Bifurcation currents in one-dimensional holomorphic dynamics

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Introduction

The aim of these lectures is the study of bifurcations within holomorphic families of polynomials or rational functions by mean of pluripotential-theoretic and ergodic tools.

The starting point of the subject is a discovery made by DeMarco [DeM2] who shown that the bifurcation locus of any such family is the support of a $(1, 1)$ closed positive current which admits both the Lyapunov exponent function and the sum of the Green function evaluated on critical points as a global potential. This current, denoted T_{bif}, is called the bifurcation current.
In the recent years, several authors have investigated the geometry of the bifurcation locus using the current T_{bif} and its powers $T_{\text{bif}} \wedge T_{\text{bif}} \wedge \cdots \wedge T_{\text{bif}}$ [DeM1], [DeM2], [BB1], [P], [DF], [Du], [BE], [BB2], [BB3], [G]. The approach followed by these papers enlights a certain stratification of the bifurcation locus which corresponds to the degree of self-intersection of T_{bif}. The main results are exposed in these notes, they go from laminarity statements for certain regions of the bifucation locus to Hausdorff dimension estimates and includes precise density (or equidistribution) properties relative to various classes of specific parameters.
We have not discussed bifurcation theory for families of endomorphisms of higher dimensional complex projective spaces but have mentionned, among the techniques presented in these notes, those which also work in this more general context. This aspects appear in the papers [BB1], [BDM], [P] and in the survey [DS].

We have tried to give a synthetic and self-contained presentation of the subject. In most cases, we have given complete and detailed proofs and, sometimes, have substantially simplify those available in the litterature. Although basics about ergodic theory are discussed in the first chapter, we have not treated the elements of pluripotential theory. The other lectures delivered during this week will provide most of the needed knowledge. For that we also refer the reader to the appendix about pluripotential theory available in Sibony's [Sib] and Dinh and Sibony's [DS] surveys or to Demailly's book [Dem].

Contents

1 Rational functions as ergodic dynamical systems **7**

1.1 Potential theoretic aspects . . . 7
1.1.1 The Fatou-Julia dichotomy . . . 7
1.1.2 The Green measure of a rational map . . . 9
1.2 Ergodic aspects . . . 12
1.2.1 Equidistribution towards the Green measure, mixing . . . 12
1.2.2 The natural extension . . . 14
1.3 The Lyapunov exponent . . . 16
1.3.1 Definition, formulas and some properties . . . 16
1.3.2 Lyapunov exponent and multipliers of repelling cycles . . . 18

2 Holomorphic families **23**

2.1 Generalities . . . 23
2.1.1 Holomorphic families and the space Rat_d . . . 23
2.1.2 The space of degree d polynomials . . . 24
2.1.3 Moduli spaces and the case of degree two rational maps . . . 25
2.2 The hypersurfaces $\mathrm{Per}_n(w)$ . . . 26
2.2.1 Defining $\mathrm{Per}_n(w)$ using dynatomic polynomials . . . 26
2.2.2 The construction of dynatomic polynomials . . . 29
2.2.3 In the moduli space of degree two rational maps . . . 31
2.3 The connectedness locus in polynomial families . . . 32
2.3.1 Connected and disconnected Julia sets of polynomials . . . 32
2.3.2 Polynomials with a bounded critical orbit . . . 33

3 The bifurcation current **37**

3.1 Stability versus bifurcation . . . 37
3.1.1 Motion of repelling cycles and Julia sets . . . 37
3.1.2 Stability of critical orbits . . . 39
3.1.3 Some remarkable parameters . . . 42
3.2 Potential theoretic approach . . . 44
3.2.1 Przytycki's generalized formula . . . 44
3.2.2 Lyapunov exponent and bifurcation current . . . 47
3.2.3 DeMarco's formula . . . 49

4 Equidistribution towards the bifurcation current **51**

4.1 Distribution of critically periodic parameters 51

4.1.1 Statement and a general strategy 51

4.1.2 Proof of Theorem 4.1.1 54

4.2 Distribution of rational maps with cycles of a given multiplier 56

4.2.1 The case of attracting cycles 56

4.2.2 Averaging the multipliers 59

4.2.3 The case of neutral cycles in polynomial families 63

4.3 Laminated structures in bifurcation loci 65

4.3.1 Holomorphic motion of the Mandelbrot set 65

4.3.2 Further laminarity statements for T_{bif} 69

5 The bifurcation measure **71**

5.1 A Monge-Ampère mass related with strong bifurcations 71

5.1.1 Basic properties 71

5.1.2 Some concrete families 73

5.2 Density statements 75

5.2.1 Misiurewicz parameters 75

5.2.2 Shishikura or hyperbolic parameters 77

5.2.3 Shishikura parameters with chosen multipliers 79

5.3 The support of the bifurcation measure 79

5.3.1 A transversality result 80

5.3.2 The bifurcation measure and strong-bifurcation loci 81

5.3.3 Hausdorff dimension estimates 84

Chapter 1

Rational functions as ergodic dynamical systems

1.1 Potential theoretic aspects

1.1.1 The Fatou-Julia dichotomy

A rational function f is a holomorphic map of the Riemann sphere to itself and may be repesented as the ratio of two polynomials

$$f = \frac{a_0+a_1z+a_2z^2+\cdots+a_dz^d}{b_0+b_1z+b_2z^2+\cdots+b_dz^d}$$

where at least one of the coefficents a_d and b_d is not zero. The number d is the algebraic degree of f. In the sequel we shall more likely speak of *rational map*. Such a map may also be considered as a holomorphic ramified self-cover of the Riemann sphere whose topological degree is equal to d. Among these maps, polynomials are exactly those for which ∞ is totally invariant: $f^{-1}\{\infty\} = f\{\infty\} = \infty$.

It may also be convenient to identify the Riemann sphere with the one-dimensional complex projective space $\mathbf{P}^1$ that is the quotient of $\mathbf{C}^2 \setminus \{0\}$ by the action $z \mapsto u \cdot z$ of $\mathbf{C}^*$. Let us recall that the Fubini-Study form ω on $\mathbf{P}^1$ satisfies $\pi^*(\omega) = dd^c \ln \| \ \|$ where the norm is the euclidean one on $\mathbf{C}^2$.
In this setting, the map f can be seen as induced on $\mathbf{P}^1$ by a non-degenerate and d-homogenous map of $\mathbf{C}^2$

$$F(z_1, z_2) := \left(a_0z_2^d + a_1z_1z_2^{d-1} + \cdots + a_dz_1^d, b_0z_2^d + b_1z_1z_2^{d-1} + \cdots + b_dz_1^d\right)$$

through the canonical projection $\pi : \mathbf{C}^2 \setminus \{0\} \to \mathbf{P}^1$

The homogeneous map F is called a lift of f; all other lifts are proportional to F.

A point around which a rational map f does not induce a local biholomorphism is called *critical*. The image of such a point is called a *critical value*. A degree d rational map has exactly $(2d-2)$ critical points counted with multiplicity. The critical set of f is the collection of all critical points and is denoted $\mathcal{C}_f$.

As for any self-map, we may study the dynamics of rational ones, that is trying to understand the behaviour of the sequence of iterates

$$f^n := f \circ \cdots \circ f.$$

The Fatou-Julia dynamical dichotomy consists in a splitting of $\mathbf{P}^1$ into two disjoint subsets on which the dynamics of f is radically different. The *Julia set* of a rational map f is the subset of $\mathbf{P}^1$ on which the dynamics of f may drastically change under a small perturbation of initial conditions while the *Fatou set* is the complement of the Julia set.

Definition 1.1.1 *The Julia set $\mathcal{J}_f$ and the Fatou set $\mathcal{F}_f$ of a rational map f are respectively defined by:*

$$\mathcal{J}_f := \{z \in \mathbf{P}^1 \; / \; (f^n)_n \textit{ is not equicontinuous near } z\}$$
$$\mathcal{F}_f := \mathbf{P}^1 \setminus \mathcal{J}_f.$$

Both the Julia and the Fatou set are totally invariant: $\mathcal{J}_f = f\big(\mathcal{J}_f\big) = f^{-1}\big(\mathcal{J}_f\big)$ and $\mathcal{F}_f = f\big(\mathcal{F}_f\big) = f^{-1}\big(\mathcal{F}_f\big)$. In particular f induces two distinct dynamical systems on $\mathcal{J}_f$ and $\mathcal{F}_f$.
The dynamical system $f : \mathcal{J}_f \to \mathcal{J}_f$ is chaotical. However, as it results from the Sullivan non-wandering theorem and the Fatou-Cremer classification, the dynamics of a rational map is totally predictible on its Fatou set.

The periodic orbits are called *cycles* and play a very important role in the understanding of the dynamics.

Definition 1.1.2 *A n-cycle is a set of n distinct points $z_0, z_1, \cdots, z_{n-1}$ such that $f(z_i) = z_{i+1}$ for $0 \leq i \leq n-2$ and $f(z_{n-1}) = z_0$. One says that n is the exact period of the cycle.*

Each point z_i is fixed by f^n. The *multiplier* of the cycle is the derivative of f^n at some point z_i of the cycle and computed in a local chart: $\big(\chi \circ f \circ \chi^{-1}\big)'(\chi(z_i))$. It is easy to see that this number depends only on the cycle and neither on the point z_i or the chart χ. By abuse we shall denote it $(f^n)'(z_i)$.
The local dynamic of f near a cycle is governed by the multiplier m. This leads to the following

Definition 1.1.3 *The multiplier of a n-cycle is a complex number m which is equal to the deivative of f^n computed in any local chart at any point of the cycle.*

When $|m| > 1$ the cycle is said repelling *when* $|m| < 1$ *the cycle is said* attracting

when $|m| = 1$ *the cycle is called* neutral.

Repelling cycles belongs to the Julia set and attracting one to the Fatou set. For neutral cycles this depends in a very delicate way on the diophantine properties of the argument of m.

The first fundamental result about Julia sets is the following.

Theorem 1.1.4 *Repelling cycles are dense in the Julia set.*

It is possible to give an elementary proof of that result using the Brody-Zalcman renormalization technique (see [BM]). We shall see later that repelling cycles actually equidistribute a measure whose support is exactly the Julia set.

1.1.2 The Green measure of a rational map

Our goal is to endow the dynamical system $f : \mathcal{J}_f \to \mathcal{J}_f$ with an ergodic structure capturing most of its chaotical nature. This is done by exhibiting an invariant measure μ_f on $\mathcal{J}_f$ which is of constant Jacobian. Such a measure was first constructed by Lyubich [L]. For our purpose it will be extremely important to use a potential-theoretic approach which goes back to Brolin [Br] for the case of polynomials. We follow here the presentation given by Dinh and Sibony in their survey [DS] which also covers *mutatis mutandis* the construction of Green currents for holomorphic endomorphisms of $\mathbf{P}^k$.

The following Lemma is the key of the construction. It relies on the fundamental fact that

$$d^{-1} f^\star \omega = \omega + dd^c v$$

for some smooth function v on $\mathbf{P}^1$. This follows from a standard cohomology argument or may be seen concretely by setting $v := d^{-1} \ln \frac{\|F(z)\|}{\|z\|^d}$ for some lift F of f.

Lemma 1.1.5 *Up to some additive constant, there exists a unique continuous function g on $\mathbf{P}^1$ such that $d^{-n} f^{n\star}\nu \to dd^c g + \omega$ for all positive measure ν which is given by $\nu = \omega + dd^c u$ where u is continuous.*

Proof. Let us set $g_n := v + \cdots + d^{-n+1} v \circ f^{n-1}$. One sees by induction that $d^{-n} f^{n\star}\nu = \omega + dd^c g_n + dd^c(d^{-n} u \circ f)$. As the sequence $(g_n)_n$ is clearly uniformly converging, the conclusion follows by setting $g := \lim_n g_n$. $\square$

It might be useful to see how the function g can be obtained by using lifts.

Lemma 1.1.6 *Let F be a lift of a degree d rational map f. The sequence $d^{-n} \ln \|F^n(z)\|$ converges uniformly on compact subsets of $\mathbf{C}^2 \setminus \{0\}$ to a function G_F which satisfies the following invariance and homogeneity properties:*

i) $G_F \circ F = dG_F$

ii) $G_F(tz) = G_F(z) + ln|t|, \forall t \in \mathbf{C}$.

Moreover, $G_F - \ln \| \; \| = g \circ \pi$ *where g is given by Lemma 1.1.5.*

Proof. Let us set $G_n(z) := d^{-n} \ln \|F^n(z)\|$. As F is homogeneous and non-degenerate there exists a constant $M > 1$ such that

$$\tfrac{1}{M}\|z\|^d \leq \|F(z)| \leq M\|z\|^d.$$

Thus $\frac{1}{M}\|F^n(z)\|^d \leq \|F^{n+1}(z)\| \leq M\|F^n(z)\|^d$ which, taking logarithms and dividing by d^{n+1} yields $|G_{n+1}(z) - G_n(z)| \leq \frac{\ln M}{d^{n+1}}$. This shows that G_n is uniformly converging to G_F. The properties i) and ii) follows immediately from the definition of G_F.

According to the proof of Lemma 1.1.5, $g = \lim_n \left(v + \cdots + d^{-n+1} v \circ f^{n-1}\right)$ where $v \circ \pi = d^{-1} \ln \frac{\|F(z)\|}{\|z\|^d}$. To get the last assertion, it suffices to observe that $d^{-k} v \circ f^k \circ \pi = G_{k+1} - G_k$. □

The two above lemmas lead us to coin the following

Definition 1.1.7 *Let F be a lift of a degree d rational map f. The Green function G_F of F on $\mathbf{C}^2$ is defined by*

$$G_F := \lim_n d^{-n} \ln \|F^n(z)\|.$$

The Green function of g_F of F on $\mathbf{P}^1$ is defined by

$$G_F - \ln \| \; \| = g_F \circ \pi.$$

We will sometimes use the notation g_f instead of g_F. The function g_f is defined modulo an additive constant.

The function G_F is $p.s.h$ on $\mathbf{C}^2$ with a unique pole at the origin.

It is worth emphasize that both g and G_F are uniform limits of smooth functions. In particular, these functions are continuous. One may actually prove more (see [DS] Proposition 1.2.3 or [BB1] Proposition 1.2):

Proposition 1.1.8 *The Green functions $G_F(z)$ and $g_F(z)$ are Hölder continuous in F and z.*

We are now ready to define the measure μ_f and verify its first properties.

Theorem 1.1.9 *Let f be a degree $d \geq 2$ rational map and g be a Green function of f. Let $\mu_f := \omega + dd^c g$. Then μ_f is a f-invariant probability measure whose support is equal to $\mathcal{J}_f$. Moreover μ_f has constant Jacobian: $f^\star \mu_f = d\mu_f$.*

Proof. As a weak limit of probability measures, μ_f is a probability measure.

We shall use Lemma 1.1.5 for showing that $f^\star \mu_f = d\mu_f$. By construction $v + d^{-1} g \circ f = \lim_n v + d^{-1} g_n \circ f = \lim_n g_{n+1} = g$ and thus $d^{-1} f^\star \mu_f = \omega + dd^c v + d^{-1} dd^c (g \circ f) = \omega + dd^c(v + d^{-1} g \circ f) = \mu_f$.

The invariance property $f_\star \mu_f = \mu_f$ follows immediately from $f^\star \mu_f = d\mu_f$ by using the fact that $f_\star f^\star = d\, Id$.

Let us show that the support of μ_f is equal to $\mathcal{J}_f$. If $U \subset \mathcal{F}_f$ is open then $f^{n\star}\omega$ is uniformly bounded on U and therefore $\mu_f(U) = \lim \int_U d^{-n} f^{n\star}\omega = 0$, this shows that $Supp\ \mu_f \subset \mathcal{J}_f$. Conversaly the identity $f^\star \mu_f = d\mu_f$ implies that $(Supp\ \mu_f)^c$ is invariant by f which, by Picard-Montel's theorem, implies that $(Supp\ \mu_f)^c \subset \mathcal{F}_f$.□

It is sometimes useful to use the Green function G_F for defining local potentials of μ_f.

Proposition 1.1.10 *Let f be a rational map and F be a lift. For any section σ of the canonical projection π defined on some open subset U of $\mathbf{P}^1$, the function $G_F \circ \sigma$ is a potential for μ_f on U.*

Proof. On U one has $dd^c G_F \circ \sigma = dd^c g_F + dd^c \ln \|\sigma\| = dd^c g_F + \omega = \mu_f$. □

The measure μ_f is the image by the canonical projection $\pi : \mathbf{C}^2 \setminus \{0\} \to \mathbf{P}^1$ of a Monge-Ampère measure associated to the Green function G_F.

Proposition 1.1.11 *Let F be a lift of a degree d rational map f and G_F a Green function of F. The measure $\mu_F := dd^c G_F^+ \wedge dd^c G_F^+$ is supported on the compact set $\{G_F = 0\}$ and satisfies $F^\star \mu_F = d^2 \mu_F$ and $\pi_\star \mu_F = \mu_f$.*

This construction will be used only once in this text and we therefore skip its proof. Observe that the support of μ_F is contained in the boundary of the compact set $K_F := \{G_F \leq 0\}$ which is precisely the set of points z with bounded forward orbits by F.

The case of polynomials presents interesting features, in particular the Green measure coincides with the harmonic measure of the filled-in Julia set.

Proposition 1.1.12 *let P be a degree d polynomial on $\mathbf{C}$. The Green function g_P of P is the subharmonic function defined by*

$$g_P := \lim_n d^{-n} \ln^+ |P^n|$$

and is a global potential of the Green measure of P.

Proof. We may take $F := \left(z_2^d, P(\frac{z_1}{z_2}), z_2^d\right)$ as a lift of P. Then $F^n := \left(z_2^{d^n} P^n(\frac{z_1}{z_2}), z_2^{d^n}\right)$ and

$$G_F(z_1, 1) = \frac{1}{2} \lim_n d^{-n} \ln\left(1 + |P^n(z_1)|^2\right) = \lim_n d^{-n} \ln^+ |P^n(z_1)|.$$

The conclusion then follows from Proposition 1.1.10. □

1.2 Ergodic aspects

1.2.1 Equidistribution towards the Green measure, mixing

Definition 1.2.1 *Let (X, f, μ) be a dynamical system. One says that the measure μ is mixing if and only if*

$$\lim_n \int_X (\varphi \circ f^n)\ \psi\ \mu = \int_X \varphi\ \mu\ \int_X \psi\ \mu$$

for any test functions φ and ψ.

This means that the events $\{f^n(x) \in A\}$ and $\{x \in B\}$ are asymptotically independants for any pair of Borel sets A, B.

As we shall see, the constant Jacobian property implies that Green measures are mixing.

Theorem 1.2.2 *The Green measure μ_f of any degree d rational map f is mixing.*

Proof. Let us set $c_\varphi := \int \varphi\ \mu_f$ and $c_\psi := \int \psi\ \mu_f$ where φ and ψ are two test functions. We may assume that $c_\varphi = 1$.

Since μ_f and $\varphi\mu_f$ are two probability measures, there exists a smooth function u_φ on $\mathbf{P}^1$ such that:

$$\varphi\mu_f = \mu_f + \Delta u_\varphi. \tag{1.2.1}$$

On the other hand, by the constant Jacobian property $f^*\mu_f = \mu_f$ we have:

$$d^{-n} f^{n*}\left((\varphi - c_\varphi)\mu_f\right) = \left(\varphi \circ f^n - c_\varphi\right)\mu_f \tag{1.2.2}$$

Now, combining 1.2.1 and 1.2.2 we get:

$$\int(\varphi\circ f^n)\psi\,\mu_f-\Big(\int\varphi\,\mu_f\Big)\Big(\int\psi\,\mu_f\Big)=\int(\varphi\circ f^n)\psi\,\mu_f-c_\varphi c_\psi=$$

$$\int\psi\big(\varphi\circ f^n-c_\varphi\big)\,\mu_f=\int\psi\;d^{-n}f^{n*}\big((\varphi-c_\varphi)\mu_f\big)=\int\big(d^{-n}f^n_*\psi\big)\,(\varphi-1)\,\mu_f=$$

$$\int\big(d^{-n}f^n_*\psi\big)\Delta u_\varphi=\int\psi\;d^{-n}f^{n*}(\Delta u_\varphi)=\int\psi\;d^{-n}\Delta(u_\varphi\circ f^n)=d^{-n}\int(u_\varphi\circ f^n)\;\Delta\psi.$$

This shows that $\lim_n\int(\varphi\circ f^n)\psi\,\mu_f=\big(\int\varphi\,\mu_f\big)\big(\int\psi\,\mu_f\big)$. □

It is not hard to show that a mixing measure is also ergodic.

Definition 1.2.3 *Let (X,f,μ) be a dynamical system. One says that the measure μ is ergodic if and only if all integrable f-invariant functions are constants.*

In particular this allows to use the classical Birkhoff ergodic theorem which says that time-averages along typical orbits coincide with the spatial-average:

Theorem 1.2.4 *Let (X,f,μ) be an ergodic dynamical system and $\varphi\in L^1(\mu)$. Then for μ almost every x one has:*

$$\lim_n\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1}\varphi(f^k(z))=\int_X\ln\varphi\;\mu$$

The measure-theoretic counterpart of Fatou-Julia theorem 1.1.4 is the following equidistribution result which has been first proved by Lyubich [L]. The content of subsection 1.3.2 will provide another proof which exploits the mixing property.

Theorem 1.2.5 *Let f be a rational map of degree d. Let $R_n^\star$ denote the set of n periodic repelling points of f. Then $d^{-n}\sum_{R_n^\star}\delta_z$ is weakly converging to μ_f.*

Let us finally mention another classical equidistribution result. We refer to [DS] for a potential theoretic proof.

Theorem 1.2.6 *Let f be a degree d rational map. Then*

$$\lim_n d^{-n}\textstyle\sum_{\{f^n(x)=a\}}\delta_x=\mu_f$$

for any $a\in\mathbf{P}^1$ which is not exceptional for f.

We recall that a is exceptional for f if and only if $a=\infty$ and f is a polynomial or $a\in\{0,\infty\}$ and f is of the form $z^{\pm d}$.

Although this will not be used in this text, we mention that the Green measure μ_f of any degree d rational map f is the unique measure of maximal entropy for f. This means that the entropy of μ_f is maximal and, according to the variational principle, equals $\ln d$ which is the value of the topological entropy of f.

