Then we can consider the truncated potential energy

$$\Phi_N((\theta_n)_{n<|N|}) = \sum_{-N\leq n<N} [\frac{1}{2} C(\theta_{n+1} - \theta_n)^2 + V(\theta_n))].$$

The critical points of Φ_N satisfy (9) for $|n| < N$.

Let us now assume that the potential V is in $C^2(\mathbb{R})$ and let us interpret the function

$$h(\theta, \theta_1) = -\frac{C}{2}(\theta_1 - \theta)^2 - V(\theta)$$

as the generating function of a symplectic twist map. This makes sense since $h_{\theta\theta_1} = C > 0$ and so Exercise 8 is applicable. The associated map is defined by

$$r = \frac{\partial h}{\partial \theta}(\theta, \theta_1), \ \ r_1 = -\frac{\partial h}{\partial \theta_1}(\theta, \theta_1)$$

or

$$f: \quad \theta_1 = \theta + \frac{1}{C} r + \frac{1}{C} V'(\theta), \ \ r_1 = r + V'(\theta).$$

The previous discussion leads to an interesting conclusion: given a "critical point" $(\theta_n^*)_{n\in\mathbb{Z}}$ of Φ, the sequence $\{(\theta_n^*, r_n^*)\}_{n\in\mathbb{Z}}$ with $r_n^* = C(\theta_{n+1}^* - \theta_n^*) + V'(\theta_n^*)$ is an f-orbit. The process can be reversed.

Exercise 9 *Prove that the map f defined above is E-symplectic in $\mathbb{R}^2$ when the potential is bounded. Under what conditions is there an induced exact symplectic map $\overline{f}$ in the cylinder?*

Exercise 10 *Prove that f is conjugate to the "standard map" $\theta_1 = \theta + \frac{1}{C} r$, $r_1 = r + V'(\theta_1)$. Hint: $r \mapsto r + V'(\theta)$.*

3.2 A general framework

Assume now that $f \in \mathcal{E}^1(\Sigma)$ is a twist symplectic map and let $h = h(\theta, \theta_1)$ denote its generating function. Given $N \geq 1$ consider the function

$$\Phi_N((\theta_n)_{|n|\leq N}) = \sum_{-N\leq n<N} h(\theta_n, \theta_{n+1})$$

where $\theta_{\pm N} = A_{\pm N}$ are fixed numbers lying in $]a, b[$. This function is of class C^2 on the domain

$$\Omega_N = \{(\theta_n)_{|n|<N} : \ (\theta_n, \theta_{n+1}) \in \Omega, \ -N \leq n < N\}.$$

Exercise 11 *Prove that Ω_N is an open and connected subset of $\mathbb{R}^{2N-1}$.*

Critical points of Φ_N are solutions of

$$\partial_1 h(\theta_n, \theta_{n+1}) + \partial_2 h(\theta_{n-1}, \theta_n) = 0, \ |n| < N, \ \theta_{\pm N} = A_{\pm N}. \qquad (10)$$

The sequence $(\theta_n)_{|n|\leq N}$ obtained in this way leads to a segment of f-orbit with the definition $r_N = \partial_2 h(\theta_{N-1}, \theta_N)$ and $r_n = \partial_1 h(\theta_n, \theta_{n+1})$ if $-N \leq n < N$. Actually, from the definition of the function $R(\theta, \theta_1)$ and (7) we deduce that $a < r_n < b$ if $-N \leq n < N$. This inequality also holds for $n = N$, as follows from (10). Putting together (7), (10) and the definition of R it is easy to deduce that $f(\theta_n, r_n) = (\theta_{n+1}, r_{n+1})$ if $-N \leq n < N$.

The previous process can be reversed in order to obtain critical points of Φ_N from f-orbits. Our goal is to construct complete f-orbits with certain additional properties. To this end we will prove the existence of critical points of Φ_N and let $N \to \infty$. This is clarified by the following result, valid for the general second order difference equation

$$E(\theta_{n-1}, \theta_n, \theta_{n+1}) = 0 \qquad (11)$$

where $E : S \to \mathbb{R}$ is a continuous function defined on

$$S = \{(\theta_{-1}, \theta_0, \theta_1) \in \mathbb{R}^3 : \ \delta \leq \theta_0 - \theta_{-1} \leq \Delta \text{ and } \delta \leq \theta_1 - \theta_0 \leq \Delta\}$$

with $\Delta > \delta \geq 0$.

Lemma 12 *Assume that for $N \geq N^*$ there exists a finite sequence $(\theta_n^{[N]})_{|n|\leq N}$ satisfying (11) for $|n| < N$. Moreover, assume that*

$$\lim_{N\to+\infty} \theta_{\pm N}^{[N]} = \pm\infty.$$

Then there exists a complete solution of (11). This means that there is a sequence $(\theta_n)_{n\in\mathbb{Z}}$ satisfying (11) for all $n \in \mathbb{Z}$.

Proof. Let us consider the space of sequences $\mathbb{R}^{\mathbb{Z}}$ endowed with the product topology. We recall that this space is metrizable and the associated convergence is just the convergence of each coordinate. Inside $\mathbb{R}^{\mathbb{Z}}$ we consider the space

$$K_\infty = \{\Theta = (\theta_n)_{n\in\mathbb{Z}} : \ \delta \le \theta_{n+1} - \theta_n \le \Delta \ \forall n \in \mathbb{Z}, \ |\theta_0| \le \Delta\}.$$

This space is compact because it can be viewed as a closed subset of

$$\hat{K}_\infty = \{\Theta = (\theta_n)_{n\in\mathbb{Z}} : \ \theta_n \in [n\delta - \Delta, (n+1)\Delta] \ \forall n \in \mathbb{Z}\}$$

and $\hat{K}_\infty$ is compact by Tichonoff's Theorem on the product of compact spaces. We will look for a solution of (11) lying in K_∞.

For large N we can assume $\theta_N^{[N]} > 0 > \theta_{-N}^{[N]}$ and so there exists an integer $\nu = \nu(N)$, $-N \le \nu < N$, such that $\theta_\nu^{[N]} < 0 < \theta_{\nu+1}^{[N]}$. Since $0 < \theta_{\nu+1}^{[N]} - \theta_\nu^{[N]} \le \Delta$ we conclude that

$$|\theta_\nu^{[N]}| \le \Delta. \tag{12}$$

Also we notice that

$$\lim_{N\to+\infty} [\pm N - \nu(N)] = \pm\infty. \tag{13}$$

For the sign $+$ this limit is justified using the estimates

$$\theta_N^{[N]} - \Delta \le \theta_N^{[N]} - \theta_\nu^{[N]} = \sum_{n=\nu}^{N-1} (\theta_{n+1}^{[N]} - \theta_n^{[N]}) \le (N-\nu)\Delta.$$

The case of the sign $-$ is treated similarly.

