PLURIPOTENTIAL THEORY AND MONGE-AMPÈRE FOLIATIONS

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The existence of regular solutions for the complex homogeneous Monge-Ampère equation with the least possible degeneracy defines a reach geometry: there exists a foliation in complex curves of the domain of existence of the solution such that the restriction of the solution to one of the leaf is a harmonic. In fact, whenever such solutions exist, their construction turns out to be based on the determination of suitable foliations. As it was shown in the seminal work of Lempert on convex domains the existence of regular pluricomplex Green functions of constant least possible degeneracy is connected to a very good behavior of the Kobayashi metric: in particular to the existence of extremal disks which coincide with the notion of stationary disks. Many of these ideas have been exploited in questions of classification and characterization of special complex manifolds.

In this framework, we will report on recent joint work with Andrea Spiro which deals with two lines of research. Following and simplifying ideas that go back to papers of Lempert for strictly convex domains and of Bland-Duchamp for domains that are small deformations of the unit ball, the first is the construction of (complete) modular data for a large family of complex manifolds which carry regular pluricomplex Green functions. This class of manifolds naturally includes all smoothly bounded, strictly linearly convex domains and all smoothly bounded, strongly pseudoconvex circular domains of $\mathbb{C}^n$.

The modular data for this class of manifolds and, even more, the methods used, suggest naturally to ask similar questions for almost complex manifolds and to investigate the possibility of defining a useful notion of almost complex pluricomplex Green function. The generality of the setting poses new difficulties. The abundance of J-holomorphic curves, which is an advantage in many geometrical considerations, turns into a drawback in considering objects as the Kobayashi metric. In particular the notions of stationary and extremal disks are in general different. As for the construction of Green pluripotentials, it is necessary to cope with the behavior of plurisubharmonic functions in the non-integrable case which may be is rather unexpected even for arbitrarily small deformations of the standard complex structure. Finally the kernel distribution of (natural candidate of) almost-complex Monge-Ampère operator, even if appropriate non-degenericity conditions is assumed, is usually neither integrable, nor J-invariant. All this is in clear contrast with the classical setting and hence it cannot be expected that, for completely arbitrary non-integrable structures, one can reproduce the whole pattern of fruitful properties relating regular solutions of complex Monge-Ampère equations, foliations in disks and Kobayashi metric.

Nevertheless it is possible to determine sufficient conditions on the almost complex structure, which ensure the existence of almost complex Green pluripotential and the equality between the two notions of stationary disks and of extremal disks. The class of such structures is very large in many regards, in fact determined by a finite set of conditions (it is finite-codimensional) in an infinite dimensional space.

References

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