1.2.2 The natural extension

To any ergodic dynamical system, it is possible to associate a new system which is invertible and contains all the information of the original one. It is basically obtained by considering the set of all complete orbits on which is acting a shift. This general construction is the so-called *natural extension* of a dynamical system; here is a formal definition.

Definition 1.2.7 *The natural extension of a dynamical system (X, f, μ) is the dynamical system $\left(\widehat{X}, \hat{f}, \hat{\mu}\right)$ where*

$$\widehat{X} := \{\hat{x} := (x_n)_{n\in\mathbf{Z}} \ / \ x_n \in X, \ f(x_n) = x_{n+1}\}$$
$$\hat{f}(\hat{x}) := (x_{n+1})_{n\in\mathbf{Z}}$$
$$\hat{\mu}\{(x_n) \ s.t. \ x_0 \in B\} = \mu(B).$$

The canonical projection $\pi_0 : \hat{X} \to X$ is given by $\pi_0(\hat{x}) = x_0$. One sets τ for $(\hat{f})^{-1}$.

Let us stress that $\pi_0 \circ \hat{f} = f \circ \pi_0$ and $(\pi_0)_*(\hat{\mu}) = \mu$. The measure $\hat{\mu}$ inherits most of the ergodic properties of μ.

Proposition 1.2.8 *The measure $\hat{\mu}$ is ergodic (resp. mixing) if and only if μ is ergodic (rep. mixing).*

We refer the reader to the chapter 10 of [CFS] for this construction and its properties.

A powerful way to control the behaviour of inverse branches along typical orbits of the system $(\mathcal{J}_f, f, \mu_f)$ is to apply standard ergodic theory to its natural extension. This is what we shall do now. The first point is to observe that one may work with orbits avoiding the critical set of f. To this purpose one considers

$$\widehat{X}_{reg} = \{\hat{x} \in \hat{\mathcal{J}}_f \ / \ x_n \notin \mathcal{C}_f \ ; \ \forall n \in \mathbf{Z}\}.$$

As $\hat{\mu}$ is $\hat{f}$-invariant and μ does not give mass to points, one sees that $\hat{\mu}(\widehat{X}_{reg}) = 1$.

Definition 1.2.9 *Let $\hat{x} \in \widehat{X}_{reg}$ and $p \in \mathbf{Z}$. The injective map induced by f on some neighbourhood of x_p is denoted f_{x_p}. The inverse of f_{x_p} is defined on some neighbourhood of x_{p+1} and is denoted $f_{x_p}^{-1}$. We then set*

$$f_{\hat{x}}^{-n} := f_{x_{-n}}^{-1} \circ \cdots \circ f_{x_{-1}}^{-1}.$$

The map $f_{\hat{x}}^{-n}$ is called "iterated inverse branch of f along $\hat{x}$ and of depth n".

Proposition 1.2.10 *For any sufficently small and strictly positive ϵ, there exists a function $\alpha_\epsilon : \widehat{X}_{reg} \to]0,1[$ such that*

$$\alpha_\epsilon(\tau(\hat{x})) \geq e^{-\epsilon}\alpha_\epsilon(\hat{x}) \text{ and}$$

$$f^{-1}_{x_{-k-1}} \text{ is defined on } D(x_{-k}, \alpha_\epsilon(\tau^k(\hat{x})))$$

for $\hat{\mu}$-a.e. $\hat{x} \in \widehat{X}_{reg}$ and every $k \in \mathbf{Z}$.

The function α_ϵ is a so-called *slow function*. The interest of such a function is, that in some situations, its decreasing is negligeable with respect to other datas. For instance, in some circumstances, Proposition 1.2.10 will tell us that the the local inverses $f^{-1}_{x_{-k}}$ are defined on discs of essentially fixed radius along the orbit $\hat{x}$.

Proof. We need the following quantitative version of the inverse mapping theorem (see [BD] lemme 2).

Lemma 1.2.11 *Let $\rho(x) := |f'(x)|$, $r(x) := \rho(x)^2$. There exists $\epsilon_0 > 0$ and, for $\epsilon \in]0, \epsilon_0]$, $0 < C_1(\epsilon), C_2(\epsilon)$ such that for every $x \in J$:*

1- f is one-to-one on $D(x, C_1(\epsilon)\rho(x))$,

2- $D(f(x), C_2(\epsilon)r(x)) \subset f\big(D(x, C_1(\epsilon)\rho(x))\big)$,

3- Lip $f^{-1}_x \leq e^{\frac{\epsilon}{3}}\rho(x)^{-1}$ on $D(f(x), C_2(\epsilon)r(x))$.

Let us set $\beta_\epsilon(\hat{x}) :=$ Min $(1, C_2(\epsilon)r(x_{-1}))$. According to the two first assertions of the above Lemma, $f^{-1}_{x_{-1}} = f^{-1}_{\hat{x}}$ is defined on $D(x_0, \beta_\epsilon(\hat{x}))$ and, similarly, $f^{-1}_{x_{-k-1}} = f^{-1}_{\tau^k(\hat{x})}$ is defined on $D(x_{-k}, \beta_\epsilon(\tau^k(\hat{x})))$. All we need is to find a function α_ϵ such that $0 < \alpha_\epsilon < \beta_\epsilon$ and $\alpha_\epsilon(\tau(\hat{x})) \geq e^{-\epsilon}\alpha_\epsilon(\hat{x})$.
As μ admits continuous local potentials, the function $\ln \beta_\epsilon$ is $\hat{\mu}$-integrable. Then, by Birkhoff ergodic theorem, $\int_{\widehat{X}} \ln \beta_\epsilon \ \hat{\mu} = \lim_{|n| \to +\infty} \frac{1}{|n|} \sum_{k=1}^n \ln \beta_\epsilon(\tau^k(\hat{x}))$ and, in particular

$$\lim_{|n| \to +\infty} \frac{1}{|n|} \ln \beta_\epsilon(\tau^n(\hat{x})) = 0 \text{ for } \hat{\mu}\text{-a.e. } \hat{x} \in \widehat{X}.$$

In other words, for $\hat{\mu}$-a.e. $\hat{x} \in \widehat{X}_{reg}$ there exists $n_0(\epsilon, \hat{x}) \in \mathbf{N}$ such that $\beta_\epsilon(\tau^n(\hat{x})) \geq e^{-|n|\epsilon}$ for $|n| \geq n_0(\epsilon, \hat{x})$. Setting then $V_\epsilon := \inf_{|n| \leq n_0(\epsilon,\hat{x})} \big(\beta_\epsilon(\tau^n(\hat{x}))e^{|n|\epsilon}\big)$ we obtain a measurable function $V_\epsilon : \widehat{X}_{reg} \to]0,1]$ such that: $\beta_\epsilon(\tau^n(\hat{x})) \geq e^{-|n|\epsilon}V_\epsilon(\hat{x})$ for $\hat{\mu}$-a.e. $\hat{x} \in \widehat{X}_{reg}$ and every $n \in \mathbf{Z}$. It suffices to take $\alpha_\epsilon(\hat{x}) := \text{Inf}_{n \in \mathbf{Z}}\{\beta_\epsilon(\tau^n(\hat{x}))e^{|n|\epsilon}\}$. □

1.3 The Lyapunov exponent

1.3.1 Definition, formulas and some properties

Let us consider the ergodic dynamical system $(\mathcal{J}_f, f, \mu_f)$ which has been constructed in the last section. As the measure μ_f has continuous local potentials, the function $\ln|f'|$ belongs to $L^1(\mu_f)$ for any choice of a metric $|\ |$ on $\mathbf{P}^1$. We may therefore apply the Birkhoff ergodic theorem to get:

$$\lim_n \frac{1}{n} \ln|(f^n)'(z)| = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \ln|f'(f^k(z))| = \int_{\mathbf{P}^1} \ln|f'|\, \mu_f, \quad \mu_f\text{-a.e. .} \tag{1.3.1}$$

This identity shows that the integral $\int_{\mathbf{P}^1} \ln|f'|\, \mu_f$ does not depend on the choice of the metric $|\ |$ and leads to the following definition.

Definition 1.3.1 *The Lyapunov exponent of the ergodic dynamical system $(\mathcal{J}_f, f, \mu_f)$ is the number*

$$L(f) = \int_{\mathbf{P}^1} \ln|f'|\, \mu_f.$$

For simplicity we shall usualy say that $L(f)$ is the Lyapunov exponent of f.

As the identity 1.3.1 shows, the Lyapunov exponent $L(f)$ is the exponential rate of growth of $|(f^n)'(z)|$ for a typical $z \in \mathcal{J}_f$.

Remark 1.3.2 *Using the invariance property $f_*\mu_f = \mu_f$ one immediately sees that $L(f^n) = nL(f)$.*

We shall need an expression of $L(F)$ which uses the formalism of line bundles. In order to prove it we first compare the Lyapunov exponents of f with the sum of Lyapunov exponents of one of its lifts F.

Proposition 1.3.3 *Let F be a lift of some rational map f of degree d. Then the sum of Lyapunov exponents of F with respect to μ_F is given by $L(F) := \int \ln|\det F'|\, \mu_F$ and is equal to $L(f) + \ln d$.*

Proof. let F be a polynomial lift of f. Using the fact that $f^*\omega = |f'|_\sigma^2 \omega$, it is not difficult to check that

$$|f'(\xi)|_\sigma = \frac{1}{d} \frac{\|z\|^2}{\|F(z)\|^2} |\det F'(z)|$$

for any z such that $\pi(z) = \xi$. We thus have

$$\frac{1}{n} \ln|(f^n)'(\xi)|_\sigma + \ln d = \frac{1}{n} \ln \frac{\|z\|^2}{\|F^n(z)\|^2} + \frac{1}{n} \ln|\det(F^n)'(z)|\,.$$

Then the conclusion follows by Birkhoff theorem since $\|F^n(z)\|$ stays away from 0 and $+\infty$ when z is in the support of μ_F and $\pi_\star\mu_F = \mu_f$. □

For any integer D the line bundle $\mathcal{O}_{\mathbf{P}^1}(D)$ over $\mathbf{P}^1$is the quotient of $(\mathbf{C}^2\setminus\{0\})\times\mathbf{C}$ by the action of $\mathbf{C}^*$ defined by $(z,x)\mapsto(uz,u^Dx)$. We denote by $[z,x]$ the elements of this quotient.

The canonical metric on $\mathcal{O}_{\mathbf{P}^1}(D)$ may be written

$$\|[z,x]\|_0 = e^{-D\ln\|z\|}|x|.$$

The homogenity property of G_F allows us to define another metric on $\mathcal{O}_{\mathbf{P}^1}(D)$ by setting

$$\|[z,x]\|_{G_F} = e^{-DG_F(z)}|x|.$$

Let us underline that, according to Definition 1.1.7, $\|\cdot\|_{G_F} = e^{-Dg_F}\|\cdot\|_0$.

The following Lemma will turn out to be extremely useful when we shall relate the Lyapunov exponent with bifurcations.

Lemma 1.3.4 *Let f be a rational map of degree $d\geq 2$ and F be one of its lifts. Let $D := 2(d-1)$ and Jac_F be the holomorphic section of $\mathcal{O}_{\mathbf{P}^1}(D)$ induced by* $\det F'$*. Then*

$$L(f) + \ln d = \int_{\mathbf{P}^1}\ln\|Jac_F\|_{G_F}\,\mu_f.$$

Proof. The section Jac_F is defined by $Jac_F(\pi(z)) := [z,\det F']$ for any $z\in\mathbf{C}^2\setminus\{0\}$. Using Proposition 1.3.3, the fact that G_F vanishes on the support of μ_F and $\pi_*\mu_F = \mu_f$ we get

$$L(f)+\ln d = \int_{\mathbf{C}^2}\ln|\det F'|\,\mu_F = \int_{G_F^{-1}(\{0\})}\ln|\det F'|\,\mu_F =$$
$$\int_{G_F^{-1}(\{0\})}\ln\left(e^{-DG_F(z)}|\det F'|\right)\,\mu_F = \int_{\mathbf{C}^2}\ln\|Jac_F\circ\pi\|_{G_F}\,\mu_F =$$
$$\int_{\mathbf{P}^1}\ln\|Jac_F\|_{G_F}\,\mu_f.$$

□

It is an important and not obvious fact that $L(f)$ is stricly positive. It actually follows from the Margulis-Ruelle inequality that $L(f)\geq\frac{1}{2}\ln(d)$ where d is the degree of f. We will presnt later a simple argument which shows that this bound is equal to $\ln d$ for polynomials. Zdunik and Mayer ([Z], [May]) have proved that the bound $\frac{1}{2}\ln(d)$ is taken if and only if the map f is a Lattès example. Let us summarize these results in the following statement.

Theorem 1.3.5 *The Lyapunov exponent of a degree d rational map is always greater than $\frac{1}{2}\ln(d)$ and the equality occurs if and only if the map is a Lattès example.*

We recall that a Lattès map is, by definition, induced on the Riemann sphere from an expanding map on a complex torus by mean of some elliptic function. We refer to the survey paper of Milnor [Mi2] for a detailed discussion of these maps.

A remarkable consequence of the positivity of $L(f)$ is that the iterated inverse branch $f_{\hat{x}}^{-n}$ (see definition 1.2.9) are approximately e^{-nL}-Lipschiptz and are defined on a disc whose size only depends on $\hat{x}$.

Proposition 1.3.6 *There exists $\epsilon_0 > 0$ and, for $\epsilon \in]0, \epsilon_0]$, two measurable functions $\eta_\epsilon : \widehat{X}_{reg} \to]0, 1]$ and $S_\epsilon : \widehat{X}_{reg} \to]1, +\infty]$ such that the maps $f_{\hat{x}}^{-n}$ are defined on $D(x_0, \eta_\epsilon(\hat{x}))$ and Lip $f_{\hat{x}}^{-n} \leq S_\epsilon(\hat{x})e^{-n(L-\epsilon)}$ for every $n \in \mathbf{N}$ and $\hat{\mu}$-a.e. $\hat{x} \in \widehat{X}_{reg}$.*

Proof. We may assume that $0 < \epsilon_0 < \frac{L}{3}$. Since $f_{\hat{x}}^{-n} = f_{x_{-n}}^{-1} \circ \cdots \circ f_{x_{-1}}^{-1}$, the third assertion of Lemma 1.2.11 yields $\ln \mathrm{Lip}\, f_{\hat{x}}^{-n} \leq n\frac{\epsilon}{3} - \sum_{k=1}^{n} \ln \rho(x_{-k})$. By Birkhoff ergodic theorem we thus have

$$\limsup \frac{1}{n} \ln \mathrm{Lip}\, f_{\hat{x}}^{-n} \leq -L + \frac{\epsilon}{3} \text{ for } \hat{\mu}\text{-a.e. } \hat{x} \in \widehat{X}.$$

Then there exists $n_0(\hat{x})$ such that Lip $f_{\hat{x}}^{-n} \leq e^{-n(L-\epsilon)}$ for $n \geq n_0(\hat{x})$ and it suffices to set $S_\epsilon := \max_{0 \leq n \leq n_0(\hat{x})} \left(e^{n(L-\epsilon)}\mathrm{Lip}\, f_{\hat{x}}^{-n}\right)$ to get the estimate

$$\mathrm{Lip}\, f_{\hat{x}}^{-n} \leq S_\epsilon(\hat{x})e^{-n(L-\epsilon)} \text{ for every } n \in \mathbf{N} \text{ and } \hat{\mu}\text{-a.e. } \hat{x} \in \widehat{X}_{reg}.$$

We now set $\eta_\epsilon := \frac{\alpha_\epsilon}{S_\epsilon}$ where α_ϵ is the given by Proposition 1.2.10. Let us check by induction on $n \in \mathbf{N}$ that $f_{\hat{x}}^{-n}$ is defined on $D(x_0, \eta_\epsilon(\hat{x}))$ for $\hat{\mu}$-a.e. $\hat{x} \in \widehat{X}$ and every $n \in \mathbf{N}$. Here we will use the fact that the function α_ϵ is slow: $\alpha_\epsilon(\tau(\hat{x})) \geq e^{-\epsilon}\alpha_\epsilon(\hat{x})$. Assume that $f_{\hat{x}}^{-n}$ is defined on $D\big(x_0, \eta_\epsilon(\hat{x})\big)$. Then, by our estimate on Lip $f_{\hat{x}}^{-n}$, we have

$$f_{\hat{x}}^{-n}\Big(D\big(x_0, \eta_\epsilon(\hat{x})\big)\Big) \subset D\big(x_{-n}, e^{-n(L-\epsilon)}\alpha_\epsilon(\hat{x})\big).$$

On the other hand, by Proposition 1.2.10, the branch $f_{x_{-n-1}}^{-1}$ is defined on the disc $D\big(x_{-n}, \alpha_\epsilon(\tau^{n+1}(\hat{x}))\big)$ which, as α_ϵ is slow, contains $D\big(x_{-n}, e^{-(n+1)\epsilon}\alpha_\epsilon(\hat{x})\big)$. Now, since $0 < \epsilon_0 < \frac{L}{3}$ one has $e^{-(n+1)\epsilon} \geq e^{-n(L-\epsilon)}$ and thus $f_{\hat{x}}^{-(n+1)} = f_{x_{-n-1}}^{-1} \circ f_{\hat{x}}^{-n}$ is defined on $D\big(x_0, \eta_\epsilon(\hat{x})\big)$. □

1.3.2 Lyapunov exponent and multipliers of repelling cycles

The following approximation property will play an important role in our study of bifurcation currents. We would like to mention that Deroin and Dujardin have recently used similar ideas to study the bifurcation in the context of kleinean groups (see [DD]).

Theorem 1.3.7 *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map of degree $d \geq 2$ and L the Lyapunov exponent of f with respect to its Green measure. Then:*

$$L = \lim_n d^{-n} \sum_{p \in R_n^*} \frac{1}{n} \ln |(f^n)'(p)|$$

where $R_n^* := \{p \in \mathbf{P}^1 \ / \ p$ has exact period n and $|(f^n)'(p)| > 1\}$.

Observe that the Lyapunov exponent $\lim_k \frac{1}{k} \ln |(f^k)'(p)|$ of f along the orbit of a point p is precisely equal to $\frac{1}{n} \ln |(f^n)'(p)|$ when p is n periodic. The above Theorem thus shows that the Lyapunov exponent L of f is the limit, when $n \to +\infty$, of the averages of Lyapunov exponents of repelling n-cycles.

To establish the above Theorem, we will prove that the repelling cycles equidistribute the Green measure μ_f in a somewhat constructive way and control the multipliers of the cycles which appear. For proving the equidistribution, we follow the approach used by Briend-Duval [BD] in the context of endomorphisms of $\mathbf{P}^k$, the positivity of the Lyapunov exponent plays a crucial role there. This strategy actually yields to a version of Theorem 1.3.7 for endomorphisms of $\mathbf{P}^k$; this has been done in [BDM]. Okuyama has given a different proof of Theorem 1.3.7 in [O], his proof actually does not use the positivity of the Lyapunov exponent. The proof we present here is that of [Be] with a few more details.

Proof. For the simplicity of notations we consider polynomials and therefore work on $\mathbf{C}$ with the euclidean metric. We shall denote $D(x, r)$ the open disc centered at $x \in \mathbf{C}$ and radius $r > 0$. From now on, f is a degree $d \geq 2$ polynomial whose Julia set is denoted J and whose Green measure is denoted μ.

We shall use the natural extension (see subsection 1.2.2) and exploit the positivity of L through Proposition 1.3.6. Let us add a few notations to those already introduced in Propositions 1.2.10 and 1.3.6 . Let $0 < \epsilon_0$ be given by Proposition 1.2.10.

For $0 < \epsilon \leq \epsilon_0$ and $n, N \in \mathbf{N}$ we set:

$$\widehat{X}_N^\epsilon := \{\hat{x} \in \widehat{X} \ / \ \eta_\epsilon(\hat{x}) \geq \tfrac{1}{N} \text{ and } S_\epsilon(\hat{x}) \leq N\}$$
$$\hat{\nu}_N^\epsilon := 1_{\widehat{X}_N^\epsilon} \hat{\mu}$$
$$\nu_N^\epsilon := \pi_* \hat{\nu}_N^\epsilon.$$

For $0 < \epsilon \leq L$ and $n, N \in \mathbf{N}$ we set:

$$R_n^\epsilon := \{p \in \mathbf{C} \ / \ f^n(p) = p \text{ and } |(f^n)'(p)| \geq e^{n(L-\epsilon)}\}$$
$$\mu_n^\epsilon := d^{-n} \sum_{R_n^\epsilon} \delta_p$$

$$R_n := R_n^L = \{p \in \mathbf{C} \ / \ f^n(p) = p \text{ and } |(f^n)'(p)| \geq 1\}$$
$$\mu_n := \mu_n^L = d^{-n} \sum_{R_n} \delta_p.$$

We have to show that $\mu_n^L \to \mu$ and $L = \lim_n \frac{d^{-n}}{n} \sum_{R_n^*} \ln |(f^n)'(p)|$.

The following Lemma reduces the problem to some estimates on Radon-Nikodym derivatives.

Lemma 1.3.8 *If any weak limit σ of $\left(\mu_n^\epsilon\right)_n$ for $\epsilon \in]0, \epsilon_0[$ satisfies $\frac{d\sigma}{d\nu_N^{\epsilon'}} \geq 1$ for some $\epsilon' > 0$ and every $N \in \mathbf{N}$ then $\mu_n^L \to \mu$ and $L = \lim_n \frac{d^{-n}}{n} \sum_{R_n^*} \ln |(f^n)'(p)|$.*

Proof. We start by showing that $\mu_n^\epsilon \to \mu$ for any $\epsilon \in]0, L]$. Let σ be a weak limit of $\left(\mu_n^\epsilon\right)_n$. Since all the μ_n^ϵ are probability measures, it suffices to show that $\sigma = \mu$.

Assume first that $0 < \epsilon < \epsilon_0$. By assumption $\frac{d\sigma}{d\nu_N^{\epsilon'}} \geq 1$ and therefore $\sigma \geq \nu_N^{\epsilon'}$ for every $N \in \mathbf{N}$. Letting $N \to +\infty$ one gets $\sigma \geq \mu$. This actually implies that $\sigma = \mu$ since

$$\sigma(J) \leq \limsup_n \mu_n^\epsilon(J) \leq \lim_n \tfrac{d^n+1}{d^n} = 1 = \mu(J).$$

We have shown that $\mu_n^\epsilon \to \mu$ for $0 < \epsilon < \epsilon_0$. Let us now assume that $\frac{\epsilon_0}{2} =: \epsilon_1 \leq \epsilon$. From $\mu_n^\epsilon \geq \mu_n^{\epsilon_1}$ and $\mu_n^{\epsilon_1} \to \mu$ one gets $\sigma \geq \mu$. Just as before this implies that $\sigma = \mu$.