The equation (11) is autonomous and so the shifted sequence $\tilde{\theta}_n^{[N]} = \theta_{n-\nu}^{[N]}$, satisfies (11) for $|n-\nu| < N$. Next we complete the finite sequence $(\tilde{\theta}_n^{[N]})$ so that it becomes a point $\tilde{\Theta}^{[N]}$ of K_∞. A simple way to achieve this is to define

$$\tilde{\theta}_n^{[N]} = \omega(n - N - \nu) + \tilde{\theta}_N^{[N]}, \ \text{ if } n > N + \nu$$

and

$$\tilde{\theta}_n^{[N]} = \omega(n + N - \nu) + \tilde{\theta}_{-N}^{[N]}, \ \text{ if } n < -N + \nu$$

with $\omega = \frac{1}{2}(\delta + \Delta)$. Using (12) it is easy to check that the family $\tilde{\Theta}^{[N]}$ is contained in K_∞. By compactness we can extract a convergent subsequence $(\tilde{\Theta}^{[\sigma(N)]})$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is increasing. We claim that the limit $\Theta =$

$(\theta_n)_{n\in\mathbb{Z}}$ is a complete solution (11). To check this assertion we observe that for fixed n and large N, $\tilde{\theta}_k^{[N]} = \theta_{k-\nu}^{[N]}$ if $|n-k| \le 1$. This is a consequence of (13). Then

$$E(\tilde{\theta}_{n-1}^{[\sigma(N)]}, \tilde{\theta}_n^{[\sigma(N)]}, \tilde{\theta}_{n+1}^{[\sigma(N)]}) = 0, \quad N \text{ large},$$

and we can pass to the limit $(N \to +\infty)$ using the continuity of E.■

We will apply the previous lemma to the difference equation

$$\partial_1 h(\theta_n, \theta_{n+1}) + \partial_2 h(\theta_{n-1}, \theta_n) = 0, \quad \delta \le \theta_{n+1} - \theta_n \le \Delta,$$

where δ and Δ are positive numbers such that

$$F(\theta, a) < \delta < \Delta < F(\theta, b) \quad \forall \theta \in \mathbb{R}.$$

After finding solutions for $|n| < N$ as critical points of Φ_N, we will pass to the limit.

This section will be finished with a discussion of the nature of the critical points of Φ_N in the simplest instance. Consider, as in Example 3 of Section 1, the map $\theta_1 = \theta + r$, $r_1 = r$. The generating function is

$$h(\theta, \theta_1) = -\frac{1}{2}(\theta_1 - \theta)^2.$$

After fixing $A_{\pm N} \in \mathbb{R}$ we observe that the function $-\Phi_N$ is coercive on $\mathbb{R}^{2N-1}$. For if $\Theta_N = (\theta_n)_{|n|<N}$ is a generic point in $\mathbb{R}^{2N-1}$ and n_0 is an integer with $|n_0| < N$ and $|\theta_{n_0}| = \max_{|n|<N} |\theta_n| = ||\Theta_N||_\infty$, then

$$-4\Phi_N(\Theta_N) \ge \; 2\sum_{n_0 \le n < N} (\theta_{n+1} - \theta_n)^2 \; \ge \left(\; \sum_{n_0 \le n < N} |\theta_{n+1} - \theta_n| \; \right)^2.$$

The last term dominates $|\theta_{n_0} - A_N|^2$ and, since $|\theta_{n_0} - A_N| \ge ||\Theta_N||_\infty - |A_N|$, we deduce that $\Phi_N(\Theta_N) \to -\infty$ as $||\Theta_N||_\infty \to \infty$. The conditions $\frac{\partial \Phi_N}{\partial \theta_n} = 0$ lead to the discrete Dirichlet problem

$$\begin{cases} \theta_{n+1} + \theta_{n-1} - 2\theta_n = 0, \\ \theta_{-N} = A_{-N}, \; \theta_N = A_N. \end{cases}$$

This problem has the unique solution $\theta_n^{[N]} = n\omega + c$ with $\omega = \frac{A_N - A_{-N}}{2N}$, $c = \frac{1}{2}(A_N + A_{-N})$. As a consequence $\Theta_N^* = (\theta_n^{[N]})_{|n|<N}$ is the unique critical point of Φ_N and

$$\max \Phi_N = \Phi_N(\Theta_N^*).$$

Let us fix $\delta < \Delta$ and assume that $\delta < \frac{A_N - A_{-N}}{2N} < \Delta$. Then Θ_N^* is in the interior of the compact set

$$S_N = \{\Theta_N = (\theta_n)_{|n|<N} : \; \delta \le \theta_{n+1} - \theta_n \le \Delta, \text{ if } -N \le n < N\}.$$

This observation will be relevant later.

4 Existence of complete orbits

In this section we fix two positive numbers $\Delta > \delta > 0$ and consider the strip

$$S = \{(\theta, \theta_1) \in \mathbb{R}^2 : \; \delta \le \theta_1 - \theta \le \Delta\}$$

and a given function $h = h(\theta, \theta_1)$ in $C^1(S)$. Our goal is to prove the existence of a complete orbit of the difference equation

$$\partial_1 h(\theta_n, \theta_{n+1}) + \partial_2 h(\theta_{n-1}, \theta_n) = 0, \;\; n \in \mathbb{Z}. \tag{14}$$

Notice that this setting implicitly implies that the complete solution satisfies $(\theta_n, \theta_{n+1}) \in S$ for each n. The prototype of function h will be $h_*(\theta, \theta_1) = -\alpha(\theta - \theta_1)^2$ with α a positive constant. We will impose two conditions that roughly say that h is close to h_* and the strip S is sufficiently wide. Width is measured by the quotient Δ/δ.

Theorem 13 *Assume that $h \in C^1(S)$ and there are two positive numbers $\overline{\alpha}$, $\underline{\alpha}$ with $\overline{\alpha} < 2\underline{\alpha}$ and*

$$-\overline{\alpha}(\theta_1 - \theta)^2 \le h(\theta_1, \theta) \le -\underline{\alpha}(\theta_1 - \theta)^2 \;\;\; \forall (\theta, \theta_1) \in S. \tag{15}$$

Then there exists a number $\sigma \ge 1$, depending only on the quotient $\overline{\alpha}/\underline{\alpha}$, such that if $\sigma^2\delta \le \Delta$ then the equation (14) has a complete solution.

As will be seen from the proof, the number σ can be computed explicitly. To obtain results of qualitative nature it is enough to interpret $\sigma = \sigma(q)$ as an increasing function depending on $q = \overline{\alpha}/\underline{\alpha} \in [1, 2[$. This is illustrated by the following consequence on the existence of equilibria for the Frenkel-Kontorowa model.

Corollary 14 *Assume that the potential V is bounded and of class C^1. Then the equation*

$$\theta_{n+1} + \theta_{n-1} - 2\theta_n = V'(\theta_n), \;\; n \in \mathbb{Z}$$

has infinitely many complete solutions.