We now want to show that $L = \lim_n \frac{d^{-n}}{n} \sum_{R_n^*} \ln |(f^n)'(p)|$. Let us set $\varphi_n(p) := \frac{1}{n} \ln |(f^n)'(p)|$. For $M > 0$ one has

$$\mu_n^\epsilon(J)(L - \epsilon) \leq d^{-n} \sum_{R_n^\epsilon} \varphi_n(p) \leq d^{-n} \sum_{R_n} \varphi_n(p) = \int_J \ln |f'| \mu_n \leq$$
$$\leq \int_J Max\big(\ln |f'|, -M\big) \mu_n$$

since $\mu_n^\epsilon \to \mu$ and $\mu_n = \mu_n^L \to \mu$ we get

$$(L - \epsilon) \leq \liminf d^{-n} \sum_{R_n} \varphi_n(p) \leq \limsup d^{-n} \sum_{R_n} \varphi_n(p) \leq$$
$$\int_J Max\big(\ln |f'|, -M\big) \mu.$$

To obtain $\lim d^{-n} \sum_{R_n} \varphi_n(p) = L$ it suffices to make first $M \to +\infty$ and then $\epsilon \to 0$. Since there are less than $2nd^{\frac{n}{2}}$ periodic points whose period strictly divides n, one may replace R_n by $R_n^* := \{p \in \mathbf{P}^1 \ / \ p \text{ has exact period } n \text{ and } |(f^n)'(p)| \geq 1\}$. $\square$

Let us now finish the proof of Theorem 1.3.7. We assume here that $0 < \epsilon < \frac{\epsilon_0}{2}$. Let $\hat{a} \in \widehat{X}_N^\epsilon$ and $a := \pi(\hat{a})$. For every $r > 0$ we denote by D_r the closed disc centered at a of radius r. According to Lemma 1.3.8, it suffices to show that *any weak limit* σ of $\left(\mu_n^{2\epsilon}\right)_n$ satisfies

$$\sigma(D_{r'}) \geq \nu_N^\epsilon(D_{r'}), \text{ for any integer } N \text{ and all } 0 < r' < \frac{1}{N}. \tag{1.3.2}$$

Let us pick $r' < r < \frac{1}{N}$. We set $\widehat{D}_r := \pi^{-1}(D_r)$ and :

$$\widehat{C}_n := \{\hat{x} \in \widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N} \ / \ f_{\hat{x}}^{-n}(D_r) \cap D_{r'} \neq \emptyset\}.$$

Let also consider the collection S_n of sets of the form $f_{\hat{x}}^{-n}(D_r)$ where $\hat{x}$ runs in $\widehat{C}_n$. As $f_{\hat{x}}^{-n}$ is an inverse branch on D_r of the ramified cover f^n, one sees that the sets of the collection S_n are mutually disjoint.

Let us momentarily admit the two following estimates:

$$d^{-n}\big(\text{Card } S_n\big) \leq \mu_n^{2\epsilon}(D_r) \ \text{ for } n \ \text{ big enough} \tag{1.3.3}$$

$$d^{-n}\big(\text{Card } S_n\big)\ \mu(D_r) \geq \hat{\mu}\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}) \cap \widehat{D}_{r'}\big). \tag{1.3.4}$$

Combining 1.3.3 and 1.3.4 yields:

$$\hat{\mu}\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}) \cap \widehat{D}_{r'}\big) \leq \mu(D_r)\mu_n^{2\epsilon}(D_r)$$

which, by the mixing property of $\hat{\mu}$, implies

$$\nu_N^{\epsilon}(D_r)\mu(D_{r'}) = \hat{\mu}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N})\hat{\mu}(\widehat{D}_{r'}) \leq \mu(D_r)\sigma(D_r)$$

since $\mu(D_{r'}) > 0$, one gets 1.3.2 by making $r \to r'$.

Let us now prove the estimate 1.3.3. We have to show that D_r contains at least $\big(\text{Card } S_n\big)$ elements of $R_n^{2\epsilon}$ when n is big enough. Here we shall use Proposition 1.3.6. For every $\hat{x} \in \widehat{C}_n \subset \widehat{X}^{\epsilon}_{reg\ N}$ one has $\eta_\epsilon(\hat{x}) \geq \frac{1}{N}$ and $S_\epsilon(\hat{x}) \leq N$ and thus the map $f_{\hat{x}}^{-n}$ is defined on D_r ($r < \frac{1}{N}$) and Diam $f_{\hat{x}}^{-n}(D_r) \leq 2r$ Lip $f_{\hat{x}}^{-n} \leq 2rS_\epsilon(\hat{x})e^{-n(L-\epsilon)} \leq 2rNe^{-n(L-\epsilon)}$.
As moreover $f_{\hat{x}}^{-n}(D_r)$ meets $D_{r'}$, there exists n_0, which depends only on ϵ, r and r', such that $f_{\hat{x}}^{-n}(D_r) \subset D_r$ for every $\hat{x} \in \widehat{C}_n$ and $n \geq n_0$. Thus, by Brouwer theorem, $f_{\hat{x}}^{-n}$ has a fixed point $p_n \in f_{\hat{x}}^{-n}(D_r)$ for every $\hat{x} \in \widehat{C}_n$ and $n \geq n_0$. Since the elements of S_n are mutually disjoint sets, we have produced $\big(\text{Card } S_n\big)$ fixed points of f^n in D_r for $n \geq n_0$. It remains to check that these fixed points belong to $R_n^{2\epsilon}$. This actually follows immediately from the estimates on Lip $f_{\hat{x}}^{-n}$. Indeed:

$$|(f^n)'(p_n)| = |(f_{\hat{x}}^{-n})'(p_n)|^{-1} \geq \big(\text{Lip } f_{\hat{x}}^{-n}\big)^{-1} \geq N^{-1}e^{n(L-\epsilon)} \geq e^{n(L-2\epsilon)}$$

for n big enough.

Finally we prove the estimate 1.3.4. Let us first observe that

$$\pi\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}) \cap \widehat{D}_{r'}\big) \subset \bigcup_{\hat{x} \in \widehat{C}_n} f_{\hat{x}}^{-n}(D_r). \tag{1.3.5}$$

This can be easily seen : if $\hat{u} \in \hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}) \cap \widehat{D}_{r'}$ then $u_0 = \pi(\hat{u}) \in D_{r'} \cap f_{\hat{x}}^{-n}(D_r)$ where $\hat{x} := \hat{f}^n(\hat{u}) \in \widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}$.
By the constant Jacobian property we have $\mu\big(f_{\hat{x}}^{-n}(D_r)\big) = d^{-n}\mu(D_r)$ and, since the sets $f_{\hat{x}}^{-n}(D_r)$ of the collection S_n are mutually disjoint, we obtain

$$\mu\Big(\bigcup_{\hat{x} \in \widehat{C}_n} f_{\hat{x}}^{-n}(D_r)\Big) = \big(\mathrm{Card}\ S_n\big)\ d^{-n}\mu(D_r). \tag{1.3.6}$$

Combining 1.3.5 with 1.3.6 yields 1.3.4:

$$\big(\mathrm{Card}\ S_n\big)\ d^{-n}\mu(D_r) \geq \mu\big[\pi\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}\big) \cap \widehat{D}_{r'}\big)\big] =$$
$$\hat{\mu}\big[\pi^{-1} \circ \pi\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}\big) \cap \widehat{D}_{r'}\big)\big] \geq \hat{\mu}\big(\hat{f}^{-n}(\widehat{D}_r \cap \widehat{X}^{\epsilon}_{reg\ N}\big) \cap \widehat{D}_{r'}\big).$$

□

Chapter 2

Holomorphic families

We introduce here the main spaces in which we shall work in the subsequent chapters and present some of their structural properties.

2.1 Generalities

2.1.1 Holomorphic families and the space Rat_d

Let us a sart with a formal definition.

Definition 2.1.1 *Let M be a complex manifold. A holomorphic map*

$$f : M \times \mathbf{P}^1 \to \mathbf{P}^1$$

such that all rational maps $f_\lambda := f(\lambda, \cdot) : \mathbf{P}^1 \to \mathbf{P}^1$ have the same degree $d \geq 2$ is called holomorphic family *of degree d rational maps parametrized by M. For short, any such family will be denoted $(f_\lambda)_{\lambda \in M}$.*

Any degree d rational map $f := \frac{a_d z^d + \cdots + a_1 z + a_0}{b_d z^d + \cdots + b_1 z + b_0}$ is totally defined by the point $[a_d : \cdots : a_0 : b_d : \cdots : b_0]$ in the projective space $\mathbf{P}^{2d+1}$. This allows to identify the space Rat_d of degree d rational maps with a Zariski dense open subset of $\mathbf{P}^{2d+1}$. We can be more precise by looking at the space of homogeneous polynomial maps of $\mathbf{C}^2$ which is identified to $\mathbf{C}^{2d+2}$ by the correspondance

$$(a_d, \cdots, a_0, b_d : \cdots, b_0) \mapsto \left(\sum_{i=1}^d a_i z_1^i z_2^{d-i}, \sum_{i=1}^d b_i z_1^i z_2^{d-i} \right).$$

Indeed, Rat_d is precisely the image by the canonical projection $\pi : \mathbf{C}^{2d+2} \to \mathbf{P}^{2d+1}$ of the subspace H_d of $\mathbf{C}^{2d+2}$ consisting of non-degenerate polynomials. As H_d is the complement in $\mathbf{C}^{2d+2}$ of the projective variety defined by the vanishing of the resultant $Res(\left(\sum_{i=1}^d a_i z_1^i z_2^{d-i}, \sum_{i=1}^d b_i z_1^i z_2^{d-i} \right)$, one sees that $Rat_d = \mathbf{P}^{2d+1} \setminus \Sigma_d$ where Σ_d is an (irreducible) algebraic hypersurface of $\mathbf{P}^{2d+1}$.

From now on, we will always consider Rat_d as a quasi-projective manifold. We may therefore also see any holomorphic family of degree d rational maps with parameter space M as a holomorphic map f from M to Rat_d. In particular we may take for M any submanifold of Rat_d; this is especially interresting when M is dynamically defined as are, for instance, the hypersurfaces $Per_n(w)$ which will be defined in the next section.

The simplest example of holomorphic family is the family of quadratic polynomials. Up to affine conjugation, any degree 2 polynomial is of the form $z^2 + a$. To understand quadratic polynomials it is therefore sufficent to work with the family $(z^2 + a)_{a\in\mathbf{C}}$.

In most cases, when considering a holomorphic family $(f_\lambda)_{\lambda\in M}$, we shall make the two following mild assumptions.

Assumptions 2.1.2 *Let $(f_\lambda)_{\lambda\in M}$ be any holomorphic family of degree d rational maps.*

- *A1 The* marked critical points assumption *means that the critical set C_λ of f_λ is given by $2d-2$ graphs: $C_\lambda = \cup_1^{2d-2}\{c_i(\lambda)\}$ where the maps $M \ni \lambda \mapsto c_i(\lambda) \in \mathbf{P}^1$ are holomorphic.*
- *A2 The* no persistent neutral cycles assumption *means that if f_{λ_0} has a neutral cycle then this cycle becomes attracting or repelling under a suitable small perturbation of λ_0.*

2.1.2 The space of degree d polynomials

As for quadratic polynomials, there exists a nice parametrization of the space of degree d polynomials.

Let $\mathcal{P}_d$ be the space of polynomials of degree $d \geq 2$ with $d-1$ marked critical points up to conjugacy by affine transformations. Although this space has a natural structure of affine variety of dimension $d-1$, we may actually work with a specific parametrization of $\mathcal{P}_d$ which we shall now present.

For every $(c, a) := (c_1, c_2, \cdots, c_{d-2}, a) \in \mathbf{C}^{d-1}$ we denote by $P_{c,a}$ the polynomial of degree d whose critical points are $(0, c_1, \cdots, c_{d-2})$ and such that $P_{c,a}(0) = a^d$. This polynomial is explicitely given by:

$$P_{c,a} := \frac{1}{d}z^d + \sum_2^{d-1} \frac{(-1)^{d-j}}{j}\sigma_{d-j}(c)z^j + a^d$$

where $\sigma_i(c)$ is the symmetric polynomial of degree i in $(c_1, \cdots, c_{d-2})$. For convenience we shall set $c_0 := 0$.

Thus, when considering degree d polynomials, instead of working in $\mathcal{P}_d$ we may consider the holomorphic family

$$\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$$

whose parameter space M is simply $\mathbf{C}^{d-1}$. Using this parametrization, one may exhibit a finite ramified cover $\pi : \mathbf{C}^{d-1} \to \mathcal{P}_d$ (see [DF] Proposition 5.1).

It will be crucial to consider the projective compactification $\mathbf{P}^{d-1}$ of $\mathbf{C}^{d-1} = M$. This is why we wanted the expression of $P_{c,a}$ to be homogeneous in (c, a) and have used the parameter a^d instead of a. In this context, we shall denote by $\mathbf{P}_\infty$ the projective space at infinity : $\mathbf{P}_\infty := \{[c : a : 0]\ ; (c, a) \in \mathbf{C}^{d-1} \setminus \{0\}\}$.

2.1.3 Moduli spaces and the case of degree two rational maps

The group of Möbius transformations, which is isomorphic to $PSL(2,\mathbf{C})$, acts by conjugation on the space Rat_d of degree d rational maps. The dynamical properties of two conjugated rational maps are clearly equivalent and it is therefore natural to work within the quotient of Rat_d resulting from this action.

The moduli space Mod_d is, by definition, the quotient of Rat_d under the action of $PSL(2,\mathbf{C})$ by conjugation. We shall denote as follows the canonical projection:

$$\begin{aligned} \Pi : Rat_d &\longrightarrow Mod_d \\ f &\longmapsto \bar{f} \end{aligned}$$

We shall usually commit the abuse of language which consists in considering an element of Mod_d as a rational map. For instance, "$\bar{f}$ has a n-cycle of multiplier w" means that every element of $\bar{f}$ posseses such a cycle. We shall also sometimes write f instead of $\bar{f}$.

Although the action of $PSL(2,\mathbf{C})$ is not free, it may be proven that Mod_d is a normal quasi-projective variety [Sil].

Remark 2.1.3 *The following property is helpful for working in Mod_d. Every element f of Rat_d belongs to a local submanifold T_f whose dimension equals $2d-2$ and which is transversal to the orbit of f under the action of $PSL(2,\mathbf{C})$. Moreover, T_f is invariant under the action of the stabilizer $Aut(f)$ of f which is a finite subgroup of $PSL(2,\mathbf{C})$. Finally, $\Pi(T_f)$ is a neighborhood of $\bar{f}$ in Mod_d and Π induces a biholomorphism between $T_f/Aut(f)$ and $\Pi(T_f)$.*

In his paper [Mi1], Milnor has given a particularly nice description of Mod_2 which we will briefly present. The reader may also consult the fourth chapter of book of Silverman [Sil].

A generic $f \in Rat_2$ has 3 fixed points with multipliers μ_1, μ_2, μ_3. The symmetric functions

$$\sigma_1 := \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 := \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3, \quad \sigma_3 := \mu_1\mu_2\mu_3$$

are clearly well defined on Mod_2 and it follows from the holomorphic index formula $\sum \frac{1}{1-\mu_i} = 1$ that

$$\sigma_3 - \sigma_1 + 2 = 0. \tag{2.1.1}$$

Milnor has actually shown that (σ_1, σ_2) induces a good parametrization of Mod_2 ([Mi1]).

Theorem 2.1.4 *The map $Mod_2 \to \mathbf{C}^2$ defined by $\bar{f} \mapsto (\sigma_1, \sigma_2)$ is a biholomorphism.*

It will be extremely useful to consider the projective compactification of Mod_2 obtained through the above Theorem:

$$Mod_2 \ni \bar{f} \longmapsto (\sigma_1 : \sigma_2 : 1) \in \mathbf{P}^2$$

whose corresponding line at infinity will be denoted by $\mathcal{L}$

$$\mathcal{L} := \{(\sigma_1 : \sigma_2 : 0);\ (\sigma_1, \sigma_2) \in \mathbf{C}^2 \setminus \{0\}\}.$$

It is important to stress that this compactification is actually natural in the sense that the "behaviour near $\mathcal{L}$" captures a lot of dynamically meaningful information.

2.2 The hypersurfaces $\mathbf{Per}_n(w)$

We give some geometrical properties of dynamically defined subsets of the parameter space which will play a central role in our study.

2.2.1 Defining $\mathbf{Per}_n(w)$ using dynatomic polynomials

For any holomorphic family of rational maps, the following result describes precisely the set of maps having a cycle of given period and multiplier.

Theorem 2.2.1 *Let $f : M \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of degree $d \geq 2$ rational maps. Then for every integer $n \in \mathbf{N}^*$ there exists a holomorphic function p_n on $M \times \mathbf{C}$ which is polynomial on $\mathbf{C}$ and such that:*

1- *for any* $w \in \mathbf{C} \setminus \{1\}$*, the function* $p_n(\lambda, w)$ *vanishes if and only if* f_λ *has a cycle of exact period* n *and multiplipler* w

2- $p_n(\lambda, 1) = 0$ *if and only if* f_λ *has a cycle of exact period* n *and multiplier* 1 *or a cycle of exact period* m *whose multiplier is a primitive* r^{th} *root of unity with* $r \geq 2$ *and* $n = mr$

3- *for every* $\lambda \in M$*, the degree* $N_d(n)$ *of* $p_n(\lambda, \cdot)$ *satisfies* $d^{-n} N_d(n) \sim \frac{1}{n}$.

This leads to the following

Definition 2.2.2 *Under the assumptions and notations of Theorem 2.2.1, one sets*

$$Per_n(w) := \{\lambda \in M / \ p_n(\lambda, w) = 0\}$$

for any integer n *and any complex number* w.

According to Theorem 2.2.1, $Per_n(w)$ is (at least when $w \neq 1$) the set of parameters λ for which f_λ has a cycle of exact period n ad multiplier w. Moreover, $\mathrm{Per}_n(w)$ is an hypersurface in the parameter space M or coincides with M. We also stress that the estimate on the degree $N_d(n)$ of $p_n(\lambda, \cdot)$ will be important in some of our applications.

We now start to explain the construction of the functions p_n. It clearly suffices to treat the case of the family Rat_d and then set $p_n(\lambda, w) := p_n(f_\lambda, w)$ for any holomorphic family $M \ni \lambda \mapsto f_\lambda \in \mathrm{Rat}_d$. Our presentation borrows to the fourth chapter of the book of Silverman [Sil] and the paper [Mi1] of Milnor.

We will consider polynomial families; to deal with the general case one may adapt the proof by using lifts to $\mathbf{C}^2$. According to the discussion we had in subsection 2.1.2, any degree d polynomial φ will be identified to a point in $\mathbf{C}^{d-1}$.

The key point is to associate to any integer n and any polynomial φ of degree $d \geq 2$ a polynomial $\Phi^*_{\varphi,n}$ whose roots, for a generic φ, are exactly the periodic points of φ with exact period n. Such polynomials are called *dynatomic* since they generalize cyclotomic ones, they are defined as follows.

Definition 2.2.3 *For a degree* d *polynomial* φ *and an integer* n *one sets*

$$\Phi_{\varphi,n}(z) := \varphi^n(z) - z.$$

The associated dynatomic polynomials *are then defined by setting*

$$\Phi^*_{\varphi,n}(z) := \prod_{k|n} \left(\Phi_{\varphi,k}(z)\right)^{\mu(\frac{n}{k})}$$

where $\mu : \mathbf{N}^* \to \{\mathbf{-1}, \mathbf{0}, \mathbf{1}\}$ *is the classical Möbius function.*

It is clear that $\Phi_{\varphi,n}$ is a polynomial whose roots are all periodic points of φ with exact period *dividing* n, and that $\Phi^*_{\varphi,n}$ is a fraction whose roots and poles belong to the same set. Actually $\Phi^*_{\varphi,n}$ is still a polynomial but this is not at all obvious. This will be shown by studying its valuation at any m-periodic point of φ for $m|n$ and deeply relies on the fact that the sum $\sum_{k|n} \mu(\frac{k}{n})$ vanishes if $n > 1$. We shall obtain a precise description of the roots of $\Phi^*_{\varphi,n}$:

Theorem 2.2.4 *Let φ be a polynomial of degree $d \geq 2$. Then $\Phi^*_{\varphi,n}$ is a polynomial whose roots are the periodic points of φ with exact period m dividing n and multiplier w satisfying $w^r = 1$ when $2 \leq r := \frac{n}{m}$.*

The proof of the above theorem will be given in the next subsection, for the moment we admit it and prove Theorem 2.2.1. Let us note that the degree $\nu_d(n)$ of $\Phi^*_{\varphi,n}$ is given by $\nu_d(n) = \sum_{k|n} d^k \mu(\frac{n}{k})$ and is clearly equivalent to d^n since $|\mu| \in \{0, 1\}$ and $\mu(1) = 1$. We may also observe that $(\varphi^n)'(z) = 1$ for any root z of $\Phi^*_{\varphi,n}$ whose period strictly divides n.

The construction of $p_n(\varphi, w)$ requires to understand the structure of the zero set of $(\varphi, z) \mapsto \Phi^*_{\varphi,n}(z)$. We recall that φ is seen as a point in $\mathbf{C}^{d-1}$. Here are the informations we need.

Proposition 2.2.5 *The set $Per_n := \{(\varphi, z) \ / \ \Phi^*_{\varphi,n}(z) = 0\}$ is an algebraic subset of $\mathbf{C}^{d-1} \times \mathbf{C}$. The roots of $\Phi^*_{\varphi,n}$ are simple and have exact period n when $\varphi \in \mathbf{C}^{d-1} \setminus X_n$ for some proper algebraic subset X_n of $\mathbf{C}^{d-1}$.*

Proof. One sees on 2.2.3 that $\Phi^*_{\varphi,n}(z)$ is rational in φ. On the other hand, $\Phi^*_{\varphi,n}(z)$ is locally bounded as it follows from the description of its roots given by Theorem 2.2.4. Thus $\Phi^*_{\varphi,n}(z)$ is actually polynomial in φ.
Let us set $\Delta(\varphi) := \prod_{i \neq j} (\alpha_i(\varphi) - \alpha_j(\varphi))$ where the α_i are the roots of $\Phi^*_{\varphi,n}$ counted with multiplicity. This is a well defined function which vanishes exactly when $\Phi^*_{\varphi,n}$ has a multiple root. This function is holomorphic outside its zero set and therefore everywhere by Rado's theorem. Then $\{\Delta = 0\}$ is an analytic subset of $\mathbf{C}^{d-1}$ which is proper since $\varphi_0 := z^d \notin \{\Delta = 0\}$.
Let Y_n be the projection of $Per_n \cap \{(\varphi^n)'(z) = 1\}$ onto $\mathbf{C}^{d-1}$. By Remmert mapping theorem Y_n is an analytic subset of $\mathbf{C}^{d-1}$. Using φ_0 again one sees that $Y_n \neq \mathbf{C}^{d-1}$. One may take $X_n := \{\Delta = 0\} \cup Y_n$. □

Let $\mathrm{Z}(\Phi^*_{\varphi,n})$ be the set of roots of $\Phi^*_{\varphi,n}$ counted with multiplicity. If $z \in \mathrm{Z}(\Phi^*_{\varphi,n})$ has exact period m with $n = mr$, we denote by $w_n(z)$ the r-th power of the multiplier of z (that is $(\varphi^n)'(z)$). As Theorem 2.2.4 tells us :

a point z is periodic of exact period n and $w_n(z) \neq 1$ if and only if
$z \in \mathrm{Z}(\Phi^*_{\varphi,n})$ and $w_n(z) \neq 1$.

Let us now consider the sets

$$\Lambda_n^*(\varphi) := \{w_n(z);\ z \in \mathrm{Z}(\Phi_{\varphi,n}^*)\}$$

and let us denote by $\sigma_i^{*(n)}(\varphi)$, $1 \leq i \leq \nu_d(n)$, the associated symmetric functions. The symmetric functions $\sigma_i^{*(n)}$ are globally defined and continuous on $\mathbf{C}^{d-1}$ and, according to Proposition 2.2.5, are holomorphic outside X_n. These functions are therefore holomorphic on $\mathbf{C}^{d-1}$. We set

$$q_n(\varphi, w) := \prod_{i=0}^{\nu_d(n)} \sigma_i^{*(n)}(\varphi)(-w)^{\nu_d(n)-i}.$$

By construction $q_n(\varphi, w) = 0$ if and only if $w \in \Lambda_n^*(\varphi)$ and q_n is holomorphic in (φ, w) and polynomial in w. As Proposition 2.2.5 shows, the elements of $\mathrm{Z}(\Phi_{\varphi,n}^*)$ are cycles of exact period n and therefore each element of $\Lambda_n^*(\varphi)$ is repeated n times when $\varphi \notin X_n$. This means that there exists a polynomial $p_n(\varphi, \cdot)$ such that $q_n(\varphi, \cdot) = (p_n(\varphi, \cdot))^n$ when $\varphi \notin X_n$. As $p_n(\varphi, w)$ is holomorphic where it does not vanish one sees that p_n extends to all $\mathbf{C}^{d-1} \times \mathbf{C}$. In other words, p_n may be defined by

$$(p_n(\varphi, w))^n := q_n(\varphi, w) = \prod_{i=0}^{\nu_d(n)} \sigma_i^{*(n)}(\varphi)(-w)^{\nu_d(n)-i}.$$

The degree $N_d(n)$ of $p_n(\lambda, \cdot)$ is equal to $\frac{1}{n}\nu_d(n) = \frac{1}{n}\sum_{k|n} \mu(\frac{n}{k})d^k$. In particular $d^{-n}N_d(n) \sim \frac{1}{n}$. □

2.2.2 The construction of dynatomic polynomials

We aim here to prove Theorem 2.2.4. For this purpose, let us recall that the Möbius function $\mu : \mathbf{N}^* \to \{-\mathbf{1}, \mathbf{0}, \mathbf{1}\}$ enjoys the following fundamental property:

$$\sum_{k|n} \mu\left(\frac{n}{k}\right) = 0 \text{ for any } n \in \mathbf{N}^*. \tag{2.2.1}$$

Let us also adopt a few more notations. The valuation of $\Phi_{\varphi,n}$ (resp. $\Phi_{\varphi,n}^*$) at some point z will be denoted $a_z(\varphi, n)$ (resp. $a_z^*(\varphi, n)$). The set of m-periodic points of φ will be denoted $Per(\varphi, m)$.

The following lemma summarizes elementary facts.

Lemma 2.2.6 *Let ψ be a polynomial and $z \in Per(\psi, 1)$. Let $\lambda := \psi'(z)$, then for $q \geq 2$ one has:*

i) $\lambda^q \neq 1 \Rightarrow a_z(\psi, q) = a_z(\psi, 1) = 1$

ii) $\lambda \neq 1$ *and* $\lambda^q = 1 \Rightarrow a_z(\psi, q) > a_z(\psi, 1) = 1$

iii) $\lambda = 1 \Rightarrow a_z(\psi, q) = a_z(\psi, 1) \geq 2$.

Proof. We may assume $z = 0$ and set $\psi = \lambda X + \alpha X^e + o(X^e)$. Then $\Phi_{\psi,q}$ is equal to $(\lambda^q - 1)\, X + o(X)$ when $(\lambda^q - 1) \neq 0$ and to $q\alpha X^e + o(X^e)$ if $\lambda = 1$. The assertions $i)$ to $iii)$ then follow immediately. $\square$

We have to compute $a_z^*(\varphi, n)$ for $z \in Per_n(\varphi, m)$ and $m|n$. We denote by λ the multiplier of z (i.e. $\lambda = (\varphi^m)'(z)$) and set $N := \frac{n}{m}$. When λ is a root of unity we denote by r its order. Clearly, Theorem 2.2.4 will be proved if we establish the following three facts:

F1 : $N = 1 \Rightarrow a_z^*(\varphi, n) > 0$

F2 : $N \geq 2$ and $\lambda^N \neq 1$ or $\lambda = 1 \Rightarrow a_z^*(\varphi, n) = 0$

F3 : $N \geq 2,\ \lambda^N = 1$ and $\lambda \neq 1 \Rightarrow a_z^*(\varphi, n) \geq 0$ and $a_z^*(\varphi, n) > 0$ iff $r = N$.

Besides definitions, the following computation will use the obvious facts that $a_z(\varphi, k) = 0$ when $k \neq qm$ for some $q \in \mathbf{N}^*$ and that $\varphi^{qm} = (\varphi^q)^m$:

$$a_z^*(\varphi, n) = \sum_{k|n} \mu\left(\frac{n}{k}\right) a_z(\varphi, k) = \sum_{qm|n} \mu\left(\frac{n}{qm}\right) a_z(\varphi, qm) =$$
$$\sum_{q|N} \mu\left(\frac{N}{q}\right) a_z(\varphi, qm) = \sum_{q|N} \mu\left(\frac{N}{q}\right) a_z(\varphi^m, q).$$

We now proceed Fact by Fact, always starting with the above identity.

F1) $a_z^*(\varphi, n) = \mu(1) a_z(\varphi^n, 1) > 0$.

F2) Since $\lambda^q \neq 1$ when $q|N$ although $\lambda = 1$, the assertions i) and iii) of Lemma 2.2.6 tell us that $a_z(\varphi^m, q) = a_z(\varphi^m, 1)$. Then $a_z^*(\varphi, n) = \left(\sum_{q|N} \mu\left(\frac{N}{q}\right)\right) a_z(\varphi^m, 1)$ which, according to 2.2.1, equals 0.

F3) When r is not dividing q then $\lambda^q \neq 1$ and, by the assertion i) of Lemma 2.2.6 we have $a_z(\varphi^m, q) = a_z(\varphi^m, 1)$. We may therefore write

$$a_z^*(\varphi, n) = \left(\sum_{q|N} \mu\left(\frac{N}{q}\right)\right) a_z(\varphi^m, 1) + \sum_{q|N,\, r|q} \mu\left(\frac{N}{q}\right) \left[a_z(\varphi^m, q) - a_z(\varphi^m, 1)\right].$$

By 2.2.1, the first term in the above expression vanishes and we get

$$a_z^*(\varphi, n) = \sum_{k|\frac{N}{r}} \mu\left(\frac{N/r}{k}\right) \left[a_z(\varphi^m, rk) - a_z(\varphi^m, 1)\right] =$$

$$\sum_{k|\frac{N}{r}} \mu\left(\frac{N/r}{k}\right) \left[a_z(\varphi^{mr}, k) - a_z(\varphi^m, 1)\right] = a_z^*(\varphi^{mr}, \frac{N}{r}) - a_z(\varphi^m, 1) \sum_{k|\frac{N}{r}} \mu\left(\frac{N/r}{k}\right).$$

We finally consider two subcases.

If $N \neq r$ then, by 2.2.1, $\sum_{k|\frac{N}{r}} \mu\left(\frac{N/r}{k}\right) = 0$ and thus $a_z^*(\varphi, n) = a_z^*(\varphi^{mr}, \frac{N}{r})$. Since z is a fixed point of φ^{mr} whose multiplier equals $\lambda^r = 1$, Fact F2 shows that $a_z^*(\varphi^{mr}, \frac{N}{r}) = 0$.
If $N = r$ we get $a_z^*(\varphi, n) = a_z^*(\varphi^{mr}, 1) - a_z(\varphi^m, 1) = a_z(\varphi^{mr}, 1) - a_z(\varphi^m, 1)$. Since z is a fixed point of φ^m whose multiplier λ satisfies $\lambda^r = \lambda^N = 1$ and $\lambda \neq 1$, the assertion ii) of Lemma 2.2.6 shows that this quantity is strictly positive. $\square$

Remark 2.2.7 *It follows easily from the above construction that the growth of $p_n(\lambda, w)$ is polynomial in λ when $\lambda \in \mathbf{C}^{d-1}$. This shows that $p_n(\lambda, w)$ is actually a polynomial function on $\mathbf{C}^{d-1} \times \mathbf{C}$.*

2.2.3 In the moduli space of degree two rational maps

Theorem 2.2.8 *The map $Mod_2 \to \mathbf{C}^2$ defined by $\bar{f} \mapsto (\sigma_1, \sigma_2)$ is a biholomorphism. Using this identification, $p_n(\lambda, w)$ is a polynomial on $\mathbf{C}^2 \times \mathbf{C}$. Moreover, for every fixed $w \in \mathbf{C}$, the degree of $p_n(\cdot, w)$ is equal to $\frac{\nu_2(n)}{2}$ which is the number of hyperbolic components of period n in the Mandelbrot set.*

Let us recall the projective compactification $Mod_2 \ni \bar{f} \longmapsto (\sigma_1 : \sigma_2 : 1) \in \mathbf{P}^2$ whose corresponding line at infinity $\mathcal{L} = \{(\sigma_1 : \sigma_2 : 0);\ (\sigma_1, \sigma_2) \in \mathbf{C}^2 \setminus \{0\}\}$.

Any $Per_n(w)$ may be seen as a curve in $\mathbf{P}^2$. We shall use the following facts which are also due to Milnor ([Mi1]).

Proposition 2.2.9 *1) For all $w \in \mathbf{C}$ the curve $Per_1(w)$ is actually a line whose equation in $\mathbf{C}^2$ is $(w^2+1)\lambda_1 - w\lambda_2 - (w^3+2) = 0$ and whose point at infinity is $(w : w^2+1 : 0)$. In particular, $Per_1(0) = \{\lambda_1 = 2\}$ is the line of quadratic polynomials, its point at infinity is $(0 : 1 : 0)$.*

2) For $n > 1$ and $w \in \mathbf{C}$ the points at infinity of the curves $Per_n(w)$ are of the form $(u : u^2+1 : 0)$ with $u^q = 1$ and $q \leq n$.

The following Proposition, also du to Milnor (see Theorem 4.2 in [Mi1]), implies that the curves $Per_n(w) = \{p_n(\cdot, w) = 0\}$ have no multiplicity.

Proposition 2.2.10 *Let $N_2(n) := Card\,(Per_n(0) \cap Per_1(0))$ be the number of hyperbolic components of period n in the Mandelbrot set. Then $N_2(n) = \frac{\nu_2(n)}{2}$ where $\nu_2(n)$ is defined inductively by $\nu_2(1) = 2$ and $2^n = \sum_{k|n} \nu_2(k)$. Moreover, for any $w \in \Delta$ and any $\eta \in \Delta$ we have $Deg\, p_n(\cdot, w) = N_2(n) = Card\,(Per_n(w) \cap Per_1(\eta))$.*

Working with the same compactification, Epstein [Eps1] has proved the boundedness of certain hyperbolic components of Mod_2:

Theorem 2.2.11 *Let H be a hyperbolic component of Mod_2 whose elements admit two distinct attracting cycles. If neither attractor is a fixed point then H is relatively compact in Mod_2*

2.3 The connectedness locus in polynomial families

2.3.1 Connected and disconnected Julia sets of polynomials

Among rational functions, polynomials are characterized by the fact that ∞ is a totally invariant critical point. For any polynomial P the super-attractive fixed point ∞ determines a basin of attraction

$$\mathcal{B}_P(\infty) := \{z \in \mathbf{C} \text{ s.t. } \lim_n P^n(z) = \infty\}.$$

This basin is always connected and its boundary is precisely the Julia set J_P of P. The complement of $\mathcal{B}_P(\infty)$ is called the filled-in Julia set of P.

Another nice feature of polynomials is the possibility to define a *Green function* g_P by setting

$$g_P(z) := \lim_n \frac{1}{deg(P)^n} \ln^+ |P^n(z)|.$$

The Green function g_P is a subharmonic function on the complex plane which vanishes exactly on the filled-in Julia set of P.

Any degree d polynomial P is locally conjugated at infinity with the polynomial z^d. This means that there exists a local change of coordinates φ_P (which is called Böttcher function) such that $\varphi_P \circ P = (\varphi_P)^d$ on a neighbourhood of ∞. It is important to stress the following relation between the Böttcher and Green functions:

$$\ln |\varphi_P| = g_P.$$

The only obstruction to the extension of the Böttcher function φ_P to the full basin $\mathcal{B}_P(\infty)$ is the presence of other critical points than ∞ in $\mathcal{B}_P(\infty)$. This leads to the following important result:

Theorem 2.3.1 *For any polynomial P of degree $d \geq 2$ the folllowing conditions are equivalent:*

i) $\mathcal{B}_P(\infty)$ *is simply connected*

ii) J_P *is connected*

iii) $\mathcal{C}_P \cap \mathcal{B}_P(\infty) = \{\infty\}$

iv) P is conformally conjugated to z^d on $\mathcal{B}_P(\infty)$.

The above theorem gives a nice characterization of polynomials having a connected Julia set. Let us apply it to the quadratic family $(z^2+a)_{a\in\mathbf{C}}$. The Julia set J_a of $P_a := z^2 + a$ is connected if and only if the orbit of the critical point 0 is bounded. In other words, the set of parameters a for which J_a is connected is the famous

Definition 2.3.2 Mandelbrot set. *Let P_a denote the quadratic polynomial z^2+a. The Mandelbrot set $\mathcal{M}$ is defined by*

$$\mathcal{M} := \{a \in \mathbf{C} \ s.t. \ \sup_n |P_a^n(0)| < \infty\}.$$

The Mandelbrot set is thus the connectedness locus of the quadratic family. It is not difficult to show that $\mathcal{M}$ is compact. The compacity of the connectedness locus in the polynomial families of degree $d \geq 3$ is a much more delicate question which has been solved by Branner and Hubbard [BH]. We shall treat it in the two next subsections and also present a somewhat more precise result which will turn out to be very useful later.

2.3.2 Polynomials with a bounded critical orbit

We work here with the parametrization $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$ of $\mathcal{P}_d$ and will use the projective compactification $\mathbf{P}^{d-1}$ introduced in the subsection 2.1.2.
We aim to show that the subset of parameters (c, a) for which the polynomial $P_{c,a}$ has at least one bounded critical orbit can only cluster on certain hypersurfaces of $\mathbf{P}_\infty$. The ideas here are essentially those used by Branner and Hubbard for proving the compactness of the connectedness locus (see [BH] Chapter 1, section 3) but we also borrow from the paper ([DF]) of Dujardin and Favre.

We shall use the following

Definition 2.3.3 *The notations are those introduced in subsection 2.1.2. For every $0 \leq i \leq d-2$, the hypersurface Γ_i of $\mathbf{P}_\infty$ is defined by:*

$$\Gamma_i := \{[c : a : 0] /\ \alpha_i(c, a) = 0\}$$

where α_i is the homogeneous polynomial given by:

$$\alpha_i(c, a) := P_{c,a}(c_i) = \frac{1}{d}c_i^d + \sum_{j=2}^{d-1} \frac{(-1)^{d-j}}{j}\sigma_{d-j}(c)c_i^j + a^d.$$

We denote by $\mathcal{B}_i$ the set of parameters (c, a) for which the critical point c_i of $P_{c,a}$ has a bounded forward orbit (recall that $c_0 = 0$):

$$\mathcal{B}_i := \{(c,a) \in \mathbf{C}^{d-1} \ s.t. \ \sup_n |P^n_{c,a}(c_i)| < \infty\}.$$

The key point is the following

Lemma 2.3.4 *The intersection $\Gamma_0 \cap \Gamma_1 \cap \cdots \cap \Gamma_{d-2}$ is empty and $\Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k}$ has codimension k in $\mathbf{P}_\infty$ if $0 \le i_1 < \cdot\cdot < i_k \le d-2$.*

Proof. A simple degree argument shows that $P_{c,a}(0) = P_{c,a}(c_1) = \cdots = P_{c,a}(c_{d-2}) = 0$ implies that $c_1 = \cdots = c_{d-2} = a = 0$. Thus $\Gamma_0 \cap \Gamma_1 \cap \cdots \cap \Gamma_{d-2} = \emptyset$. Then the conclusion follows from Bezout's theorem. □

Since the connectedness locus coincides with $\cap_{0 \le i \le d-2} \mathcal{B}_i$, the announced result can be stated as follows.

Theorem 2.3.5 *For every $0 \le i \le d-2$, the cluster set of $\mathcal{B}_i$ in $\mathbf{P}_\infty$ is contained in Γ_i. In particular, the connectedness locus is compact in $\mathbf{C}^{d-1}$.*

Let us mention the following interesting consequence. We recall that, according to Remark 2.2.7, the sets $Per_m(\eta)$ may be seen as algebraic subsets of the projective space $\mathbf{P}^{d-1}$.

Corollary 2.3.6 *If $1 \le k \le d-1$, $m_1 < m_2 < \cdots < m_k$ and $\sup_{1 \le i \le k} |\eta_i| < 1$ then $Per_{m_1}(\eta_1) \cap \cdots \cap Per_{m_k}(\eta_k)$ is an algebraic subset of codimension k whose intersection with $\mathbf{C}^{d-1}$ is not empty.*

Proof. By Bezout's theorem, $\mathrm{Per}_{m_1}(\eta_1) \cap \cdots \cap \mathrm{Per}_{m_k}(\eta_k)$ is a non-empty algebraic subset of $\mathbf{P}^{d-1}$ whose dimension is bigger than $(d-1-k)$.
Any cycle of attracting basins capture a critical orbit. Therefore, Theorem 2.3.5 implies that the intersection of $\mathbf{P}_\infty$ with $\mathrm{Per}_{m_1}(\eta_1) \cap \cdots \cap \mathrm{Per}_{m_k}(\eta_k)$ is contained in some $\Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k}$ since the m_i are mutually distinct and the $|\eta_i|$ strictly smaller than 1. Then, according to Lemma 2.3.4, $\mathbf{P}_\infty \cap \mathrm{Per}_{m_1}(\eta_1) \cap \cdots \cap \mathrm{Per}_{m_k}(\eta_k)$ has codimension k in $\mathbf{P}_\infty$. The conclusion now follows from obvious dimension considerations. □

As the Green function $g_{c,a}$ of the polynomial $P_{c,a}$ is defined by

$$g_{c,a}(z) := \lim_n d^{-n} \ln^+ |P^n_{c,a}(z)|$$

one sees that

$$\mathcal{B}_i = \{(c,a) \in \mathbf{C}^{d-1} \text{ s.t. } g_{c,a}(c_i) = 0\}.$$

This is why the proof of Theorem 2.3.5 will rely on estimates on the Green functions and, more precisely, on the following result.

Proposition 2.3.7 *Let $g_{c,a}$ be the Green function of $P_{c,a}$ and G be the function defined on $\mathbf{C}^{d-1}$ by: $G(c,a) := \max\{g_{c,a}(c_k);\ 0 \le k \le d-2\}$. Let $\delta := \frac{\sum_{k=0}^{d-2} c_k}{d-1}$. Then the following estimates occur:*

1) $G(c,a) \leq \ln \max\{|a|, |c_k|\} + O(1)$ *if* $\max\{|a|, |c_k|\} \geq 1$

2) $\max\{g_{c,a}(z), G(c,a)\} \geq \ln |z - \delta| - \ln 4$.

Let us first show how Theorem 2.3.5 may be deduced from Proposition 2.3.7.