Proof. To prove the corollary we select the number $\sigma_0 = \sigma(3/2)$ corresponding to $\overline{\alpha}/\underline{\alpha} = 3/2$ and work on the region $S : \; \delta \le \theta_1 - \theta \le \Delta$, where $\delta > 0$ is a parameter to be adjusted and $\Delta = \sigma_0^2 \delta$. Our equation is just (14) for

$$h(\theta, \theta_1) = -\frac{1}{2}(\theta - \theta_1)^2 + V(\theta).$$

Moreover, if $(\theta, \theta_1) \in S$,

$$h(\theta, \theta_1) \leq -\frac{1}{2}(\theta - \theta_1)^2 + ||V||_\infty \leq -\frac{1}{2}(\theta - \theta_1)^2 + \frac{||V||_\infty}{\delta^2}(\theta - \theta_1)^2.$$

A similar lower estimate can be obtained to show that the condition (15) holds with

$$\underline{\alpha} = \frac{1}{2} - \frac{||V||_\infty}{\delta^2}, \;\; \overline{\alpha} = \frac{1}{2} + \frac{||V||_\infty}{\sigma_0^4 \delta^2}.$$

For large δ the inequality $\overline{\alpha}/\underline{\alpha} \leq 3/2$ holds and so the constant $\sigma = \sigma(\overline{\alpha}/\underline{\alpha})$ given by the theorem satisfies $\sigma \leq \sigma_0$. Then $\sigma^2\delta \leq \sigma_0^2\delta = \Delta$ and the theorem is applicable. This shows the existence of an equilibrium for the Frenkel-Kontorowa model $(\theta_n^\delta)_{n\in\mathbb{Z}}$ with $\delta \leq \theta_{n+1} - \theta_n \leq \sigma^2\delta$, $n \in \mathbb{Z}$. Letting $\delta \to +\infty$ we obtain infinitely many equilibria.■

Later we will present other applications of the theorem or of some variant of it. In all cases h will be the generating function of a twist symplectic map f. Indeed the condition (15) automatically implies that f is E-symplectic.

Proof of Theorem 13. For each $N \geq 3$ we select two numbers $A_{\pm N}$ satisfying

$$A_{-N} = -A_N, \;\; N\delta \leq A_N \leq N\Delta, \tag{16}$$

and consider the subset of $\mathbb{R}^{2N-1}$,

$$S_N = \{\Theta_N = (\theta_n)_{|n|\leq N} : \; \delta \leq \theta_{n+1} - \theta_n \leq \Delta \text{ for each } n = -N, \ldots, N-1\}$$

with the convention $\theta_{\pm N} = A_{\pm N}$. This set is non-empty since it contains at least the point $(\frac{n}{N} A_N)$. It is easily proved that S_N is closed and contained in the ball $||\Theta_N||_\infty \leq A_N$ and, since we are in finite dimension, we can deduce that this set is compact. The continuous function

$$\Phi_N : S_N \to \mathbb{R}, \;\; \Phi_N(\Theta_N) = \sum_{-N\leq n<N} h(\theta_n, \theta_{n+1})$$

reaches its maximum at some point $\Theta_N^* \in S_N$,

$$\Phi_N(\Theta_N^*) = \max_{S_N} \Phi_N.$$

We will prove that, for an appropriate choice of the sequence A_N, the point Θ_N^* is in the interior of S_N. Hence this is a critical point of Φ_N that can be also interpreted as a solution of (10). Finally we can apply Lemma 12 to complete the proof. From now on we will concentrate on the claim

$$\Theta_N^* \in \text{int}(S_N). \tag{17}$$

To this end we make a couple of observations on the configuration of the atoms of Θ_N^*.

(i) *There exists* $L > 1$ *such that*

$$\frac{1}{L}(\theta_{n+1}^* - \theta_n^*) \leq \theta_{n+2}^* - \theta_{n+1}^* \leq L(\theta_{n+1}^* - \theta_n^*)$$

for each $n = -N, \dots, N-2$. *Moreover* L *only depends on the quotient* $\overline{\alpha}/\underline{\alpha}$.

To prove this assertion we modify Θ_N^* replacing θ_{n+1}^* by the mid-point between θ_n^* and θ_{n+2}^*; that is,

$$\hat{\Theta}_N = (\hat{\theta}_n)_{|n| \leq N}, \;\; \hat{\theta}_m = \theta_m^* \text{ if } m \neq n+1, \;\; \hat{\theta}_{n+1} = \frac{1}{2}(\theta_n^* + \theta_{n+2}^*).$$

The new point $\hat{\Theta}_N$ also belongs to S_N. Indeed

$$\hat{\theta}_{n+2} - \hat{\theta}_{n+1} = \hat{\theta}_{n+1} - \hat{\theta}_n = \frac{1}{2}(\theta_{n+2}^* - \theta_n^*) = \frac{1}{2}(\theta_{n+2}^* - \theta_{n+1}^*) + \frac{1}{2}(\theta_{n+1}^* - \theta_n^*)$$

and these differences remain between δ and Δ. The maximizing property of Θ_N^* implies that $\Phi_N(\Theta_N^*) \geq \Phi_N(\hat{\Theta}_N)$, leading to

$$h(\theta_n^*, \theta_{n+1}^*) + h(\theta_{n+1}^*, \theta_{n+2}^*) \geq h(\theta_n^*, \hat{\theta}_{n+1}) + h(\hat{\theta}_{n+1}, \theta_{n+2}^*).$$

As a consequence

$$-\underline{\alpha}[(\theta_{n+1}^* - \theta_n^*)^2 + (\theta_{n+2}^* - \theta_{n+1}^*)^2] \geq -\overline{\alpha}\frac{(\theta_{n+2}^* - \theta_n^*)^2}{2}.$$

Assume that $\ell := \frac{\theta_{n+2}^* - \theta_{n+1}^*}{\theta_{n+1}^* - \theta_n^*} \geq 1$, otherwise we would define ℓ as the inverse fraction. Using that $\theta_{n+2}^* - \theta_n^* = (1+\ell)(\theta_{n+1}^* - \theta_n^*)$ we are lead to the inequality

$$\varphi(\ell) := \frac{2(1+\ell^2)}{(1+\ell)^2} \leq \overline{\alpha}/\underline{\alpha}.$$

The function $\varphi : [1, +\infty[\to [1, 2[$ is an increasing homeomorphism and so $\ell \leq \varphi^{-1}(\overline{\alpha}/\underline{\alpha})$. This implies that (i) holds with $L = \varphi^{-1}(\overline{\alpha}/\underline{\alpha})$. Notice that at this point we are using $\overline{\alpha} < 2\underline{\alpha}$.