Proof of Theorem 2.3.5. Let $\|(c,a)\|_\infty := \max\big(|a|, |c_k|\big)$. We simply have to check that $\alpha_i\big(\frac{(c,a)}{\|(c,a)\|_\infty}\big)$ tends to 0 when $\|(c,a)\|_\infty$ tends to $+\infty$ and $g_{c,a}(c_i)$ stays equal to 0. As $P_{c,a}(c_i) = \alpha_i(c,a)$ and $g_{c,a}(c_i) = 0$, the estimates given by Proposition 2.3.7 yield:

$$\ln \|(c,a)\|_\infty + O(1) \geq \max\big(d g_{c,a}(c_i), G(c,a)\big) = \max\big(g_{c,a} \circ P_{c,a}(c_i), G(c,a)\big) \geq \geq \ln \frac{1}{4}|\alpha_i(c,a) - \delta|$$

since α_i is d-homogeneous we then get:

$$(1-d)\ln \|(c,a)\|_\infty + O(1) \geq \ln \frac{1}{4}|\alpha_i\big(\frac{(c,a)}{\|(c,a)\|_\infty}\big) - \frac{\delta}{\|(c,a)\|_\infty^d}|$$

and the conclusion follows since $\frac{\delta}{\|(c,a)\|_\infty^d}$ tends to 0 when $\|(c,a)\|_\infty$ tends to $+\infty$. $\square$

Let us end this subsection by giving a

Proof of Proposition 2.3.7. The first estimate is a standard consequence of the uniform growth of $P_{c,a}$ at infinity. Let us however prove it with care. We will set $A := \|(c,a)\|_\infty$ and $M_A(z) := \max(A, |z|)$. From

$$|P_{c,a}(z)| \leq \tfrac{1}{d}|z|^d\big(1 + d\max\{\tfrac{|\sigma_{d-j}(c)|}{j|z|^{d-j}}, \tfrac{|a|^d}{|z|^d}\}\big)$$

we get $|P_{c,a}(z)| \leq C_d|z|^d$ for $|z| \geq A$ where the constant C_d only depends on d. We may assume that $C_d \geq 1$. By the maximum modulus principle this yields

$$|P_{c,a}(z)| \leq C_d M_A(z)^d.$$

It is easy to check that

$$M_A(Cz) \leq C M_A(z) \text{ if } C \geq 1$$
$$M_A\big(M_A(z)^N\big) = M_A(z)^N \text{ if } A \geq 1.$$

From now on we shall assume that $A \geq 1$. By induction one gets

$$|P_{c,a}^n(z)| \leq C_d^{1+d+\cdots+d^{n-1}} M_A(z)^{d^n}$$

which implies

$$g_{c,a}(z) \leq \tfrac{\ln C_d}{d-1} + \ln \max\big(\|(c,a)\|_\infty, |z|\big) \text{ if } \|(c,a)\|_\infty \geq 1$$

and in particular

$$G(c,a) \leq \frac{\ln C_d}{d-1} + \ln \|(c,a)\|_\infty \text{ if } \|(c,a)\|_\infty \geq 1.$$

The second estimate is really more subtle. It exploits the fact that the Green function g_P coincides with the log-modulus of the Böttcher coordinate function φ_P and relies on a sharp control of the distorsions of this holomorphic function.

The Böttcher coordinate function is a univalent function $\varphi_{c,a} : \{g_{c,a} > G(c,a)\} \to \mathbf{C}$ such that $\varphi_{c,a} \circ P_{c,a} = \varphi_{c,a}^d$. It is easy to check that $\ln|\varphi_{c,a}| = g_{c,a}$ where it makes sense and that $\varphi_{c,a}(z) = z - \delta + O(\frac{1}{z})$ where $\delta := \frac{\sigma_1(c)}{d-1} = \frac{\sum c_k}{d-1}$.

One thus sees that $\varphi_{c,a} : \{g_{c,a} > G(c,a)\} \to \mathbf{C} \setminus \overline{D}\big(0, e^{G(c,a)}\big)$ is a univalent map whose inverse $\psi_{c,a}$ satisfies $\psi_{c,a}(z) = z + \delta + O(\frac{1}{z})$ at infinity. We shall now apply the following result, which is a version of the Koebe $\frac{1}{4}$-theorem (see [BH], Corollary 3.3), to $\psi_{c,a}$.

Theorem 2.3.8 *If $F : \widehat{\mathbf{C}} \setminus \overline{D}_r \to \widehat{\mathbf{C}}$ is holomorphic and injective and*

$$F(z) = z + \textstyle\sum_{n=1}^{\infty} \frac{a_n}{z^n},\ z \in \mathbf{C} \setminus \overline{D}_r$$

then $\mathbf{C} \setminus \overline{D}_{2r} \subset F\big(\mathbf{C} \setminus \overline{D}_r\big)$.

Pick $z \in \mathbf{C}$ and set $r := 2\max\big(e^{g_{c,a}(z)}, e^{G(c,a)}\big)$. Then $z \notin \psi_{c,a}\big(\mathbf{C} \setminus \overline{D}_r\big)$ since otherwise we would have $e^{g_{c,a}}(z) = |\varphi_{c,a}(z)| > r \geq 2e^{g_{c,a}}(z)$.
Thus, according to the above distorsion theorem, $z \notin \mathbf{C} \setminus \overline{D}\big(\delta, 2r\big)$. In other words $|z - \delta| \leq 2r = 4\max\big(e^{g_{c,a}(z)}, e^{G(c,a)}\big)$ and the desired estimate follows by taking logarithms. □

Chapter 3

The bifurcation current

We consider in this chapter an arbitrary holomorphic family of degree d rational maps $(f_\lambda)_{\lambda\in M}$ with marked critical points (see 2.1.2). The Julia set of f_λ wil be denoted $\mathcal{J}_\lambda$, its critical set $\mathcal{C}_\lambda$.

Our first aim is to describe necessary and sufficient conditions for the Julia sets $\mathcal{J}_\lambda$ to move continuously (and actually holomorphically) with the parameter λ. The parameters around which such a motion does exist are called stable. We will show that the set of stable parameters is dense in the parameter space; this is the essence of the Mañé-Sad-Sullivan theory. We will then exhibit a closed positive $(1,1)$-current on the parameter space whose support is precisely the complement of the stabilty locus; this is the bifurcation current.

3.1 Stability versus bifurcation

3.1.1 Motion of repelling cycles and Julia sets

As Julia sets coincide with the closure of the sets of repelling cycles (see Theorems 1.1.4 and 1.2.5), it is natural to investigate how $\mathcal{J}_\lambda$ varies with λ through the parametrizations of such cycles.

Assume that f_{λ_0} has a repelling n-cycle $\{z_0, f_{\lambda_0}(z_0), \cdots, f_{\lambda_0}^{n-1}(z_0)\}$. Then, the implicit function theorem, applied to the equation $f_\lambda^n(z) - z = 0$ at (λ_0, z_0) shows that there exists a holomorphic map $h_\lambda(z_0) : U_0 \to \mathbf{P}^1$ such that $h_{\lambda_0}(z_0) = z_0$ and $h_\lambda(z_0)$ is a n-periodic repelling point of f_λ for all $\lambda \in U_0$. Moreover, $h_\lambda(\cdot)$ can be extended to the full cycle so that $f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$.
This shows that every repelling n-cycle of f_{λ_0} *moves holomorphically* on some neighbourhood of λ_0.

These observations lead to the following formal definition.

Definition 3.1.1 *Let us denote by* $\mathcal{R}_{\lambda,n}$ *the set of repelling n-cycles of* f_λ*. Let* Ω *be a neighbourhood of* λ_0 *in* M*. One says that* $\mathcal{R}_{\lambda_0,n}$ *moves holomorphically on* Ω *if there exists a map*

$$h : \Omega \times \mathcal{R}_{\lambda_0,n} \ni (\lambda, z) \mapsto h_\lambda(z) \in \mathcal{R}_{\lambda,n}$$

which depends holomorphically on λ *and satisfies* $h_{\lambda_0} = Id$*,* $f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$*.*

More generally, the holomorphic motion of an arbitrary subset of the Riemann sphere is defined in the following way.

Definition 3.1.2 *Let* E *be subset of the Riemann sphere and* Ω *be a complex manifold. Let* $\lambda_0 \in \Omega$*. An holomorphic motion of* E *over* Ω *centered at* λ_0 *is a map*

$$h : \Omega \times E \ni (\lambda, z) \mapsto h_\lambda(z) \in \widehat{\mathbf{C}}$$

which satisfies the following properties:

i) $h_{\lambda_0} = Id|_E$

ii) $E \ni z \mapsto h_\lambda(z)$ *is one-to-one for every* $\lambda \in \Omega$

iii) $\Omega \ni \lambda \mapsto h_\lambda(z)$ *is holomorphic for every* $z \in E$*.*

The interest of holomorphic motions relies on the fact that any holomorphic motion of a set E extends to the closure of E. This is a quite simple consequence of Picard-Montel theorem.

Lemma 3.1.3 (basic λ-lemma) *Let* $E \subset \widehat{\mathbf{C}}$ *be a subset of the Riemann sphere and* $\sigma : E \times \Omega \ni (z,t) \mapsto \sigma(z,t) \in \widehat{\mathbf{C}}$ *be a holomorphic motion of* E *over* Ω*. Then* σ *extends to a holomorphic motion* $\tilde{\sigma}$ *of* $\overline{E}$ *over* Ω*. Moreover* $\tilde{\sigma}$ *is continuous on* $\overline{E} \times \Omega$*.*

As $\mathcal{J}_\lambda$ is the closure of the set of repelling cycles of f_λ, this lemma implies that the Julia set $\mathcal{J}_{\lambda_0}$ moves holomorphically over a neighbourhood V_{λ_0} of λ_0 in M as soon as all repelling cycles of f_{λ_0} move holomorphically on V_{λ_0}. Moreover, the holomorphic motion obtained in this way clearly conjugates the dynamics: $h_\lambda(\mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda$ and $f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$ on $\mathcal{J}_{\lambda_0}$.

The above arguments lead to the following basic observation.

Lemma 3.1.4 *If there exists a neighbourhood* Ω *of* λ_0 *in the parameter space* M *such that, for all sufficebtly big* n *,* $\mathcal{R}_{\lambda_0,n}$ *moves holomorphically on* Ω*, then there exists a holomorphic motion* $h_\lambda : \mathcal{J}_{\lambda_0} \to \mathcal{J}_\lambda$ *which conjugates the dynamics.*

Let us mention here that there exists a much stronger version of the λ-lemma, which is due to Slodkowski and shows that the holomorphic motion actually extends as a quasi-conformal transformation of the full Riemann sphere (see Theorem 4.3.4). In particular, under the assumption of the above lemma, f_λ and f_{λ_0} are quasi-conformally conjugated when $\lambda \in \Omega$.

We may now define the set of stable parameters and its complement; the bifurcation locus.

Definition 3.1.5 *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of degree d rational maps. The* stable set $\mathcal{S}$ *is the set of parameters $\lambda_0 \in M$ for which there exists a neighbourhood Ω of λ_0 and a holomorphic motion h_λ of $\mathcal{J}_{\lambda_0}$ over Ω centered at λ_0 and such that $f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$ on $\mathcal{J}_{\lambda_0}$.*
The bifurcation locus B_{if} *is the complement $M \setminus \mathcal{S}$.*

By definition, $\mathcal{S}$ is an open subset of M but it is however not yet clear that it is not empty. We shall actually show that $\mathcal{S}$ is dense in M. To this purpose we will prove that the stability is characterized by the stability of the critical orbits. The next subsection will be devoted to this simple but remarkable fact.

3.1.2 Stability of critical orbits

Let us start by explaining why bifurcations are related with the instability of critical orbits.

As we saw in Lemma 3.1.4, a parameter λ_0 belongs to the bifurcation locus if for any neighbourhood Ω of λ_0 in the parameter space M there exists $n_0 \geq 0$ for which $\mathcal{R}_{\lambda_0,n_0}$ does not move holomorphically on Ω.
It is not very difficult to see that this forces one of the repelling n_0-cycles of f_{λ_0}, say R_{λ_0}, to become neutral and then attracting for a certain value $\lambda_1 \in \Omega$. Now comes the crucial point. A classical result asserts that the basin of attraction of any attracting cycle of a rational map contains a critical point (see [BM] Théorème II.5). Thus, one of the critical orbit $f_\lambda^k(c_i(\lambda))$ is uniformly converging to R_λ on a neighbourhood of λ_1. Then the sequence $f_\lambda^k(c_i(\lambda))$ cannot be normal on Ω since otherwise, by Hurwitz lemma, it should converge uniformly to R_λ which is repelling for λ close to λ_0. This arguments show that the bifurcation locus is contained in the set of parameters around which the post-critical set does not move continuously.

This is an extremely important observation because it will allow us to detect bifurcations by considering only the critical orbits. It leads to the following definitions.

Definition 3.1.6 *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of degree d rational maps. A marked critical point $c(\lambda)$ is said to be* passive *at λ_0 if the sequence $\left(f_\lambda^n(c(\lambda))\right)_n$*

is normal on some neighbourhood of λ_0. *If* $c(\lambda)$ *is not passive at* λ_0 *one says that it is* active. *The* activity locus *of* $c(\lambda)$ *is the set of parameters at which* $c(\lambda)$ *is active.*

The key result may now be given.

Lemma 3.1.7 *In a holomorphic family with marked critical points the bifurcation locus coincide with the union of the activity loci of the critical points.*

Proof. According to our previous arguments, the bifurcation locus is contained in the union of the activity loci. It remains to show that, for any marked critical point $c(\lambda)$, the sequence $\left(f_\lambda^n\left(c(\lambda)\right)\right)_n$ is normal on $\mathcal{S}$. Assume that $\lambda_0 \in \mathcal{S}$. As $\mathcal{J}_{\lambda_0}$ is a perfect compact set, we may find three distinct points a_1, a_2, a_3 on $\mathcal{J}_{\lambda_0}$ which are avoided by the orbit of $c(\lambda_0)$. Since the holomorphic motion h_λ of $\mathcal{J}_{\lambda_0}$ conjugates the dynamics, the orbit of $c(\lambda)$ avoids $\{h_\lambda(a_j);\ 1 \leq j \leq 3\}$ for all λ in a small neighbourhood of λ_0. The conclusion then follows from Picard-Montel's Theorem. $\square$

The following lemma is quite useful.

Lemma 3.1.8 *If* λ_0 *belongs to the activity locus of some marked critical point* $c(\lambda)$ *then there exists a sequence of parameters* $\lambda_k \to \lambda_0$ *such that* $c(\lambda_k)$ *belongs to some super-attracting cycle of* f_{λ_k} *or is strictly preperiodic to some repelling cycle of* f_{λ_k}.

Proof. Since $c(\lambda)$ is active at λ_0 it cannot be persistently fixed. After an arbitrarily small perturbation of λ_0 we may asume that there exists holomorphic maps $c_{-2}(\lambda),\ c_{-1}(\lambda)$ near λ_0 such that $f_\lambda\left(c_{-2}(\lambda)\right) = c_{-1}(\lambda),\ f_\lambda\left(c_{-1}(\lambda)\right) = c(\lambda)$ and $Card\ \{c_{-2}(\lambda),\ c_{-1}(\lambda), c(\lambda)\} = 3$. By Picard-Montel Theorem, the sequence $\left(f_\lambda^n\left(c(\lambda)\right)\right)_n$ cannot avoid $\{c_{-2}(\lambda), c_{-1}(\lambda), c(\lambda)\}$ on any neighbourhood of λ_0.
A similar argument using Picard-Montel Theorem and a repelling cycle of period $n_0 \geq 3$ shows that $c(\lambda)$ becomes strictly preperiodic for λ arbitrarily close to λ_0. $\square$

We are now ready to state and prove Mañé-Sad-Sullivan theorem.

Theorem 3.1.9 *Let* $(f_\lambda)_{\lambda \in M}$ *be a holomorphic family of degree* d *rational maps with marked critical points* $\{c_1(\lambda), \cdots, c_{2d-2}(\lambda)\}$.
A parameter λ_0 *is stable if one of the following equivalent conditions is satisfied.*

1) $\mathcal{J}_{\lambda_0}$ *moves holomorphically around* λ_0 *(see definition 3.1.5)*

2) the critical points are passive at λ_0 *(see definition 3.1.6)*

3) f_λ *has no unpersistent neutral cycles for* λ *sufficently close to* λ_0.

The set $\mathcal{S}$ *of stable parameters is dense in* M.

Proof. The equivalence between 1) and 2) is given by Lemma 3.1.7. Similar arguments may allow to show that 3) is an equivalent statement (we will give an alternative proof of that later). It remains to show that $\mathcal{S}$ is dense in M. According to Lemma 3.1.8 we may perturb λ_0 and assume that f_{λ_0} has a superattracting cycle of period bigger than 3 which persists, as an attracting cycle, for λ close enough to λ_0. If λ_0 is still active, Picard-Montel's Theorem shows that a new perturbation guarantees that a critical point falls in the attracting cycle and, therefore, becomes passive. Since the number of critical points is finite, we may make all critical points passive after a finite number of perturbations. □

Example 3.1.10 *In the quadratic polynomial family, the bifurcation locus is the boundary of the connectivity locus (or the Mandelbrot set). Indeed, it follows immedialey from the definition 2.3.2 of the Mandelbrot set that its boundary is the activity locus.*

Example 3.1.11 *The situation is more complicated in the family* $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$ *of degree d poynomials when $d \geq 3$ (see subsection 2.1.2). Indeed, Theorem 2.3.5 shows that the bifurcation (i.e. activity) locus is not bounded since it coincides with the boundary of* $\cup_{0\leq i\leq d-2}\mathcal{B}_i$ *(where $\mathcal{B}_i$ is the set of parameters for which the orbit of the critical point c_i is bounded) while the connectedness locus* $\cap_{0\leq i\leq d-2}\mathcal{B}_i$ *is bounded.*

Although we shall not use it, we end this section by quoting a very interesting classification of the activity situations which is due to Dujardin and Favre (see [DF] Theorem 4).

Theorem 3.1.12 *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of degree d rational maps with a marked critical point $c(\lambda)$. If c is passive on some connected open subset U of M then exactly one of the following cases holds:*

1) *c is never preperiodic in U and the closure of its orbits move holomorphically on U*

2) *c is persitently preperiodic on U*

3) *the set of parameters for which c is preperiodic is a closed subvariety in U. Moreover, either there exists a persistently attracting cycle attracting c throughout U, or c lies in the interior of a linearization domain associated to a persistent irrationally neutral periodic point.*

It is worth emphasize that the proof of that result relies on purely local, and subtle, arguments.

3.1.3 Some remarkable parameters

• As we already mentionned, the basin of any attracting cycle of a rational map contains a critical point. As a consequence, any rational map of degree d has at most $2d-2$ attracting cycles. Then, using perturbation arguments, Fatou and Julia proved that the number of non-repelling cycles is bounded by $6d-6$. The sharp bound has been obtained by Shishikura [Sh] using quasiconformal surgery, Epstein has given a more algebraic proof based on quadratic differentials (see [Eps2]).

Theorem 3.1.13 *A rational map of degree d has at most* $2d-2$ *non-repelling cycles.*

In particular, any degre d rational map cannot have more than $2d-2$ neutral cycles. Shishikura has also shown that the bound $2d-2$ is sharp (we will give another proof, using bifurcation currents, in subsectiondensShiHyp). Let us mention that the Julia set of degree d maps having $2d-2$ Cremer cycles coincides with the full Riemann sphere. These results motivate the following definition.

Definition 3.1.14 *The set* $\mathcal{S}hi$ *of degree* d Shishikura rational maps *is defined by*

$$\mathcal{S}hi = \{f \in Rat_d \ / \ f \text{ has } 2d-2 \text{ neutral cycles}\}.$$

In a holomorphic family $(f_\lambda)_{\lambda \in M}$ *we shall denote by* $\mathcal{S}hi(M)$ *the set of parameters* λ *for which* f_λ *is Shishikura.*

According to Theorem 3.1.9, one has $\mathcal{S}\mathrm{hi} \subset \mathcal{B}\mathrm{if}$. In the last chapter, we will obtain some informations on the geometry of $\mathcal{S}\mathrm{hi}$ and, in particular, reprove that $\mathcal{S}\mathrm{hi}$ is not empty.

• Any repelling cycle of $f \in Rat_d$ is an invariant (compact) set on which f is uniformly expanding. Some rational map may be uniformly expanding on much bigger compact sets. Such compact sets are called *hyperbolic* and are necessarily contained in the Julia set. A rational map which is uniformly expanding on its Julia set is said to be hyperbolic. Let us give a precise definition.

Definition 3.1.15 *Let* f *be a rational map. A compact set* K *of the Riemann sphere is said to be* hyperbolic *for* f *if it is invariant* ($f(K) \subset K$) *and* f *is uniformly expanding on* K*: there exists* $C > 0$ *and* $M > 1$ *such that*

$$|(f^n)'(z)|_\sigma \geq CM^n \ ; \ \ \forall z \in K, \ \forall n \geq 0.$$

It may happen that a critical orbit is captured by some hyperbolic set and, in particular, by a repelling cycle. Such rational maps play a very important role in the study of the parameter space. The reason is that they allow to define transfer maps which carry some informations from the dynamical plane to the parameter space. A particular attention will be devoted to maps for which all critical orbits are captured by a hyperbolic set.

Definition 3.1.16 *The set* $\mathcal{M}is$ *of degree* d Misiurewicz rational maps *is defined by*

$$\mathcal{M}is = \{f \in Rat_d \ / \ \text{all critical orbits of } f \text{ are captured by a compact hyperbolic set}\}$$

When the hyperbolic set is an union of repelling cycles the map is said to be strongly Misiurewicz and the set of such maps is denoted $\mathcal{M}iss$. *In a holomorphic family* $(f_\lambda)_{\lambda\in M}$ *we shall denote by* $\mathcal{M}is(M)$ *(resp.* $\mathcal{M}iss(M)$*) the set of parameters* λ *for which* f_λ *is Misiuerewicz (resp. strongly Misiurewicz).*

Within a holomorphic family $(f_\lambda)_{\lambda\in M}$, one may show that a critical point whose orbit is captured by a hyperbolic set and leaves this set under a small perturbation is active. In particular, we have the following inclusion: $\mathcal{M}\mathrm{is} \subset \mathcal{B}\mathrm{if}$. To prove this, one first has to construct a holomorphic motion of the hyperbolic set and then linearize along its orbits (see [G]). When the hyperbolic set is a cycle, the motion is given by the implicit function theorem and the linearizability is a well known fact.

Lemma 3.1.17 *Let* $(f_\lambda)_{\lambda\in M}$ *be a holomorphic family of degree* d *rational maps with a marked critical point* $c(\lambda)$. *Assume that* f_λ *has a repelling* n*-cycle* $\mathcal{R}(\lambda) := \{z_\lambda, f_\lambda(z_\lambda), \cdots, f_\lambda^{n-1}(z_\lambda)\}$ *for* $\lambda \in U$. *If* $f_{\lambda_0}^k(c(\lambda_0) \in \mathcal{R}(\lambda_0)$ *for some* $\lambda_0 \in U$ *but not for all* $\lambda \in U$ *then* $c(\lambda)$ *is active at* λ_0.

Proof. We may assume that $n = 1$ which means that z_λ is fixed by f_λ. Shrinking U and linearizing we get a a family of local biholomorphisms ϕ_λ which depends holomorphically on λ and such that $\phi_\lambda(0) = z_\lambda$ and $f_\lambda \circ \phi_\lambda(u) = m_\lambda \phi_\lambda(u)$ (see [BM] Théorème II.1 and Remarque II.2). As z_λ is repelling, one has $|m_\lambda| > 1$. Let us set $u(\lambda) := \phi_\lambda^{-1}(f_\lambda^k(c(\lambda))$, then $f_\lambda^{p+k}(c_\lambda) = f_\lambda^p(\phi_\lambda(u(\lambda))) = m_\lambda u(\lambda)$ which shows that $f_\lambda^{p+k}(c_\lambda)$ is not normal at λ_0 since, by assumption, $u(\lambda_0) = 0$ but u does not vanish identically on U.

□

• As we already mentionned, a rational map which is uniformly expanding on its Julia set is called hyperbolic. From a dynamical point of view, the study of such maps turns out to be much easier.

Definition 3.1.18 *The set* $\mathcal{H}yp$ *of degree* d hyperbolic rational maps *is defined by*

$$\mathcal{H}yp = \{f \in Rat_d \ / \ f \text{ is uniformly expanding on its Julia set}\}.$$

In a holomorphic family $(f_\lambda)_{\lambda\in M}$ *we shall denote by* $\mathcal{H}yp(M)$ *the set of parameters* λ *for which* f_λ *is hyperbolic.*

There are several characterizations of hyperbolicity. One may show that a rational map f is hyperbolic if and only if its postcitical set does not contaminate its Julia set:

$$f \text{ is hyperbolic} \Leftrightarrow \overline{\cup_{n\geq 0} f^n(\mathcal{C}_f)} \cap \mathcal{J}_f = \emptyset.$$

As a consequence, in any holomorphic family $(f_\lambda)_{\lambda\in M}$, hyperbolic parameters are stable : $H_{\rm yp}(M) \subset \mathcal{S}$. This characterization also implies that a hyperbolic map has only attracting or repelling cycles.

The Fatou's conjecture asserts that the hyperbolic parameters are dense in any holomorphic family, it is an open problem even for quadratic polynomials. According to Mañé-Sad-Sullivan Theorem it can be rephrased as follows

Fatou's conjecture 3.1.19 *$H_{yp}(M) = \mathcal{S}$ for any holomorphic family $(f_\lambda)_{\lambda\in M}$.*

let us also mention that Mañé, Sad and Sullivan have shown that hyperbolic and non-hyperbolic parameters cannot coexist in the same stable component (i.e connected component of $\mathcal{S}$).

Although the bifurcation locus of the quadratic polynomial family is clearly accumulated by hyperbolic parameters, this is far from being clear in other families and seems to be an interesting question. In the last chapter we will show that parameters which, in some sense, produce the strongest bifurcations, are accumulated by hyperbolic parameters.

3.2 Potential theoretic approach

3.2.1 Przytycki's generalized formula

We want here to establish a fundamental formula which relates the Lyapunov exponent (see definition 1.3.1) and the critical points. The model is a formula due to Przytycki which deals with the case of polynomials.

Theorem 3.2.1 (Przytycki's formula) *Let P be a unitary degree d polynomial, $L(P)$ its Lyapunov exponent and g_P its Green function. Then*

$$L(P) = \ln d + \sum g_P(c)$$

where the sum is taken over the critical points of P counted with multiplicity.

Proof. Let us write $c_1, c_2, \cdots, c_{d-1}$ the critical points of P. Then

$$L(P) = \int_{\mathbf{C}} \ln|P'|\,\mu_P = \int_{\mathbf{C}} \ln|d\prod_{j=1}^{d-1}(z-c_j)|\,\mu_P = \ln d + \sum_{j=1}^{d-1}\int_{\mathbf{C}} \ln|z-c_j|\,\mu_P.$$

Now, since $\mu_P = dd^c g_P$, the formula immediately follows by an integration by parts. □

Remark 3.2.2 *As the Green function g_P of a polynomial P is positive (see Proposition 1.1.12) the above formula implies that $L(P) \geq \ln d$.*

It is more delicate to perform such an integration by part in the case of a rational map f. To this purpose we will work in the line bundle $\mathcal{O}_{\mathbf{P}^1}(D)$ for $D := 2(d-1)$ which we endow with two metrics, the flat one $\|[z,x]\|_0$ and the Green metric $\|[z,x]\|_{G_F}$ whose potential is the Green function G_F of some lift F of f (see subsection 1.3.1).

In this general situation the integration by part yields the following formula.

Proposition 3.2.3 *Let f be a rational map of degree $d \geq 2$ and F be one of its lifts. Let $D := 2(d-1)$ and Jac_F be the holomorphic section of $\mathcal{O}_{\mathbf{P}^1}(D)$ induced by $\det F'$. Let g_F be the Green function of F on $\mathbf{P}^1$ and μ_f be the Green measure of f. Then*

$$L(f) + \ln d = \int_{\mathbf{P}^1} g_F[\mathcal{C}_f] \ - \ 2(d-1) \int_{\mathbf{P}^1} g_F(\mu_f + \omega) \ + \ \int_{\mathbf{P}^1} \ln \|Jac_F\|_0 \, \omega.$$

Proof. Let us recall that $\| \cdot \|_G = e^{-Dg_F} \| \cdot \|_0$. According to Lemma 1.3.4 we have:

$$L(f) + \ln d = \int_{\mathbf{P}^1} \ln \|Jac_F\|_{G_F} \, \mu_f = \int_{\mathbf{P}^1} \ln \|Jac_F\|_{G_F} \, dd^c g_F + \int_{\mathbf{P}^1} \ln \|Jac_F\|_{G_F} \, \omega$$

which, after integrating by parts, yields

$$L(f) + \ln d = \int_{\mathbf{P}^1} g_F \left(dd^c \ln \|Jac_F\|_{G_F}\right) + \int_{\mathbf{P}^1} \ln \|Jac_F\|_{G_F} \, \omega$$

and by Poincaré-Lelong formula:

$$L(f) + \ln d = \int_{\mathbf{P}^1} g_F \left([\mathcal{C}_f] - D\mu_f\right) + \int_{\mathbf{P}^1} \left(\ln \|Jac_F\|_0 - Dg_F\right) \, \omega. \qquad (3.2.1)$$

□

When working with a holomorphic family of polynomials $(P_\lambda)_M$, Przytycki's formula says that the Lyapunov function $L(P_\lambda)$ and the sum of values of the Green function on critical points differ from a constant. In particular, $L(P_\lambda)$ is a *p.s.h* function on M and these two functions induce the same $(1,1)$ current on M. It is then rather clear, using Mañé-Sad-Sullivan Theorem 3.1.9, that this current is exactly supported by the bifurcation locus.

We aim to generalize this to holomorphic families of rational maps and will therefore compute the dd^c of the left part of formula 3.2.1. The following formula has been established in [BB1].

Theorem 3.2.4 *Let $(f_\lambda)_M$ be a holomorphic family of degree d rational maps which admits a holomorphic family of lifts $(F_\lambda)_M$. Let p_M (resp $p_{\mathbf{P}^1}$) be the canonical projection from $M \times \mathbf{P}^1$ onto M (resp.$\mathbf{P}^1$). Then*

$$dd^c L(\lambda) = (p_M)_\star \left((dd^c_{\lambda,z} g_\lambda(z) + \hat{\omega}) \wedge [C] \right) \tag{3.2.2}$$

where $C := \{(\lambda, z) \in M \times \mathbf{P}^1 \ / \ z \in \mathcal{C}_\lambda\}$, g_λ is the Green function F_λ on $\mathbf{P}^1$, $L(\lambda)$ the Lyapunov exponent of f_λ and $\hat{\omega} := p^\star_{\mathbf{P}^1}\omega$.
Moreover, the function $\lambda \mapsto \int_{\mathbf{P}^1} g_\lambda(\mu_\lambda + \omega)$ is pluriharmonic on M.

Proof. We may work locally and define a holomorphic section Jac_λ of $\mathcal{O}_{\mathbf{P}^1}(D)$ induced by $\det F'_\lambda$. Then $\widetilde{Jac}(\lambda, [z]) := (\lambda, Jac_\lambda([z]))$ is a holomorphic section of the line bundle $M \times \mathcal{O}_{\mathbf{P}^1}(D)$ over $M \times \mathbf{P}^1$.

Let us rewrite Proposition 3.2.3 on the form $L(\lambda) + \ln d = H(\lambda) - D\ B(\lambda)$ where $H(\lambda) := \int_{\mathbf{P}^1} g_\lambda[\mathcal{C}_\lambda] \ + \ \int_{\mathbf{P}^1} \ln \|Jac_\lambda\|_0\, \omega$, $B(\lambda) := \int_{\mathbf{P}^1} g_\lambda(\mu_\lambda + \omega)$ and $D := 2(d-1)$.

We first compute $dd^c H$. Let Φ denote a $(m-1, m-1)$ test form where m is the dimension of M. Then $\langle dd^c H, \Phi\rangle = I_1 + I_2$ where $I_1 := \int_M dd^c\Phi \int_{\mathbf{P}^1} g_\lambda[\mathcal{C}_\lambda]$ and $I_2 := \int_M dd^c\Phi \int_{\mathbf{P}^1} \ln \|Jac_\lambda\|_0\, \omega$.

Slicing and then integrating by parts, we get

$$I_1 = \int_M dd^c\Phi \int_{\mathbf{P}^1} (g[C])_\lambda = \int_{M\times\mathbf{P}^1} (p_M)^\star\, (dd^c\Phi) \wedge (g[C]) =$$
$$= \int_{M\times\mathbf{P}^1} (p_M)^\star\Phi \wedge dd^c g \wedge [C].$$

Similarly and then using Poincaré-Lelong identity $dd^c \ln \|\widetilde{Jac}\|_0 = [C] - D\hat{\omega}$ one gets

$$I_2 = \int_M dd^c\Phi \int_{\mathbf{P}^1} \left(\ln \|\widetilde{Jac}\|_0\, \hat{\omega}\right)_\lambda = \int_{M\times\mathbf{P}^1} (p_M)^\star\, (dd^c\Phi) \wedge \ln \|\widetilde{Jac}\|_0\, \hat{\omega} =$$
$$= \int_{M\times\mathbf{P}^1} (p_M)^\star\Phi \wedge dd^c \ln \|\widetilde{Jac}\|_0 \wedge \hat{\omega} = \int_{M\times\mathbf{P}^1} (p_M)^\star\Phi \wedge \hat{\omega} \wedge [C].$$

This shows that $dd^c H = (p_M)_\star \left((dd^c_{\lambda,z} g_\lambda(z) + \hat{\omega}) \wedge [C] \right)$.

It remains to show that $dd^c B = 0$. It seems very difficult to prove this by calculus, we will use a trick which exploits the dynamical situation. If one replace the family $(f_\lambda)_M$ by the family $(f^2_\lambda)_M$ then L becomes $2L$ and $dd^c H$ becomes $2dd^c H$ while B is unchanged. Thus, applying Proposition 3.2.3 to $(f_\lambda)_M$ and taking dd^c yields $dd^c L = dd^c H - 2(d-1)dd^c B$ but, with the family $(f^2_\lambda)_M$, this yields $2dd^c L = 2dd^c H - 2(d^2-1)dd^c B$. By comparison one obtains $dd^c B = 0$. □

Remark 3.2.5 *In [BB1] the formula 3.2.2 has been generalized to the case of holomorphic families of endomorphisms of* $\mathbf{P}^k$*. It becomes*

$$dd^c L(\lambda) = (p_M)_\star \left((dd^c_{\lambda,z} g_\lambda(z) + \hat{\omega})^k \wedge [C] \right)$$

where $L(\lambda)$ *is now the sum of Lyapunov exponents of* f_λ *with respect to the Green measure* μ_λ*.*

3.2.2 Lyapunov exponent and bifurcation current

We have seen with Mañé-Sad-Sullivan Theorem 3.1.9 that the bifurcations, within a holomorphic family of degree d rational maps $(f_\lambda)_{\lambda\in M}$, are due to the activity of the critical points (see in particular Lemma 3.1.7). We will use this and the formula given by Theorem 3.2.4, to define a closed positive $(1,1)$-current T_{bif} on M whose support is the bifurcation locus and which admits the Lyapunov function as global potential. Besides, we will also introduce a collection of $2d-2$ closed positive $(1,1)$-currents which detect the activity of each critical point and see that the bifurcation current T_{bif} is the sum of these currents. Here are the formal definitions.

Definition 3.2.6 *Let* $(f_\lambda)_{\lambda\in M}$ *be any holomorphic family of degree d rational maps. with marked critical points* $\{c_i(\lambda);\ 1 \le i \le 2d-2\}$*. The* activity current T_i *of the marked critical point* c_i *is defined by*

$$T_i := (p_M)_\star \left((dd^c_{\lambda,z} g_\lambda(z) + \hat{\omega}) \wedge [C_i] \right).$$

where C_i *is the graph* $\{(\lambda, c_i(\lambda);\ \lambda \in M\}$ *in* $M \times \mathbf{P}^1$ *and* g_λ *is the Green function on* $\mathbf{P}^1$ *of* F_λ *for some (local) holomorphic family of lifts* (F_λ)*.*
The bifurcation current T_{bif} *is defined by*

$$T_{bif} := dd^c L$$

where L *is the Lyapunov exponent of the family* $(f_\lambda)_{\lambda\in M}$*.*

We start by giving local potentials for the activity currents.

Lemma 3.2.7 *Let* F_λ *be a local holomorphic family of lifts of* f_λ *and* G_λ *be the Green function of* F_λ*. Let* $\hat{c}_i(\lambda)$ *be a local lift of* $c_i(\lambda)$*. Then* $G_\lambda(\hat{c}_i(\lambda))$ *is a local potential of* T_i*.*

Proof. This is a straightforward computation using the fact that, for any local section σ of the canonical projection $\pi : \mathbf{C}^2 \setminus \{0\} \to \mathbf{P}^1$, the function $G_\lambda(\sigma(z))$ is a local potential of $dd^c_{\lambda,z} g_\lambda(z) + \hat{\omega}$ (see Proposition 1.1.6). □

The following result has been originally proved by DeMarco in [DeM1], [DeM2].

Theorem 3.2.8 *Let $(f_\lambda)_{\lambda\in M}$ be any holomorphic family of degree d rational maps with marked critical points $\{c_i(\lambda);\ 1\le i\le 2d-2\}$. The support of the activity current T_i is the activity locus of the marked critical point c_i.*
The support of the bifurcation current T_{bif} is the bifurcation locus of the family $(f_\lambda)_{\lambda\in M}$ and $T_{bif}=\sum_1^{2d-2}T_i$.

Proof. Let us first show that c_i is passive on the complement of $Supp\, T_i$. If $T_i=0$ on a small ball $B\subset M$ then, by Lemma 3.2.7, $G_\lambda(\hat c_i(\lambda))$ is pluriharmonic and therefore equal to $\ln|h_i(\lambda)|$ for some non-vanishing holomorphic function h_i on B. Replacing $\hat c_i(\lambda)$ by $\frac{\hat c_i(\lambda)}{h_i(\lambda)}$ one gets, thanks to the homogeneity property of G_λ (see Proposition 1.1.6), $G_\lambda(\hat c_i(\lambda))=0$. This implies that

$$\{F_\lambda^n(\hat c_i(\lambda))\ /\ n\ge 1,\lambda\in B\}\subset \cup_{\lambda\in B}G_\lambda^{-1}\{0\}$$

which, after reducing B, is a relatively compact subset of $\mathbf{C}^2$. Montel's theorem then tells us that $(F_\lambda^n(\hat c_i(\lambda)))_n$ and thus $(f_\lambda^n(c_i(\lambda)))_n$ are normal on B.

Let us now show that T_i vanishes where c_i is passive. Assume that a subsequence $(f_\lambda^{n_k}(c_i(\lambda)))_k$ is uniformly converging on a small ball $B\subset M$. Then we may find a local section σ of $\pi:\mathbf{C}^2\setminus\{0\}\to\mathbf{P}^1$ such that $F_\lambda^{n_k}(\hat c_i(\lambda))=h_{n_k}(\lambda)\cdot\sigma\circ f_\lambda^{n_k}(c_i(\lambda))$ where h_{n_k} is a non-vanishing holomorphic function on B. As $G_\lambda\circ F_\lambda=dG_\lambda$ (see Proposition 1.1.6), this yields

$$G_\lambda(\hat c_i(\lambda))=d^{-n_k}\big(\ln|h_{n_k}(\lambda)|+G_\lambda\circ\sigma\circ f_\lambda^{n_k}(c_i(\lambda))\big)$$

which, after taking dd^c and making $k\to+\infty$, implies that T_i vanishes on B.

That $dd^cL=T_{\rm bif}=\sum_1^{2d-2}T_i$ follows immediately from Theorem 3.2.4. Then, Mañé-Sad-Sullivan Theorem 3.1.9 implies that the support of $T_{\rm bif}$ is the bifurcation locus. □

It is important to stress here that the identity $dd^cL=T_{\rm bif}=\sum_1^{2d-2}T_i$ may be seen as a potential-theoretic expression of Mañé-Sad-Sullivan theory. More concretely, the expression $T_{\rm bif}=dd^cL$ will be used to investigate the set of parameters $S_{\rm hi}$ while the expression $T_{\rm bif}=\sum_1^{2d-2}T_i$ will be used for the parameters $M_{\rm is}$ (see subsection 3.1.3).
In the next chapters, we will deeply use the fact that the Lyapunov function is a potential of $T_{\rm bif}$, together with the approximation formula 1.3.7, to analyse how the hypersurfaces $Per_n(w)$ may shape the bifurcation locus.
The continuity of the Lyapunov function will turn out to be extremely useful for this study. Mañé was the first to establish the continuity of the Lyapunov function L (see [Mañ]) , using Theorem 3.2.4 and Lemma 3.2.7 one may see that this function is actually Hölder continuous.

Theorem 3.2.9 *The Lyapunov function of any holomorphic family of degree d rational maps is p.s.h and Hölder continuous.*

Proof. According to Theorem 3.2.4 and lemma 3.2.7 the functions L and $\sum G_\lambda(\hat{c}_i(\lambda))$ differ from a pluriharmonic function. The conclusion follows from the fact that $G_\lambda(z)$ is Hölder continuous in (λ, z) (see [BB1] Proposition 1.2.). $\square$

3.2.3 DeMarco's formula

Using Theorem 3.2.4 we will get an explicit version of the formula given by Proposition 3.2.3. This result was first obtained by DeMarco who used a completely different method. We refer to the paper of Okuyama [O] for yet another proof.

The key will be to compute the integral $\int_{\mathbf{P}^1} g_F\,(\mu_f + \omega)$ which appears in the formula given by Proposition 3.2.3. To this purpose we shall use the *resultant* of a homogeneous polynomial map F of degree d on $\mathbf{C}^2$. The space of such maps can be identified with $\mathbf{C}^{2d+2}$. The resultant $Res\ F$ of F polynomialy depends on F and vanishes if and only if F is degenerate. Moreover $Res\ (z_1^d, z_2^d) = 1$ and Res is $2d$-homogeneous: $Res\ aF = a^{2d}\ Res\ F$.

Lemma 3.2.10 $\int_{\mathbf{P}^1} g_F\,(\mu_f + \omega) = \frac{1}{d(d-1)} \ln |Res\ F| - \frac{1}{2}$

Proof. The function $B(F) := \int_{\mathbf{P}^1} g_F\,(\mu_f + \omega)$ is well defined on $\mathbf{C}^{2d+2} \setminus \Sigma$ where Σ is the hypersurface where Res vanishes. Moreover, according to Theorem 3.2.4, B is pluriharmonic. As B is locally bounded from above, it extends to some *p.s.h* function through Σ. Then, by Siu's theorem, there exists some positive constant c such that $dd^c B(F) = c\ dd^c \ln |ResF|$ which means that $B - c \ln |ResF|$ is pluriharmonic on $\mathbf{C}^{2d+2}$.
Let φ be a non-vanishing holomorphic function on $\mathbf{C}^{2d+2}$ such that

$$B - c \ln |ResF| = \ln |\varphi|.$$

Using the homogeneity of Res and the fact that $B(aF) = \frac{2}{d-1} \ln |a| + B(F)$ (one easily checks that $g_{aF} = \frac{1}{d-1} \ln |a| + g_F$) one gets:

$$|\varphi(aF)| = |a|^{\frac{2}{d-1} - 2cd}$$

Making $a \to 0$ one sees that $c = \frac{1}{d(d-1)}$ and φ is constant. To compute this constant one essentially tests the formula on $F_0 := (z_1^d, z_2^d)$ (see [BB1] Proposition 4.10). $\square$

We are now ready to prove the main result of this subsection.

Theorem 3.2.11 *Let f be a rational map of degree $d \geq 2$ and F be one of its lifts. Let G_F be the Green function of F on $\mathbf{C}^2$ and Res F the resultant of F. If $\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_{2d-2}$ are chosen so that $detF'(z) = \prod_{j=1}^{2d-2} \hat{c}_j \wedge z$ then*

$$L(f) + \ln d = \sum_{j=1}^{2d-2} G_F(\hat{c}_j) - \frac{2}{d} \ln |Res\ F|.$$

Proof. Taking Lemma 3.2.10 into account, the formula given by Proposition 3.2.3 becomes

$$L(f) + \ln d = \int_{\mathbf{P}^1} g_F[\mathcal{C}_f] \ - \ 2(d-1)\left(\tfrac{1}{d(d-1)} \ln |Res\ F| - \tfrac{1}{2}\right) \ + \ \int_{\mathbf{P}^1} \ln \|Jac_F\|_0\, \omega.$$

Observe that $\omega = \pi_\star m$ where $m := \left(dd^c \ln^+ \|\cdot\|\right)^2$ is the normalized Lebesgue measure on the euclidean unit sphere of $\mathbf{C}^2$. Then

$$\int_{\mathbf{C}^2} \ln |detF'|\, m = \int_{\mathbf{C}^2} \ln \left(e^{-D\|\cdot\|}|detF'|\right)\, m = \int_{\mathbf{C}^2} \ln \|J_F \circ \pi\|_0\, m = \int_{\mathbf{P}^1} \ln \|Jac_F\|_0\, \omega.$$

Let us pick U_j in the unitary group of $\mathbf{C}^2$ such that $U_j^{-1}(\hat{c}_j) = (\|\hat{c}_j\|, 0)$. Then $U_j(z) \wedge \hat{c}_j = -z_2\|\hat{c}_j\|$ and, since $\int_{\mathbf{C}^2} \ln |z_2|\, m = -\frac{1}{2}$ one gets

$$\int_{\mathbf{P}^1} \ln \|Jac_F\|_0\, \omega = \int_{\mathbf{C}^2} \ln |detF'|\, m = \sum_j \int_{\mathbf{C}^2} \ln |U_j(z) \wedge \hat{c}_j| = \sum_j \|\hat{c}_j\| - (d-1).$$

On the other hand, $\int_{\mathbf{P}^1} g_F[\mathcal{C}_f] = \sum_j g_F \circ \pi(\hat{c}_i) = \sum_j G_F(\hat{c}_j) - \sum_j \ln \|\hat{c}_j\|$ and the conclusion follows. □

Chapter 4

Equidistribution towards the bifurcation current

4.1 Distribution of critically periodic parameters

The aim of this section is to present a result due to Dujardin and Favre (see [DF]) about the asymptotic distribution of degree d polynomials which have a pre-periodic critical point. We will work in the context of polynomial families, this will allow us to modify the original proof and significantly simplify it. We refer to the paper of Dujardin-Favre for a proof working in any holomorphic family of rational maps.