(ii) *There exists* $\sigma > 1$ *such that*

$$\frac{\Delta^*}{\delta^*} \le \sigma,$$

where $\Delta^* = \max_{-N\le n<N}(\theta^*_{n+1} - \theta^*_n)$ *and* $\delta^* = \min_{-N\le n<N}(\theta^*_{n+1} - \theta^*_n)$. *Moreover* σ *only depends on the quotient* $\overline{\alpha}/\underline{\alpha}$.

Let us assume that $\Delta^* = \theta^*_{M+1} - \theta^*_M$ and $\delta^* = \theta^*_{m+1} - \theta^*_m$ with $m, M \in \{-N, \ldots, N-1\}$. If $|m-M| \le 1$ we can apply the previous step and deduce that $\Delta^*/\delta^* \le L$. From now on we assume that $|m-M| \ge 2$, say $M \ge m+2$.

We modify Θ^*_N in a new way, after eliminating θ^*_{m+1} a new atom is inserted between θ^*_M and θ^*_{M+1}. Let $\tilde\Theta_N = (\tilde\theta_n)_{|n|<N}$ be defined as $\tilde\theta_n = \theta^*_n$ if $n \le m$ or $n > M$, $\tilde\theta_n = \theta^*_{n+1}$ if $m < n < M$ and $\tilde\theta_M = \frac{1}{2}(\theta^*_M + \theta^*_{M+1})$. We prove that $\tilde\Theta_N \in S_N$ as soon as $\Delta^*/\delta^* \ge L+1$, where L is given by step (i). Actually,

$$\tilde\theta_{m+1} - \tilde\theta_m = \theta^*_{m+2} - \theta^*_m \le (L+1)\delta^* \le \Delta^* \le \Delta$$

and

$$\tilde\theta_{M+1} - \tilde\theta_M = \tilde\theta_M - \tilde\theta_{M-1} = \frac{\Delta^*}{2} \ge \frac{L+1}{2}\delta^* \ge \delta^* \ge \delta,$$

so that $\tilde\Theta_N \in S_N$. Then from $\Phi_N(\Theta^*_N) \ge \Phi_N(\tilde\Theta_N)$ we deduce that

$$\begin{aligned} &h(\theta^*_m, \theta^*_{m+1}) + h(\theta^*_{m+1}, \theta^*_{m+2}) + h(\theta^*_M, \theta^*_{M+1}) \\ &\ge h(\theta^*_m, \theta^*_{m+2}) + h(\theta^*_M, \tilde\theta_M) + h(\tilde\theta_M, \theta^*_{M+1}). \end{aligned}$$

Hence

$$\begin{aligned} -\underline{\alpha}[(\theta^*_m - \theta^*_{m+1})^2 + (\theta^*_{m+1} - \theta^*_{m+2})^2 + (\theta^*_M - \theta^*_{M+1})^2] \\ \ge -\overline{\alpha}[(\theta^*_m - \theta^*_{m+2})^2 + 2(\tilde\theta_M - \theta^*_M)^2] \end{aligned}$$

and, using again (i), we are lead to $\psi_L(\Delta^*/\delta^*) \le \overline{\alpha}/\underline{\alpha}$, where the function ψ_L is defined as

$$\psi_L(q) = \frac{1 + L^{-2} + q^2}{(1+L)^2 + \frac{1}{2}q^2}.$$

This function is strictly increasing on the interval $[1+L,\infty[$ and $\psi_L(1+L) < 1$, $\psi_L(\infty) = 2$. Hence the inequality $\psi_L(\Delta^*/\delta^*) \le \overline{\alpha}/\underline{\alpha}$ is equivalent to $\Delta^*/\delta^* \le \psi_L^{-1}(\overline{\alpha}/\underline{\alpha})$ and we have proved (ii) with

$$\sigma = \max\{1+L, \psi_L^{-1}(\overline{\alpha}/\underline{\alpha})\}.$$

Now that we have shown (ii) we can complete the proof. Define

$$A_N = \frac{1}{2}(\sigma^{-1}\Delta + \sigma\delta)N.$$

From the assumption $\sigma^2\delta < \Delta$ we observe that $\delta < \sigma\delta < \sigma^{-1}\Delta < \Delta$ and A_N is the mid point of the interval $[\sigma\delta, \sigma^{-1}\Delta]$. This implies that (16) holds. We are going to prove that for this choice of the sequence $\{A_N\}$ the claim (17) holds. By contradiction assume that $\Delta^* = \Delta$ or $\delta^* = \delta$. Then either $\delta^* \ge \frac{1}{\sigma}\Delta$ or $\Delta^* \le \sigma\delta$. To fix ideas let us consider the first case $\Delta^* = \Delta$, $\delta^* \ge \frac{1}{\sigma}\Delta$. Then

$$2A_N = \sum_{n=-N}^{N-1} (\theta^*_{n+1} - \theta^*_n) \ge \frac{2N\Delta}{\sigma}$$

and this contradicts the definition of A_N. The case $\delta^* = \delta$ is treated similarly.■

Exercise 15 *Show that the previous proof allows to compute* $\sigma = \sigma(\overline{\alpha}/\underline{\alpha})$ *explicitly. Hint:* $\sigma = \frac{4}{3}\sqrt{\frac{85}{3}}$ *if* $\overline{\alpha}/\underline{\alpha} = 5/4$.

Exercise 16 *Compute two numbers* δ *and* Δ *such that the equation*

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} = \sin\theta_n + \cos(\sqrt{2}\theta_n), \quad n \in \mathbb{Z}$$

has a solution lying on $\delta \le \theta_{n+1} - \theta_n \le \Delta$.

Exercise 17 *Prove that the conclusion of Theorem 13 still holds when the condition (15) is replaced by*

$$-\overline{\alpha}(\theta_1 - \theta)^k \le h(\theta_1, \theta) \le -\underline{\alpha}(\theta_1 - \theta)^k \quad \forall(\theta, \theta_1) \in S \tag{18}$$

with $k > 1$ *and* $\overline{\alpha} < 2^{k-1}\underline{\alpha}$. *Hint:* $\varphi(\ell) = \frac{2^{k-1}(1+\ell^k)}{(1+\ell)^2}$, $\psi_L(q) = \frac{1+L^{-k}+q^k}{(1+L)^k+2^{1-k}q^k}$.

Exercise 18 *Prove that the conclusion of Theorem 13 also holds when the condition (15) is replaced by*

$$-\overline{\alpha}(\theta_1 - \theta)^{-k} \le h(\theta_1, \theta) \le -\underline{\alpha}(\theta_1 - \theta)^{-k} \quad \forall(\theta, \theta_1) \in S \tag{19}$$

with $k > 1$ *and* $\overline{\alpha} < \frac{1+\sqrt{1+2^{1-\frac{1}{k}}}}{2}\underline{\alpha}$. *Hint:* $\varphi(\ell) = \frac{(1+\ell^{-k})(1+\ell)^k}{2^{k+1}} \ge \frac{1}{2^{k+1}}\ell^k$, $\psi_L(q) = \frac{1+L^{-k}+q^{-k}}{(1+L^{-1})^{-k}+2^{1+k}q^{-k}}$, $\psi_L(\infty) \ge (1+L^{-1})^k$.