4.1.1 Statement and a general strategy

We work here in the family $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$ of degree d polynomials which has been introduced in the subsection 2.1.2. Let us recall that $P_{c,a}$ is the polynomial of degree d whose critical points are $(0 = c_0, c_1, \cdots, c_{d-2})$ and such that $P_{c,a}(0) = a^d$.

For $0 \leq i \leq d-2$ and $0 \leq k < n$, we denote by $\mathrm{Per}(i, n, k)$ the hypersurface of $\mathbf{C}^{d-1}$ defined by

$$\mathrm{Per}(i, n, k) := \{(c, a) \in \mathbf{C}^{d-1} \ / \ P^n_{c,a}(c_i) = P^k_{c,a}(c_i)\}.$$

The results we want to establish is the following; it has been first proved by Dujardin and Favre in [DF]

Theorem 4.1.1 *In the family of degree d polynomials, for any sequence of integers* $(k_n)_n$ *such that* $0 \leq k_n < n$ *one has* $\sum_{i=0}^{d-2} \lim_n d^{-n}[Per(i, n, k_n)] = T_{bif}$.

Let us now explain the principle of the proof and fix a few notations. To simplify, we shall write λ the parameters $(c, a) \in \mathbf{C}^{d-1}$.

The bifurcation current T_{bif} is given by $T_{\text{bif}} = \sum_{i=0}^{d-2} dd^c g_\lambda(c_i)$ (see Lemma 3.2.7) where g_λ is the Green function of P_λ (see the subsection 2.3.1). It thus suffices to show that for any fixed $0 \leq i \leq d-2$ the following sequence of potentials

$$h_n(\lambda) := d^{-n} \ln |P_\lambda^n(c_i) - P_\lambda^{k_n}(c_i)|$$

converges in L^1_{loc} to $g_\lambda(c_i)$.

To this purpose we shall compare these potentials with the functions

$$g_n(\lambda) := d^{-n} \ln \max \left(1, |P_\lambda^n(c_i)|\right)$$

which do converge locally uniformly to $g_\lambda(c_i)$.

The first important point is to check that the sequence $(h_n)_n$ is locally uniformly bounded from above. We shall actually prove a little bit more.

Lemma 4.1.2 *For any compact $K \subset \mathbf{C}^{d-1}$ and any $\epsilon > 0$ there exists an integer n_0 such that $h_n|_K \leq g_n|_K + \epsilon$ for $n \geq n_0$.*

Proof. It is not difficult to see that there exists $R \geq 1$ such that

$$(1-\epsilon)|z|^{d^n} \leq |P_\lambda^n(z)| \leq (1+\epsilon)|z|^{d^n} \tag{4.1.1}$$

for every $\lambda \in K$, every $n \in \mathbf{N}$ and every $|z| \geq R$.

We now proceed by contradiction and assume that there exists $\lambda_p \in K$ and $n_p \to +\infty$ such that $h_{n_p}(\lambda_p) \geq g_{n_p}(\lambda_p) + \epsilon$. This means that

$$|P_{\lambda_p}^{n_p}(c_i) - P_{\lambda_p}^{k_{n_p}}(c_i)| \geq e^{\epsilon d^{n_p}} \max\left(1, |P_{\lambda_p}^{n_p}(c_i)|\right). \tag{4.1.2}$$

Let us set $B_p := P_{\lambda_p}^{k_{n_p}}(c_i)$. By 4.1.2 we have $\lim_p |B_p| = +\infty$ and thus $|B_p| \geq R$ for p big enough. Then, using 4.1.1, one may write

$$P_{\lambda_p}^{n_p}(c_i) = P_{\lambda_p}^{n_p - k_{n_p}}(B_p) = (u_p B_p)^{d^{n_p - k_{n_p}}}$$

where $(1-\epsilon) \leq |u_p| \leq (1+\epsilon)$ and the estimate 4.1.2 becomes

$$|(u_p B_p)^{d^{n_p - k_{n_p}}} - B_p| \geq e^{\epsilon d^{n_p}} |u_p B_p|^{d^{n_p - k_{n_p}}}.$$

This is clearly impossible when $p \to +\infty$. □

To prove that $(h_n)_n$ converges in L^1_{loc} to $g_\lambda(c_i)$ we shall use a well known compactness principle for subharmonic functions:

Theorem 4.1.3 *Let (φ_j) be a sequence of subharmonic functions which is locally uniformly bounded from above on some domain $\Omega \subset \mathbf{R}^n$. If (φ_j) does not converge to $-\infty$ then a subsequence (φ_{j_k}) converges in $L^1_{loc}(\Omega)$ to some subharmonic function φ. In particular, (φ_j) converges in $L^1_{loc}(\Omega)$ to some subharmonic function φ if it converges pointwise to φ.*

We will also need to following classical result

Lemma 4.1.4 *(**Hartogs**) Let (φ_j) be a sequence of subharmonic functions and g be a continuous function defined on some domain $\Omega \subset \mathbf{R}^n$. If $\limsup_j \varphi_j(x) \le g(x)$ for every $x \in \Omega$ then, for any compact $K \subset \Omega$ and every $\epsilon > 0$ one has $\varphi_j(x) \le g(x) + \epsilon$ on K for j big enough.*

By Lemma 4.1.2 the sequence $(h_n)_n$ is locally uniformly bounded from above and therefore, according to the Theorem 4.1.3, we have to check that $(h_n)_n$ does not converge to $-\infty$ and that $g_\lambda(c_i)$ is the only limit value of $(h_n)_n$ for the L^1_{loc} convergence. To prove this unicity statement we will combine the following maximum-type principle with our knowledge of the behaviour of the bifurcation locus at infinity in polynomials families (see Theorem 2.3.5).

Lemma 4.1.5 *Let φ, ψ be two p.s.h functions on $\mathbf{C}^k$. Assume that:*

i) ψ is continuous

ii) $\varphi \le \psi$

iii) Supp $(dd^c\varphi) \subset$ Supp $(dd^c\psi)$

iv) $\varphi = \psi$ on Supp $(dd^c\psi)$

v) for any $\lambda_0 \in \mathbf{C}^k$ there exists a complex line L through λ_0 such that $\varphi = \psi$ on the unbounded component of $L \setminus \big(L \cap \text{Supp}\,(dd^c\psi)\big)$.

Then $\varphi = \psi$.

Proof. Because of iv), we only have to show that $\varphi(\lambda_0) = \psi(\lambda_0)$ when λ_0 lies in the complement of *Supp* $(dd^c\psi)$. According to (v), we may find a complex line L in $\mathbf{C}^k$ containing λ_0 and such that ψ and φ coincide on the unbounded component Ω_∞ of $L \setminus \big(L \cap \text{Supp}\,(dd^c\psi)\big)$. We may therefore assume that $\lambda_0 \notin \Omega_\infty$. By i), iii) and iv) $\varphi|_L$ coincides with the continuous function $\psi|_L$ on *Supp* $\Delta\varphi|_L$ which, by the continuity principle, implies that $\varphi|_L$ is continuous. Let Ω_0 be the (bounded) component of $L \setminus \big(L \cap \text{Supp}\,(dd^c\psi)\big)$ containing λ_0. The continuous function $\big(\varphi - \psi\big)|_L$ vanishes on $b\Omega_0$ (see iv)), is harmonic on Ω_0 (see iii) and iv)) and negative (see ii)). The maximum principle now implies that $\varphi(\lambda_0) = \psi(\lambda_0)$. □

The following technical lemma will play an important role in the proof of Theorem 4.1.1. It shows in particular that $(h_n)_n$ doest not converge to $-\infty$.

Lemma 4.1.6 *If c_i belongs to some attracting bassin of P_{λ_0} then there exists a neighbourhood V_0 of λ_0 such that $\sup_n \sup_{V_0} \big(h_n - g_n\big) \ge 0$.*

Proof. If V_0 is a sufficently small neighbourhood of λ_0 then $P_\lambda^n(c_i) \to a_\lambda$ where a_λ is an attracting cycle of P_λ for every $\lambda \in V_0$. We will assume that $P_\lambda(a_\lambda) = a_\lambda$.
Let us now proceed by contradiction and assume that there exists $\epsilon > 0$ such that

$$|P_\lambda^n(c_i) - P_\lambda^{k_n}(c_i)| \leq e^{-\epsilon d^n} \max\big(1, |P_\lambda^n(c_i)|\big), \quad \forall \lambda \in V_0,\ \forall n. \tag{4.1.3}$$

Since $P_\lambda^n(c_i) \to a_\lambda$, 4.1.3 would imply

$$|P_\lambda^n(c_i) - P_\lambda^{k_n}(c_i)| \leq Ce^{-\epsilon d^n}, \quad \forall \lambda \in V_0,\ \forall n. \tag{4.1.4}$$

The estimate 4.1.4 implies that a_λ is a super-attracting fixed point for any $\lambda \in V_0$ which, in turn, implies that $a_\lambda = \infty$ for all $\lambda \in V_0$. But in that case we would have $|P_\lambda^{k_n}(c_i)| \leq \frac{1}{2}|P_\lambda^n(c_i)|$ for n big enough and the estimate 4.1.3 would be violated. □

4.1.2 Proof of Theorem 4.1.1

We use here the notations introduced at the beginning of the previous subsection. According to Theorem 4.1.3 we have to show that $g := g_\lambda(c_i) = \lim g_n$ is the only limit value of the sequence $(h_n)_n$ for the L^1_{loc} convergence (by Lemmas 4.1.2 and 4.1.6, this sequence of *p.s.h* functions is locally uniformly bounded and does not converge to $-\infty$). Assume that (after taking a subsequence!) h_n is converging in L^1_{loc} to h. To prove that the functions h and g coincide we shall check that they satisfy the assumptions of Lemma 4.1.5.

Our modification of the original proof essentially stays in the third step.

First step: $h \leq g$.

Let B_0 be a ball of radius r centered at λ_0 and let $\epsilon > 0$. By the mean value property we have

$$h(\lambda_0) \leq \tfrac{1}{|B_0|}\int_{B_0} h = \lim_n \tfrac{1}{|B_0|}\int_{B_0} h_n$$

but, according to Lemma 4.1.2, $h_n \leq g_n + \epsilon$ on B_0 for n big enough and thus

$$h(\lambda_0) \leq \epsilon + \lim_n \tfrac{1}{|B_0|}\int_{B_0} g_n = \epsilon + \tfrac{1}{|B_0|}\int_{B_0} g.$$

As g is continuous, the conclusion follows by making $r \to 0$ and then $\epsilon \to 0$.

Second step: $h = g$ on $Supp\, dd^c g$.

Combining Lemma 4.1.6 and the result of first step we will first see that $(h - g)$ vanishes when c_i is captured by an attracting basin.
Suppose to the contrary that c_i is captured by an attracting cycle of P_{λ_0} and

$(h-g)(\lambda_0) < 0$. As the function $(h-g)$ is upper semi-continuous, we may shrink V_0 so that $(h-g) \leq -\epsilon < 0$ on V_0. Now, as c_i is passive on V_0, the function $(h-g)$ is $p.s.h$ on V_0 and, after shrinking V_0 again, Hartogs Lemma 4.1.4 implies that $(h_n - g_n) \leq -\frac{\epsilon}{2}$ on V_0 for n big enough. This contradicts Lemma 4.1.6.

Now, if $\lambda_0 \in Supp\, dd^c g$ then $\lambda_0 = \lim_k \lambda_k$ where λ_k is a parameter for which c_i is captured by some attracting cycle (see Lemma 3.1.8). As $(h-g)$ is uper semi-continuous we get $(h-g)(\lambda_0) = 0$.

Third step: *$Supp\, dd^c h \subset Supp\, dd^c g$.*

Let Ω be a connected component of $\mathbf{C}^{d-1} \setminus Supp\, dd^c g$. We have to show that h is pluriharmonic on Ω. We proceed by contradiction. If $dd^c h$ does not vanish on Ω then there exists some n_0 for which the hypersurface

$$\mathcal{H} := \{P_\lambda^{n_0}(c_i) - P_\lambda^{k_{n_0}}(c_i) = 0\}$$

meets Ω. When $\lambda \in \mathcal{H}$ then c_i is captured by a cycle since $P_\lambda^{k_{n_0}}(c_i) =: z(\lambda)$ satisfies $P_\lambda^{m_0}(z(\lambda)) = P_\lambda^{m_0} \circ P_\lambda^{k_{n_0}}(c_i) = P_\lambda^{k_{n_0}}(c_i) = z(\lambda)$ for $m_0 := n_0 - k_{n_0} > 0$.

Let us show that $z(\lambda)$ is a neutral periodic point. We first observe that the vanishing of $dd^c g$ on Ω forces $z(\lambda)$ to be non-repelling and thus $|\big(P_\lambda^{m_0}\big)'\big(z(\lambda)\big)| \leq 1$ on $\mathcal{H} \cap \Omega$.
Let us now see why $z(\lambda)$ cannot be attracting. If this would be the case then, according to Lemma 4.1.6, we would have $h(\lambda_0) = g(\lambda_0)$ for a certain $\lambda_0 \in \mathcal{H} \cap \Omega$. As $(h-g)$ is negative and $p.s.h$ on Ω this implies, via the maximum principle, that $h = g$ on Ω. This is imposible since $dd^c h$ is supposed to be non vanishing on Ω.
We thus have $\big(P_\lambda^{m_0}\big)'\big(z(\lambda)\big) = e^{i\nu_0}$ on $\mathcal{H} \cap \Omega$ and therefore $z(\lambda)$ belongs to a neutral cycle whose period divides m_0 and whose multiplier is a root of $e^{i\nu_0}$.
In other words, $\mathcal{H} \cap \Omega$ is contained in a finite union of hypersurfaces of the form $\mathrm{Per}_n(e^{i\theta})$. This implies that

$$\mathcal{H} \subset \mathrm{Per}_{n_0}(e^{i\theta_0}).$$

for some integer n_0 and some real number θ_0.

Finally, using a global argument, we will see that this is impossible. Let us recall the following dynamical fact.

Lemma 4.1.7 *Every polynomial which has a neutral cycle also has a bounded, non-preperiodic, critical orbit.*

Thus, when $\lambda \in \mathcal{H}$, the polynomial P_λ has two distinct bounded critical orbits; the orbit of c_i which is preperiodic and the orbit of some other critical point which is given by the above lemma. This shows that $\mathcal{H}$ cannot meet the line $\{c_0 = c_1 = \cdots = c_{d-2} = 0\} := M_d$ since the corresponding polynomials (which are given by $\frac{1}{d}z^d + a^d \; a \in \mathbf{C}$) have only one critical orbit. We will now work in the projective compactification of $\mathbf{C}^{d-1}$ introduced in subsection 2.1.2. By Theorem 2.3.5, $\mathcal{H}$ and M_d cannot meet at infinity. This contradicts Bezout's theorem.

Fourth step: for any $\lambda_0 \in \mathbf{C}^{d-1}$ there exists a complex line L through λ_0 such that $h = g$ on the unbounded component of $L \setminus \big(L \cap Supp(dd^c g)\big)$.

Here one uses again Theorem 2.3.5 to pick a line L through λ_0 which meets infinity at some point $\xi_0 \notin \overline{Supp\; dd^c g}$. Then, for any λ in the unbounded component of $L \setminus \big(L \cap Supp\;(dd^c g)\big)$ the critical point c_i belongs to the super-attracting basin of ∞ and thus, as we saw in second step, $h(\lambda) = g(\lambda)$. $\square$.

4.2 Distribution of rational maps with cycles of a given multiplier

Let $f : M \times \mathbf{P}^1 \to \mathbf{P}^1$ be an arbitrary holomorphic family of degree $d \geq 2$ rational maps.

We want to investigate the asymptotic distribution of the hypersurfaces $Per_n(w)$ in M when $|w| < 1$. Concretely, we will consider the current of integration $[\mathrm{Per}_n(w)]$ or, more precisely, the currents

$$[\mathrm{Per}_n(w)] := dd^c \; \ln |p_n(\lambda, w)|$$

where $p_n(\cdot, w)$ are the canonical defining functions for the hypersurfaces $\mathrm{Per}_n(w)$ given by Theorem 2.2.1. We ask if the following convergence occurs:

$$\lim_n \tfrac{1}{d^n}[\mathrm{Per}_n(w)] = T_{\mathrm{bif}}.$$

The question is easy to handle when $|w| < 1$, more delicate when $|w| = 1$ and widely open when $|w| > 1$.

4.2.1 The case of attracting cycles

We aim to prove the following general result result (see [BB2])

Theorem 4.2.1 *For any holomorphic family of degree d rational maps* $\big(f_\lambda\big)_{\lambda \in M}$ *one has* $d^{-n}\;[Per_n(w)] \to T_{bif}$ *when* $|w| < 1$.

Proof. Let us set

$$L_n(\lambda, w) := d^{-n} \ln |p_n(\lambda, w)|.$$

Since, by definition, $T_{\text{bif}} = dd^c\ L(\lambda)$ where $L(\lambda)$ be the Lyapunov exponent of $(\mathbf{P}^1, f_\lambda, \mu_\lambda)$ and μ_λ is the Green measure of f_λ, all we have to show is that L_n converges to L in $L^1_{loc}(M)$. Here again we shall use the compactness principle for subharmonic functions (see Theorem 4.1.3).

The situation is purely local and therefore, taking charts, we may assume that $M = \mathbf{C}^k$. We write the polynomials p_n as follows :

$$p_n(\lambda, w) =: \prod_{i=1}^{N_d(n)} \big(w - w_{n,j}(\lambda)\big).$$

Using use the fact that $d^{-n}N_d(n) \sim \frac{1}{n}$ (see Theorem 2.2.1) one sees that the sequences L_n is locally uniformly bounded from above.

According to Theorem 2.2.1, the set $\{w_{n,j}(\lambda)\ /\ w_{n,j}(\lambda) \neq 1\}$ coincides with the set of multipliers of cycles of exact period n (counted with multiplicity) from which the cycles of multiplier 1 are deleted. We thus have

$$\sum_{j=1}^{N_d(n)} \ln^+ |w_{n,j}(\lambda)| = \frac{1}{n} \sum_{p \in R_n^*(\lambda)} \ln |(f^n)'(p)| \tag{4.2.1}$$

where $R_n^*(\lambda) := \{p \in \mathbf{P}^1\ /\ \ p \text{ has exact period } n \text{ and } |(f_\lambda^n)'(p)| > 1\}$. Since f_λ has a finite number of non-repelling cycles (Fatou' theorem), one sees that there exists $n(\lambda) \in \mathbf{N}$ such that

$$n \geq n(\lambda) \Rightarrow |w_{n,j}(\lambda)| > 1, \text{ for any } 1 \leq j \leq N_d(n). \tag{4.2.2}$$

By 4.2.1 and 4.2.2, one gets

$$L_n(\lambda, 0) = d^{-n} \sum_{j=1}^{N_d(n)} \ln |w_{n,j}(\lambda)| = d^{-n} \sum_{j=1}^{N_d(n)} \ln^+ |w_{n,j}(\lambda)| = \frac{d^{-n}}{n} \sum_{R_n^*(\lambda)} \ln |(f^n)'(p)|$$

for $n \geq n(\lambda)$ which, by Theorem 1.3.7, yields:

$$\lim_n L_n(\lambda, 0) = L(\lambda),\ \forall \lambda \in M. \tag{4.2.3}$$

If now $|w| < 1$, it follows from 4.2.2 that $L_n(\lambda, w) - L_n(\lambda, 0) = d^{-n} \sum_j \ln \frac{|w_{n,j}(\lambda) - w|}{|w_{n,j}(\lambda)|}$ and $\ln(1 - |w|) \leq \ln \frac{|w_{n,j}(\lambda) - w|}{|w_{n,j}(\lambda)|} \leq \ln(1 + |w|)$ for $1 \leq j \leq N_d(n)$ and $n \geq n(\lambda)$. We thus get

$$d^{-n} N_d(n) \ln(1 - |w|) \leq |L_n(\lambda, w) - L_n(\lambda, 0)| \leq d^{-n} N_d(n) \ln(1 + |w|)$$

for $n \geq n(\lambda)$. Using 4.2.3 and the fact that $d^{-n}N_d(n) \sim \frac{1}{n}$ we obtain $\lim_n L_n(\lambda, w) = L(\lambda)$ for any $(\lambda, w) \in M \times \Delta$. The L^1_{loc} convergence of $L_n(\cdot, w)$ now follows immediately from Theorem 4.1.3. □

Remark 4.2.2 *We have proved that $L_n(\lambda, w) := d^{-n} \ln |p_n(\lambda, w)|$ converges pointwise to $L(\lambda)$ on M when $|w| < 1$.*

The above discussion shows that the pointwise convergence of $L_n(\lambda, w)$ to L (and therefore the convergence $d^{-n}[\mathrm{Per}_n(w)] \to T_{\mathrm{bif}}$) is quite a straightforward consequence of Theorem 1.3.7 when $|w| < 1$. However, when $|w| \geq 1$ and λ is a non-hyperbolic parameter, the control of $L_n(\lambda, w) = d^{-n} \sum \ln |w - w_{n,j}(\lambda)|$ is very delicate because f_λ may have many cycles whose multipliers are close to w. This is why we introduce the *p.s.h* functions L_n^+ which both coincide with L_n on the hyperbolic components and are quite easily seen to converge nicely. These functions will be extremely helpfull later.