In the applications of Theorem 13 or the variants given by the previous exercises we must know how to compute h or at least how to estimate it in order to verify (15), (18) or (19). The next two sections are devoted to the computation of generating functions in two interesting mechanical situations.

5 The action functional of a Newtonian equation

Consider the differential equation

$$\ddot{x} = -U_x(t, x), \quad t \in [0, 1], \ x \in \mathbb{R} \tag{20}$$

where the potential $U : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and has two partial derivatives with respect to x, U_x and U_{xx}, which are also continuous functions in (t, x). It will be assumed that the Cauchy problem is globally well posed. This can be guaranteed if U_x has linear growth, that is

$$|U_x(t, x)| \le A|x| + B, \quad (t, x) \in [0, 1] \times \mathbb{R},$$

for some $A, B > 0$. Given $x_0, v_0 \in \mathbb{R}$, the solution satisfying $x(0) = x_0$, $\dot{x}(0) = v_0$, will be denoted by $x(t; x_0, v_0)$. If we interpret these initial conditions as coordinates, say $\theta = x_0$ and $r = v_0$, then we can define the Poincaré map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad \theta_1 = x(1; \theta, r), \quad r_1 = \dot{x}(1; \theta, r).$$

The classical theorems on the Cauchy problem[1] imply that f is a C^1-diffeomorphism. Moreover f is symplectic. This can be justified using Liouville's theorem on Hamiltonian flows but we will prove it in a different way. Consider the Lagrangian function

$$L(t, \theta, r) = \frac{1}{2}\dot{x}(t; \theta, r)^2 - U(t, x(t; \theta, r))$$

and its time average

$$H(\theta, r) = \int_0^1 L(t, \theta, r) dt.$$

This function is of class C^1 with partial derivatives

$$H_\theta = \int_0^1 L_\theta dt = \int_0^1 \{\dot{x}\frac{\partial \dot{x}}{\partial \theta} - U_x \frac{\partial x}{\partial \theta}\} dt,$$

[1] Notice that no smoothness in t has been assumed.

$$H_r = \int_0^1 L_r dt = \int_0^1 \{\dot{x}\frac{\partial \dot{x}}{\partial r} - U_x \frac{\partial x}{\partial r}\} dt.$$

Commuting ∂_t with ∂_θ and ∂_r and integrating by parts,

$$\int_0^1 \dot{x}\frac{\partial \dot{x}}{\partial \theta} dt = [\dot{x}\frac{\partial x}{\partial \theta}]_{t=0}^{t=1} - \int_0^1 \ddot{x}\frac{\partial x}{\partial \theta} dt, \quad \int_0^1 \dot{x}\frac{\partial \dot{x}}{\partial r} dt = [\dot{x}\frac{\partial x}{\partial r}]_{t=0}^{t=1} - \int_0^1 \ddot{x}\frac{\partial x}{\partial r} dt.$$

From the equation (20) we conclude that $dH = r_1 d\theta_1 - r d\theta$ and so f is symplectic.

The map f will induce a map $\overline{f}$ on the cylinder if the potential satisfies

$$U(t, x + 2\pi) = U(t, x) + p(t), \quad (t, x) \in [0, 1] \times \mathbb{R}, \tag{21}$$

for some function $p : [0, 1] \to \mathbb{R}$. This condition of generalized periodicity implies that

$$x(t; \theta + 2\pi, r) = x(t; \theta, r) + 2\pi, \quad \dot{x}(t; \theta + 2\pi, r) = \dot{x}(t; \theta, r),$$

and letting $t = 1$, $f(\theta + 2\pi, r) = f(\theta, r) + (0, 2\pi)$. Hence $\overline{f}$ is symplectic.

Exercise 19 *Prove that the Poincaré map $\overline{f}$ associated to*

$$\ddot{x} + a \sin x = p(t)$$

is exact symplectic if and only if $\int_0^1 p(t)dt = 0$. Here $a > 0$ is a parameter and $p : [0, 1] \to \mathbb{R}$ is a given continuous function.

Exercise 20 *Assume that, instead of (21), the potential satisfies*

$$U(t, x) = B(t, x) + p(t)x$$

where $p : [0, 1] \to \mathbb{R}$ is continuous and B, B_x are bounded. Prove that the Poincaré map is E-symplectic if $\int_0^1 p(t)dt = 0$. Hint: the kinetic energy $T(t) = \frac{1}{2}\dot{x}(t)^2$ satisfies $|\dot{T}| \leq CT^{1/2}$ and $\int_0^1 p(t)x(t)dt = -\int_0^1 P(t)\dot{x}(t)dt$ with $P(t) = \int_0^t p(s)ds$.

Next we are going to discuss under what conditions the Poincaré map satisfies the twist condition. The partial derivative $\frac{\partial F}{\partial r} = \frac{\partial \theta_1}{\partial r}$ can be expressed as

$$\frac{\partial \theta_1}{\partial r} = y(1)$$

where $y(t)$ is the solution of the variational equation

$$\ddot{y} + U_{xx}(t, x(t; \theta, r))y = 0$$

with $y(0) = 0$, $\dot{y}(0) = 1$. The twist condition becomes $y(1) > 0$ and can be proved using Sturm comparison theory. Actually it holds when the potential satisfies

$$U_{xx}(t,x) < \pi^2, \quad (t,x) \in [0,1] \times \mathbb{R}. \tag{22}$$

In this case our solution $y(t)$ must oscillate less than the solution of the comparison equation $\ddot{z} + \pi^2 z = 0$, $z(t) = \sin \pi t$. This implies that $y(t) > 0$ if $t \in]0,1]$ and therefore the twist holds.

Another condition implying the twist condition is

$$(2n\pi)^2 < U_{xx}(t,x) < ((2n+1)\pi)^2, \quad (t,x) \in [0,1] \times \mathbb{R} \tag{23}$$

for some $n = 1, 2, \ldots$ Now the oscillations of $y(t)$ are between those of $z_-(t) = \sin(2n\pi t)$ and $z_+(t) = \sin((2n+1)\pi t)$. Hence $y(t)$ has exactly $2n$ zeros on $]0,1[$ and $y(1)$ is positive.

Exercise 21 *Find conditions on the parameters a and ω to guarantee that the Poincaré map associated to $\ddot{x} + \omega^2 x + a \sin x = p(t)$ is a twist symplectic map.*

Once we know that the twist condition holds, to compute the generating function we solve the equation

$$x(1; \theta, r) = \theta_1$$

and find r for given θ and θ_1. This is equivalent to finding $r = \dot{x}(0)$ where $x(t)$ is the solution of the Dirichlet problem

$$\ddot{x} = -U_x(t,x), \;\; x(0) = \theta, \; x(1) = \theta_1. \tag{24}$$

The conditions (22) or (23) are sufficient to guarantee that this problem has at most one solution. This is obviously a consequence of the twist condition. However these conditions are not sufficient for the existence of solution.

Exercise 22 *Find θ and θ_1 such that*

$$\ddot{x} + \pi^2 x - \arctan x = 0, \;\; x(0) = \theta, \; x(1) = \theta_1,$$

has no solution. Hint: multiply the equation by $\sin \pi t$ *and integrate between* $t = 0$ *and* $t = 1$.

A classical result in the theory of nonlinear boundary value problems says that (24) has a unique solution when (22) or (23) are replaced by the corresponding stronger conditions

$$U_{xx}(t,x) \le \Gamma < \pi^2, \;\; (t,x) \in [0,1] \times \mathbb{R}, \tag{25}$$

$$(2n\pi)^2 < \gamma \leq U_{xx}(t,x) \leq \Gamma < ((2n+1)\pi)^2, \quad (t,x) \in [0,1] \times \mathbb{R}, \tag{26}$$

where $n = 1, 2, \ldots$ and γ and Γ are given constants. From now on we assume that (25) or (26) are satisfied and the solution of (24) will be denoted by $\xi(t;\theta,\theta_1)$. The discussions of Section 2 on the regularity of the function $R = R(\theta,\theta_1)$ together with the standard theorems on differentiability with respect to initial conditions imply that ξ and $\dot{\xi}$ are of class C^1 in $[0,1] \times \mathbb{R}^2$. Notice that

$$\xi(t;\theta,\theta_1) = x(t;\theta,R(\theta,\theta_1)) \text{ and } R(\theta,\theta_1) = \dot{\xi}(1;\theta,\theta_1).$$

The generating function $h(\theta,\theta_1) = -H(\theta,R(\theta,\theta_1))$ is well defined on the whole plane by

$$h(\theta,\theta_1) = -\int_0^1 \left[\frac{1}{2}\dot{\xi}(t;\theta,\theta_1)^2 - U(t,\xi(t;\theta,\theta_1))\right] dt.$$

The reader who is familiar with the classical theory of Calculus of Variations will recognize this expression. Up to a sign, the generating function is the restriction of the action functional to field of extremals defined by $\xi = \xi(t;\theta,\theta_1)$. More precisely, if we consider the Sobolev space $H^1(0,1)$ and the functional

$$\mathcal{A} : H^1(0,1) \to \mathbb{R}, \;\; \mathcal{A}[x] = \int_0^1 \left[\frac{1}{2}\dot{x}(t)^2 - U(t,x(t))\right] dt,$$

then

$$h(\theta,\theta_1) = -\mathcal{A}[\xi(\cdot;\theta,\theta_1)].$$

Exercise 23 *Compute the generating function associated to $\ddot{x} + \omega^2 x = 0$ with $2n\pi < \omega < 2(n+1)\pi$ for some $n = 1, 2, \ldots$*

Finally we propose a more difficult exercise dealing with an application of Theorem 13 to the framework of this section.

Exercise 24 *Prove that the equation $\ddot{x} + a\sin x = p(t)$ with $0 < a < \pi^2$ and $p(t+1) = p(t)$, $\int_0^1 p(t)dt = 0$, has a solution satisfying $\delta \leq x(t+1) - x(t) \leq \Delta$ for some $\Delta > \delta > 0$.*

6 Impact problems and generating functions

Let us consider a particle moving on the half-line $x = x(t) \geq 0$. It satisfies a Newtonian law for $x > 0$ but at the end point $x = 0$ there is an obstacle and the particle bounces elastically.

The function $x(t)$ is a solution of the impact problem

$$\begin{cases} \ddot{x} = -U_x(t,x), \quad t \in \mathbb{R}, \\ x(t) \geq 0, \\ x(\tau) = 0 \Rightarrow \dot{x}(\tau^+) = -\dot{x}(\tau^-), \end{cases} \tag{27}$$

where $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and has two partial derivatives in x, U_x and U_{xx}. Moreover it is assumed that both derivatives are continuous with respect to both variables (t, x). In short, $U \in C^{0,2}(\mathbb{R} \times \mathbb{R})$.

The exact meaning of the above impact problem is clarified by the following definition. A *bouncing solution* of (27) is a continuous function $x : \mathbb{R} \to [0, \infty[$ and a sequence of times $(t_n)_{n\in\mathbb{Z}}$ satisfying

(i) $\inf_{n\in\mathbb{Z}}(t_{n+1} - t_n) > 0$,

(ii) $x(t_n) = 0$ and $x(t) > 0$ if $t \in]t_n, t_{n+1}[$ and $n \in \mathbb{Z}$,

(iii) the restriction of $x(t)$ to $[t_n, t_{n+1}]$ is of class C^2 and satisfies the differential equation,

(iv) $\dot{x}(t_n^+) = -\dot{x}(t_n^-)$ if $n \in \mathbb{Z}$.

This solution will be *bounded* if furthermore

(v) $\sup_{t\in\mathbb{R}} |x(t)| + \operatorname{ess\,sup}_{t\in\mathbb{R}} |\dot{x}(t)| < \infty$,

(vi) $\sup_{n\in\mathbb{Z}}(t_{n+1} - t_n) < \infty$.

Notice that $\dot{x}(t)$ is well defined for $t \neq t_n$ and so the essential supremum makes sense.

Exercise 25 *Compute the bouncing motions in a linear spring with obstacle, $U(t,x) = \frac{1}{2}x^2$. Prove that all of them are bounded.*

We will present a method for the construction of bouncing solutions. The first step will be the study of a boundary value problem.

6.1 The Dirichlet problem

Let us consider the problem

$$\ddot{x} = -U_x(t,x), \quad x(t_0) = x(t_1) = 0. \tag{28}$$

From now on we will assume that the potential satisfies two additional conditions:

(**C1**) $U_{xx}(t,x) \le 0$ for each $(t,x) \in \mathbb{R}^2$

(**C2**) There exist two numbers $c_1, c_2 \in \mathbb{R}$ and two functions $\psi, \phi \in C^2(\mathbb{R})$ such that

$$\ddot{\psi}(t) + c_1 \le U_x(t,x) \le \ddot{\phi}(t) + c_2 \text{ for each } (t,x) \in \mathbb{R}^2.$$

Moreover, $\sup_{t\in\mathbb{R}} |\dot{\psi}(t)| < \infty$.

These assumptions have strong consequences for the problem (28). First of all we present a result showing that there is a unique solution.

Lemma 26 *Assume that* (**C1**) *and* (**C2**) *hold. Then problem (28) has a unique solution on the interval* $[t_0, t_1]$ *for each* $t_1 - t_0 > 0$.

Proof. The existence will be obtained via the method of upper and lower solutions. Let $\alpha(t)$ and $\beta(t)$ be the solutions of the linear problems

$$\ddot{\alpha} = -\ddot{\phi}(t) - c_2,\ \alpha(t_0) = \alpha(t_1) = 0 \text{ and } \ddot{\beta} = -c_1 - \ddot{\psi}(t),\ \beta(t_0) = \beta(t_1) = 0.$$

From (**C2**) we deduce that $-\ddot{\beta} \le -\ddot{\alpha}$ and so, by the Maximum Principle, $\alpha(t) \ge \beta(t)$ everywhere. Moreover, using (**C2**),

$$-\ddot{\alpha}(t) = c_2 + \ddot{\phi}(t) \ge U_x(t, \alpha(t)),\ -\ddot{\beta}(t) = c_1 + \ddot{\psi}(t) \le U_x(t, \beta(t)).$$

This shows that $\alpha(t)$ and $\beta(t)$ is a couple of ordered upper and lower solutions. Therefore the problem (28) has a solution lying in $\alpha \ge x \ge \beta$.

For the uniqueness we assume that $x_1(t)$ and $x_2(t)$ are two solutions of (28). We notice that the difference $y(t) = x_1(t) - x_2(t)$ satisfies the linear problem

$$\ddot{y} + \alpha(t)y = 0,\ \ y(t_0) = y(t_1) = 0, \tag{29}$$

with $\alpha(t) = \int_0^1 U_{xx}(t, \lambda x_1(t) + (1-\lambda)x_2(t))d\lambda$. The condition (**C1**) implies that $\alpha \le 0$ everywhere. By Sturm comparison theory we deduce that (29) is disconjugate and so we arrive at a contradiction unless y vanishes identically and $x_1 = x_2$. Instead of using comparison techniques we can also prove the uniqueness just by multiplying the equation by y and integrating by parts. In this way one arrives at a contradiction with the sign of the integrals.■

Exercise 27 *Prove the Maximum Principle used above: Let* $y(t)$ *be the solution of* $-\ddot{y} = p(t),\ y(t_0) = y(t_1) = 0$ *with* $p \in C[t_0, t_1]$. *If* $p(t) \ge 0$ *and* $\int_{t_0}^{t_1} p(t)dt > 0$ *then* $y(t) > 0$ *if* $t \in]t_0, t_1[$.

To guarantee the positivity of the solution of (28) it is enough to know that the lower solution $\beta(t)$ is positive. This will occur when $t_1 - t_0$ is large enough. To check this fact it is convenient to employ the explicit formula for β given by

$$\beta(t) = \frac{c_1}{2}(t_1 - t)(t - t_0) + \frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0}(t - t_0) + \psi(t_0) - \psi(t).$$

Exercise 28 *Prove that* $\beta(t) > 0$ *for each* $t \in]t_0, t_1[$ *if* $t_1 - t_0 > \frac{8}{c}||\dot{\psi}||_\infty$*. Hint: study first the interval* $]t_0, \frac{t_0+t_1}{2}[$.

To complete our study of the Dirichlet problem we present a result on differentiability with respect to the end points. The unique solution of (28) will be denoted by $x_D(t; t_0, t_1)$.

Lemma 29 *The map* $(t; t_0, t_1) \in D \mapsto (x_D(t; t_0, t_1), \dot{x}_D(t; t_0, t_1)) \in \mathbb{R}^2$ *is of class* C^1*, where* $D = \{(t; t_0, t_1) \in \mathbb{R}^3 :\ t_1 - t_0 > 0,\ t_0 \le t \le t_1\}$.

Proof. Let $x(t; t_0, x_0, v_0)$ be the solution of

$$\ddot{x} = -U_x(t, x), \quad x(t_0) = x_0,\ \dot{x}(t_0) = v_0.$$

Since $\ddot{\psi}(t) + c_1 \le U_x(t, x) \le \ddot{\phi}(t) + c_2$, this solution is well defined and smooth for $t \in]-\infty, +\infty[$ and $(t_0, x_0, v_0) \in \mathbb{R}^3$. Let us consider the equation in v_0,

$$x(t_1; t_0, 0, v_0) = 0.$$

It is equivalent to solving (28) and so we know that it has a unique solution $v_0 = v_0(t_0, t_1)$. The Implicit Function Theorem will imply that $v_0(t_0, t_1)$ is of class C^1 if we prove that

$$\frac{\partial x}{\partial v_0}(t_1; t_0, 0, v_0) > 0 \ \text{ if } t_1 > t_0 \text{ and } v_0 \in \mathbb{R}.$$

The function $y(t) = \frac{\partial x}{\partial v_0}(t; t_0, 0, v_0)$ is a solution of the initial value problem

$$\ddot{y} + U_{xx}(t, x(t; t_0, 0, v_0))y = 0, \quad y(t_0) = 0,\ \dot{y}(t_0) = 1.$$

From (**C1**) we deduce that this linear equation is disconjugate and so $y(t_1)$ has to be positive. ■

6.2 The condition of elastic bouncing

A naive approach for the construction of bouncing solutions could consist in juxtaposing solutions of Dirichlet problems for prescribed sequences of impact times. Given a sequence $(t_n)_{n\in\mathbb{Z}}$, the function

$$x(t) := x_D(t; t_n, t_{n+1}) \ \text{ if } t \in [t_n, t_{n+1}],\ n \in \mathbb{Z} \tag{30}$$

would be the candidate for bouncing solution. Indeed, if we assume that the sequence satisfies

$$t_{n+1} - t_n > \frac{8}{c_1}||\dot{\psi}||_\infty,\ \ n \in \mathbb{Z}, \tag{31}$$

then the conditions $(i), (ii)$ and (iii) of the definition are satisfied. Here we are using the previous discussions, in particular Exercise 28. In most cases this procedure does not lead to a bouncing solution because the elasticity condition given by (iv) does not necessarily hold. Next we present a method for the construction of a judicious sequence of impacts.

Consider the function

$$h(t_0, t_1) = \int_{t_0}^{t_1} L(t, x_D(t; t_0, t_1), \dot{x}_D(t; t_0, t_1))dt, \tag{32}$$

where L is the Lagrangian function associated to $\ddot{x} = -U_x$. Namely,

$$L(t, x, \dot{x}) = \frac{1}{2}\dot{x}^2 - U(t, x) + U(t, 0).$$

We recall that the Newtonian equation can be expressed in the Lagrangian framework as

$$\partial_x L - \frac{d}{dt}(\partial_{\dot{x}} L) = 0. \tag{33}$$

The function h is of class C^1 in the region $\{(t_0, t_1) \in \mathbb{R}^2 :\ t_1 - t_0 > 0\}$. This is a consequence of Lemma 29. An integration by parts leads to

$$\partial_{t_0} h(t_0, t_1)$$

$$= -L(t_0, x_D(t_0; t_0, t_1), \dot{x}_D(t_0; t_0, t_1)) + \int_{t_0}^{t_1} \{(\partial_x L)\frac{\partial x_D}{\partial t_0} + (\partial_{\dot{x}} L)\frac{\partial \dot{x}_D}{\partial t_0}\}dt$$

$$= -\frac{1}{2}\dot{x}_D(t_0; t_0, t_1)^2 + [(\partial_{\dot{x}} L)\frac{\partial x_D}{\partial t_0}]_{t=t_0}^{t=t_1} + \int_{t_0}^{t_1} [(\partial_x L) - \frac{d}{dt}(\partial_{\dot{x}} L)]\frac{\partial x_D}{\partial t_0}dt.$$

From $x_D(t_0; t_0, t_1) = x_D(t_1; t_0, t_1) = 0$ we deduce that

$$\dot{x}_D(t_0; t_0, t_1) + \frac{\partial x_D}{\partial t_0}(t_0; t_0, t_1) = \frac{\partial x_D}{\partial t_0}(t_1; t_0, t_1) = 0.$$

These identities together with (33) imply that

$$\partial_{t_0} h(t_0, t_1) = \frac{1}{2}\dot{x}_D(t_0)^2.$$

After differentiating with respect to t_1 we arrive at

$$\partial_{t_1} h(t_0, t_1) = -\frac{1}{2}\dot{x}_D(t_1)^2.$$

Assume now that (t_n) is a sequence solving

$$\partial_{t_0} h(t_n, t_{n+1}) + \partial_{t_1} h(t_{n-1}, t_n) = 0, \quad n \in \mathbb{Z}. \tag{34}$$

If the condition (31) holds, then the function defined by (30) satisfies $\dot{x}(t_n^+)^2 = \dot{x}(t_n^-)^2$. Since $x(t)$ is non-negative, $\dot{x}(t_n^-) \leq 0 \leq \dot{x}(t_n^+)$ and so the condition (iv) holds and $x(t)$ becomes a bouncing solution.

Exercise 30 *Assume that* $(t_n)_{n\in\mathbb{Z}}$ *satisfies (31), (34) and* $\sup(t_{n+1} - t_n) < \infty$*. Moreover,* $\sup_{t\in\mathbb{R}} |\ddot{\phi}(t)| < \infty$*. Prove that* $x(t)$ *is bounded.*

6.3 A bouncing ball

Let us apply the previous discussions to a concrete model. Assume that a horizontal plate (the racket) is moving according to some prescribed protocol and a particle (the ball) is in free fall until hitting the plate, when it bounces elastically. In more analytic terms assume that the unknown $z = z(t)$ is the vertical position of the particle and the given function $w(t)$ is the position of the plate. For $z > w(t)$ the free fall is modelled by $\ddot{z} = -g$, where $g > 0$ is the gravitational constant. The elastic impact is easily modelled through the relative position $x(t) = z(t) - w(t)$,

$$x(\tau) = 0 \Rightarrow \dot{x}(\tau^+) = -\dot{x}(\tau^-).$$

Assuming that $w(t)$ is of class C^2 we find that $x(t)$ is a solution of the impact problem (27) with

$$U(t, x) = (g + \ddot{w}(t))x. \tag{35}$$

From now on we assume that the position and velocity of the plate is bounded; that is,

$$w \in C^2(\mathbb{R}) \text{ and } ||w||_\infty + ||\dot{w}||_\infty < \infty. \tag{36}$$

This is sufficient to guarantee that (**C1**) and (**C2**) are satisfied with $c_1 = c_2 = g$ and $\phi = \psi = w$.

The simplicity of the potential allows in this case explicit computations.

Exercise 31 *Compute $x_D(t;t_0,t_1)$ in terms of $w(t)$. Hint: $\beta(t)$.*

Exercise 32 *Use the previous exercise together with (32) to prove that the generating function is*

$$h(t_0,t_1) = -\frac{g^2}{24}(t_1-t_0)^3 - \frac{g}{2}(w(t_1)+w(t_0))(t_1-t_0)$$
$$+\frac{(w(t_1)-w(t_0))^2}{2(t_1-t_0)} + g\int_{t_0}^{t_1} w(t)dt - \frac{1}{2}\int_{t_0}^{t_1} \dot{w}(t)^2 dt.$$

Hint: $\int_{t_0}^{t_1} \dot{x}_D^2(t)dt = -\int_{t_0}^{t_1} x_D(t)\ddot{x}_D(t)dt$.

We do not need this exact formula for h, for our purposes it is sufficient to determine the dominant term as $t_1 - t_0 \to \infty$. From the above exercise,

$$h(t_0,t_1) = -\frac{g^2}{24}(t_1-t_0)^3 + R(t_0,t_1)$$

where

$$|R(t_0,t_1)| \le C(t_1-t_0) \text{ if } t_1 > t_0.$$

Here C is a constant depending on $||w||_\infty + ||\dot{w}||_\infty$.

We are going to apply Exercise 17 with $k=3$ and fixed numbers $\underline{\alpha}$, $\overline{\alpha}$ satisfying $\underline{\alpha} < \frac{g}{24} < \overline{\alpha}$ and $\overline{\alpha} < 4\underline{\alpha}$. Then there exists $d > 0$ such that

$$-\overline{\alpha}(t_1-t_0)^3 \le h(t_1-t_0) \le -\underline{\alpha}(t_1-t_0)^3 \quad \text{if } t_1 - t_0 \ge d.$$

The number σ associated to $\underline{\alpha}$ and $\overline{\alpha}$ can be computed in order to find complete orbits of the difference equation (34) lying in $\delta \le t_{n+1} - t_n \le \sigma^2\delta$ if $\delta \ge d$. These sequences of impact times lead to bouncing solutions. Actually, the conditions (v) and (vi) are also satisfied and so these solutions are bounded. The condition (vi) is automatic from the construction of the sequence (t_n). To verify (v) we notice that $x(t)$ is a solution of the Dirichlet problem

$$\ddot{x} = -(g + w(t)), \quad x(t_n) = x(t_{n+1}) = 0$$

and, going back to Exercise 31, we obtain a bound for $||x||_\infty + ||\dot{x}||_\infty$ in terms of g, σ, δ and $||w||_\infty + ||\dot{w}||_\infty$. Alternatively we could apply Exercise 30.

We sum up the previous discussions.

Theorem 33 *Assume that $w(t)$ satisfies (36) and consider the impact problem (27) with potential given by (35). Then there exist positive constants $\sigma > 1$ and d such that for each $\delta \ge d$ there exists a bounded solution with impact times $(t_n^\delta)_{n\in\mathbb{Z}}$ satisfying*

$$\delta \le t_{n+1}^\delta - t_n^\delta \le \sigma^2\delta, \quad n \in \mathbb{Z}.$$

References

[1] V. I. Arnold, Mathematical methods of classical mechanics, Graduate Texts in Mathematics, 60, Springer-Verlag, 1978.