Definition 4.2.3 *Let $p_n(\lambda, w) =: \prod_{j=1}^{N_d(n)} (w - w_{n,j}(\lambda))$ be the polynomials associated to the family f by Theorem 2.2.1. The p.s.h functions L_n^+ are defined by:*

$$L_n^+(\lambda, w) := d^{-n} \sum_{j=1}^{N_d(n)} \ln^+ |w - w_{n,j}(\lambda)|.$$

The interest of considering these functions stays in the next lemma.

Lemma 4.2.4 *The sequence L_n^+ converges pointwise and in L^1_{loc} to L on $M \times \mathbf{C}$. For every $w \in \mathbf{C}$ the sequence $L_n^+(\cdot, w)$ converges in L^1_{loc} to L on M.*

Proof. We will show that $L_n^+(\cdot, w)$ converges pointwise to L on M for every $w \in \mathbf{C}$. As $(L_n^+)_n$ is locally uniformly bounded, this implies the convergence of $L_n^+(\cdot, w)$ in $L^1_{loc}(M)$ (Theorem 4.1.3) and the convergence of L_n^+ in $L^1_{loc}(M \times \mathbf{C})$ then follows by Lebesgue's theorem.

As $L_n(\lambda, 0) \to L(\lambda)$ (see Remark 4.2.2), we have to estimate $L_n^+(\lambda, w) - L_n(\lambda, 0) =: \epsilon_n(\lambda, w)$ on M. Let us fix $\lambda \in M$, $w \in \mathbf{C}$ and pick $R > |w|$. Since f_λ has a finite number of non-repelling cycles (Fatou' theorem), one sees that there exists $n(\lambda) \in \mathbf{N}$ such that

$$n \geq n(\lambda) \Rightarrow |w_{n,j}(\lambda)| > 1, \text{ for any } 1 \leq j \leq N_d(n).$$

We may then decompose $\epsilon_n(\lambda, w)$ in the following way:

$$\epsilon_n(\lambda, w) = d^{-n} \sum_{1\le|w_{n,j}(\lambda)|<R+1} \ln^+ |w_{n,j}(\lambda) - w| + d^{-n} \sum_{|w_{n,j}(\lambda)|\ge R+1} \ln \frac{|w_{n,j}(\lambda) - w|}{|w_{n,j}(\lambda)|}$$
$$-d^{-n} \sum_{1\le|w_{n,j}(\lambda)|<R+1} \ln |w_{n,j}(\lambda)|.$$

We may write this decomposition as $\epsilon_n(\lambda, w) =: \epsilon_{n,1}(\lambda, w) + \epsilon_{n,2}(\lambda, w) - \epsilon_{n,1}(\lambda, 0)$.

One clearly has $0 \le \epsilon_{n,1}(\lambda, w) \le d^{-n} N_d(n) \ln\big(2R+1\big)$ and thus $\lim_n \epsilon_{n,1}(\lambda, w) = 0$. Similarly, $\lim_n \epsilon_{n,2}(\lambda, w) = 0$ follows from the fact that, for $|w_{n,j}(\lambda)| > R + 1 > |w| + 1$, one has:

$$\ln(1 - \frac{R}{R+1}) \le \ln \frac{|w_{n,j}(\lambda)| - R}{|w_{n,j}(\lambda)|} \le \ln \frac{|w_{n,j}(\lambda) - w|}{|w_{n,j}(\lambda)|}$$
$$\le \ln \frac{|w_{n,j}(\lambda)| + R}{|w_{n,j}(\lambda)|} \le \ln(1 + \frac{R}{R+1}).$$

□

As the functions L_n^+ and L_n coincide on hyperbolic components, the above lemma would easily yield the convergence of $d^{-n}[\mathrm{Per}_n(w)]$ towards T_{bif} for any $w \in \mathbf{C}$ if the density of hyperbolic parameters in M was known. The remaining of this section is, in some sense, devoted to overcome this difficulty. We shall first do this in a general setting by averaging the multipiers. Then we will restrict ourself to polynomial families and, using the nice distribution of hyperbolic parameters near infinity, will show that $d^{-n}[\mathrm{Per}_n(e^{i\theta})]$ converges towards T_{bif}.

4.2.2 Averaging the multipliers

Although the convergence of $\lim_n \frac{1}{d^n}[\mathrm{Per}_n(w)]$ to T_{bif} is not clear when $|w| \ge 1$, one easily obtains the convergence by averaging over the argument of the multiplier w. The following result is due to Bassanelli and Berteloot [BB2].

Theorem 4.2.5 *For any holomorphic family of degree d rational maps $\big(f_\lambda\big)_{\lambda\in M}$ one has $\frac{d^{-n}}{2\pi}\int_0^{2\pi}[Per_n(re^{i\theta})]\, d\theta \to T_{bif}$, when $r \ge 0$.*

Proof.
One essentially has to investigate the following sequences of $p.s.h$ functions

$$L_n^r(\lambda) := \frac{d^{-n}}{2\pi} \int_0^{2\pi} \ln |p_n(\lambda, re^{i\theta})|\, d\theta.$$

We will see that $L_n^r(\lambda) \geq C\frac{\ln r}{n}$ where C only depends on the family and that L_n^r converges to L in $L^1_{loc}(M)$.

For that, we essentially will compare L_n^r with $L_n(\lambda, 0) = L_n^0$ by using the formula $\ln \max(|a|, r) = \frac{1}{2\pi}\int_0^{2\pi} \ln|a - re^{i\theta}|d\theta$. Indeed, writting

$$p_n(\lambda, w) =: \prod_{i=1}^{N_d(n)} \big(w - w_{n,j}(\lambda)\big)$$

this formula yields

$$L_n^r(\lambda) = \frac{1}{2\pi d^n}\int_0^{2\pi} \ln \prod_j |re^{i\theta} - w_{n,j}(\lambda)|d\theta = \tag{4.2.4}$$

$$d^{-n}\sum_j \ln \max(|w_{n,j}(\lambda)|, r). \tag{4.2.5}$$

According to Theorem 2.2.1, the set $\{w_{n,j}(\lambda) \;/\; w_{n,j}(\lambda) \neq 1\}$ coincides with the set of multipliers of cycles of exact period n (counted with multiplicity) from which the cycles of multiplier 1 are deleted. Using use the fact that $d^{-n}N_d(n) \sim \frac{1}{n}$ (see Theorem 2.2.1) one sees that the sequence $L_n^r(\lambda)$ is locally bounded from above and is uniformly bounded from below by $C\frac{\ln r}{n}$. Since f_λ has a finite number of non-repelling cycles (Fatou' theorem), one sees that there exists $n(\lambda) \in \mathbf{N}$ such that

$$n \geq n(\lambda) \Rightarrow |w_{n,j}(\lambda)| > 1, \text{ for any } 1 \leq j \leq N_d(n).$$

Now we deduce from 4.2.4 that for $n \geq n(\lambda)$:

$$L_n^r(\lambda) = d^{-n}\sum_j \ln|w_{n,j}(\lambda)| + d^{-n}\sum_{1\leq|w_{n,j}(\lambda)|<r} \ln\frac{r}{|w_{n,j}(\lambda)|} =$$

$$L_n(\lambda, 0) + d^{-n}\sum_{1\leq|w_{n,j}(\lambda)|<r} \ln\frac{r}{|w_{n,j}(\lambda)|}$$

and thus

$$0 \leq L_n^r(\lambda) - L_n(\lambda, 0) = d^{-n}\sum_{1\leq|w_{n,j}(\lambda)|<r} \ln\frac{r}{|w_{n,j}(\lambda)|} \leq d^{-n}N_d(n)\ln^+ r.$$

Recalling that $d^{-n}N_d(n) \sim \frac{1}{n}$ and $L_n(\lambda, 0) \to L(\lambda)$ (see Remark 4.2.2), this implies that L_n^r converges pointwise to L and, by Theorem 4.1.3, that $(L_n^r)_n$ converges to L in $L^1_{loc}(M)$.

Now, to get the conclusion, one has to justify the following identity:

$$dd^c L_n^r = \tfrac{d^{-n}}{2\pi} \int_0^{2\pi} [Per_n(re^{i\theta})] d\theta.$$

For this purpose it suffices to check that $\ln |p_n(\lambda, re^{i\theta})|$ is locally integrable. Let K be a compact subset of M and c_n be an upper bound for $\ln |p_n(\lambda, re^{i\theta})|$ on $K \times [0, 2\pi]$. Then, the negative function $\ln |p_n(\lambda, e^{i\theta})| - c_n$ is indeed integrable on $K \times [0, 2\pi]$ as it follows from the fact that $L_n^r(\lambda) \geq C \frac{\ln r}{n}$:

$$\begin{aligned} \int_K \Big(\int_0^{2\pi} \big(\ln |p_n(\lambda, re^{i\theta})| - c_n \big) d\theta \Big) dV &= 2\pi d^n \int_K L_n^r dV - 2\pi c_n \int_K dV \\ &\geq \left(d^n C \frac{\ln r}{n} - c_n \right) 2\pi \int_K dV. \end{aligned}$$

□

Remark 4.2.6 *We have proved that* $L_n^r(\lambda) := \dfrac{d^{-n}}{2\pi} \displaystyle\int_0^{2\pi} \ln |p_n(\lambda, re^{i\theta})| \, d\theta$ *is pointwise converging to* $L(\lambda)$ *on* M.

The following result is essentially a potential-theoretic consequence of the former one. It implicitely contains some information about the convergence of d^{-n} $[\mathrm{Per}_n(w)]$ for arbitrary choices of w but seems hard to improve without using further dynamical properties (see [BB3]).

Theorem 4.2.7 *For family of degree d rational maps* $(f_\lambda)_{\lambda \in M}$ *one has*

$$d^{-n} \, dd^c_{(\lambda,w)} \ln |p_n(\lambda, w)| \to dd^c L(\lambda)$$

where $p_n(\cdot, w)$ *are the canonical defining functions for the hypersurfaces* $Per_n(w)$ *given by Theorem 2.2.1.*

Proof. Let us set

$$L_n(\lambda, w) := d^{-n} \ln |p_n(\lambda, w)|.$$

As we have seen in the two last subsections (see remarks 4.2.2 and 4.2.6)

$$L_n(\lambda, 0) \to L(\lambda)$$
$$L_n^r(\lambda) := \tfrac{d^{-n}}{2\pi} \int_0^{2\pi} \ln |p_n(\lambda, re^{i\theta})| \, d\theta \to L(\lambda) \text{ for any } r \geq 0.$$

Let us also recall that the function L is continuous on M (see Theorem 3.2.9).

As the functions L_n are $p.s.h$ and the sequence $(L_n)_n$ is locally uniformly bounded from above, we shall again use the compacity properties of $p.s.h$ functions given by Theorem 4.1.3. Since $L_n(\lambda, 0)$ converges to $L(\lambda)$, the sequence $(L_n)_n$ does not converge to $-\infty$ and it therefore suffices to show that, among $p.s.h$ functions on $M \times \mathbf{C}$, the function L is the only possible limit for $(L_n)_n$ in $L^1_{loc}(M \times \mathbf{C})$.

Let φ be a $p.s.h$ function on $M \times \mathbf{C}$ and $(L_{n_j})_j$ a subsequence of $(L_n)_n$ which converges to φ in $L^1_{loc}(M \times \mathbf{C})$. Pick $(\lambda_0, w_0) \in M \times \mathbf{C}$. We have to prove that $\varphi(\lambda_0, w_0) = L(\lambda_0)$.

Let us first observe that $\varphi(\lambda_0, w_0) \leq L(\lambda_0)$. Take a ball B_ϵ of radius ϵ and centered at $(\lambda_0, w_0) \in M \times \mathbf{C}$. By the submean value property and the L^1_{loc}- convergence of L_n^+ (see Lemma 4.2.4) we have:

$$\begin{aligned}
\varphi(\lambda_0, w_0) &\leq \frac{1}{|B_\epsilon|} \int_{B_\epsilon} \varphi \, dm = \lim_j \frac{1}{|B_\epsilon|} \int_{B_\epsilon} L_{n_j} \, dm \\
&\leq \lim_j \frac{1}{|B_\epsilon|} \int_{B_\epsilon} L_{n_j}^+ \, dm = \frac{1}{|B_\epsilon|} \int_{B_\epsilon} L \, dm
\end{aligned}$$

making then $\epsilon \to 0$, one obtains $\varphi(\lambda_0, w_0) \leq L(\lambda_0)$ since L is continuous.
Let us now check that $\limsup_j L_{n_j}(\lambda_0, w_0 e^{i\theta}) = L(\lambda_0)$ for almost all $\theta \in [0, 2\pi]$. Let $r_0 := |w_0|$. By Lemma 4.2.4 L_n^+ converges pointwise to L and therefore:

$$\limsup_j L_{n_j}(\lambda_0, w_0 e^{i\theta}) \leq \limsup_j L_{n_j}^+(\lambda_0, w_0 e^{i\theta}) = L(\lambda_0).$$

On the other hand, by pointwise convergence of $L_n^{r_0}$ to L and Fatou's lemma we have:

$$\begin{aligned}
L(\lambda_0) = \lim_n L_n^{r_0}(\lambda_0) = \limsup_j \frac{1}{2\pi} \int_0^{2\pi} L_{n_j}(\lambda_0, r_0 e^{i\theta}) d\theta \leq \\
\frac{1}{2\pi} \int_0^{2\pi} \limsup_j L_{n_j}(\lambda_0, r_0 e^{i\theta}) d\theta
\end{aligned}$$

and the desired property follows immediately.
To end the proof we argue by contradiction and assume that $\varphi(\lambda_0, w_0) < L(\lambda_0)$. As φ is upper semi-continuous and L continuous, there exists a neighbourhood V_0 of (λ_0, w_0) and $\epsilon > 0$ such that

$$\varphi - L \leq -\epsilon \text{ on } V_0.$$

Pick a small ball B_{λ_0} centered at λ_0 and a small disc Δ_{w_0} centered at w_0 such that $B_0 := B_{\lambda_0} \times \Delta_{w_0}$ is relatively compact in V_0. Then, according to Hartogs Lemma 4.1.4, we have:

$$\limsup_j \left(\sup_{B_0} (L_{n_j} - L) \right) \leq \sup_{B_0} (\varphi - L) \leq -\epsilon.$$

This is impossible since, as we have seen before, we may find $(\lambda_0, r_0 e^{i\theta_0}) \in B_0$ such that $\limsup_j \left(L_{n_j}(\lambda_0, r_0 e^{i\theta_0}) - L(\lambda_0) \right) = 0$. $\square$

Remark 4.2.8 *Using standard techniques, one may deduce from the above Theorem that the set of multipliers w for which the bifurcation current T_{bif} is not a limit of the sequence $d^{-n}[Per_n(w)]$ is contained in a polar subset of the complex plane.*

4.2.3 The case of neutral cycles in polynomial families

We return to the family $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$ of degree d polynomials. Recall that $P_{c,a}$ is the polynomial of degree d whose critical points are $(0 = c_0, c_1, \cdots, c_{d-2})$ and such that $P_{c,a}(0) = a^d$ (see subsection 2.1.2).

We want to prove that, in this family, $\lim_n d^{-n}[\mathrm{Per}_n(w)] = T_{\mathrm{bif}}$ for $|w| \leq 1$. Taking the results of the previous subsection into account (see Theorem 4.2.1), it remains to treat the case $|w| = 1$ and prove the following result due to Bassanelli and Berteloot (see [BB3]).

Theorem 4.2.9 *In the family of degree d polynomials* $\lim_n d^{-n}[Per_n(e^{i\theta})] = T_{bif}$ *for any* $\theta \in [0, 2\pi]$.

We will follow a strategy similar to that used for proving Theorem 4.1.1. As we shall see, the proof would be rather simple if we would know that the bifurcation locus is accumulated by hyperbolic parameters. This is however not the case when $d \geq 3$ and is a source of technical difficulties (see the fourth step).

Proof. We denote by λ the parameter in $\mathbf{C}^{d-1}$ (i.e. $\lambda := (c, a)$) and set

$$L_n(\lambda) := d^{-n} \ln |p_n(\lambda, (e^{i\theta}))|$$

where the polynomials $p_n(\lambda, w)$ are those given by Theorem 2.2.1. We have to show that the sequence $(L_n)_n$ converges to L in L^1_{loc}.

We have already seen that$(L_n)_n$ is a uniformly locally bounded sequence of $p.s.h$ functions on $\mathbf{C}^{d-1}$. Since the family $\{P_{c,a}\}_{(c,a)\in\mathbf{C}^{d-1}}$ contains hyperbolic parameters, on which the $L_n(\lambda) = L_n^+(\lambda, e^{i\theta})$ it follows from Lemma 4.2.4 that the sequence $(L_n)_n$ does not converge to $-\infty$. Thus, according to Theorem 4.1.3, we have to show that L is the only limit value of the sequence $(L_n)_n$ for the L^1_{loc} convergence.

Assume that (after taking a subsequence!) $(L_n)_n$ is converging in L^1_{loc} to φ. To prove that the $p.s.h$ functions φ and L coincide we shall check that they satisfy the assumptions of Lemma 4.1.5.

First step: $\varphi \leq L$.

Since $L_n^+(\lambda, e^{i\theta})$ converges to L in L^1_{loc} (see Lemma 4.2.4) and $L_n(\lambda) \leq Ł_n^+(\lambda, e^{i\theta})$ we get

$$\varphi(\lambda_0) \leq \frac{1}{|B_\epsilon|} \int_{B_\epsilon} \varphi \, dm \leq \frac{1}{|B_\epsilon|} \int_{B_\epsilon} L \, dm$$

for any small ball B_ϵ centered at λ_0. The desired inequality then follows by making $\epsilon \to 0$ since the function L is continuous (see Theorem 3.2.9).

Second step: *Supp* $dd^c\varphi \subset$ *Supp* dd^cL.

Since there are no persistent neutral cycles in the family $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$, the hypersurfaces $Per_n(e^{i\theta})$ are contained in the bifurcation locus. This means that the functions L_n are pluriharmonic on $\mathbf{C}^{d-1} \setminus Supp\, dd^cL$. The same is thus true for the limit φ.

Third step: for any $\lambda_0 \in \mathbf{C}^{d-1}$ there exists a complex line D through λ_0 such that $\varphi = L$ on the unbounded component of $D \setminus \big(D \cap Supp(dd^cL)\big)$.

By Theorem 2.3.5 we may pick a line D through λ_0 which meets infinity far from the cluster set of $\mathcal{B}_i$ in $\mathbf{P}_\infty$. This means that for any λ in the unbounded component of $D \setminus \big(L \cap Supp\ (dd^cg)\big)$ all critical point c_i belongs to the super-attracting basin of ∞ and thus, λ is a hyperbolic parameter.
This implies that $L_n(\lambda) = L_n^+(\lambda, e^{i\theta})$ and, by Lemma 4.2.4, $\varphi(\lambda) = L(\lambda)$.

Fourth step: $\varphi = L$ on *Supp* dd^cL.

This is the most delicate part of the proof, it somehow proceeds by induction on d. To simplify the exposition, we will only treat the cases $d = 2$ and $d = 3$.

When $d = 2$ the parameter space is $\mathbf{C}$ and the bifurcation locus is the boundary $b\mathcal{M}$ of the Mandelbrot set. The unbounded stable component $\big(\mathcal{M}\big)^c$ is hyperbolic and thus, as we saw in the last step, $(\varphi - L) = 0$ there. Since $(\varphi - L) \leq 0$ and $(\varphi - L)$ is *u.s.c*, this implies that $\varphi = L$ on $b\mathcal{M} = Supp\, dd^cL$
Let us stress that this ends the proof when $d = 2$ (the complex line D of the third step is the parameter space itself in that case).

We now assume that $d = 3$, the parameter space is then $\mathbf{C}^2$. Let us consider the sets U_k of parameters which do admit an attracting k-cycle:

$$U_k := \bigcup_{|w|<1} Per_k(w).$$

We have to show that $(\varphi - L)$ coincide on the bifurcation locus. Since the bifurcation locus is accumulated by curves of the form $Per_k(0)$ (by Theorem 4.2.1

$\lim_k d^{-k}[\mathrm{Per}_k(0)] = T_{\mathrm{bif}}$), and the function $(\varphi - L)$ is negative and upper semi-continuous, it suffices to prove that

$$(\varphi - L) = 0 \text{ on all sets } U_k.$$

Let us first treat the problem on a curve $\mathcal{C} := \mathrm{Per}_k(\eta)$ for $|\eta| < 1$ and show that

($\star$) the sequence $\varphi_n|_{\mathcal{C}}$ converges uniformly to $L|_{\mathcal{C}}$ on the stable components.

We may assume that $\mathcal{C}$ is irreducible and desingularize it. This gives a one-dimensional holomorphic family $(P_{\pi(u)})_{u\in M}$. Keeping in mind that the elements of this family are degree 3 polynomials which do admit an attracting basin of period k and using the fact that the connectedness locus in $\mathbf{C}^2$ is compact (see Theorem 2.3.5), one sees that the family $(P_{\pi(u)})_{u\in M}$ enjoys the same properties than the quadratic polynomial family:

1- the bifurcation locus is contained in the closure of hyperbolic parameters

2- the set of non-hyperbolic parameters is compact in M.

Exactly as for the quadratic polynomial family this implies that the sequence L_n converges in L^1_{loc} to L and the convergence is locally uniform on stable components since, as we already observed, the functions L_n are pluriharmonic there.

We now want to show that $\varphi = L$ on any open subset U_k. Again, as the stable parameters are dense and $(\varphi - L)$ is *u.s.c* and negative, it suffices to show that $\varphi = L$ on any stable component of U_k. On such a component the functions L_n are pluriharmonic and thus actually converge locally uniformly to φ. Then, ($\star$) clearly implies that $\varphi = L$ on Ω.

□

4.3 Laminated structures in bifurcation loci

4.3.1 Holomorphic motion of the Mandelbrot set

We work here in the moduli space Mod_2 of degree two rational maps which, as we saw in section 2.1.3, can be identified to $\mathbf{C}^2$. Our aim is to show that the bifurcation locus in the region

$$U_1 := \{\lambda \in \mathbf{C}^2 / f_\lambda \text{ has an attracting fixed point}\}$$

can be obtained by holomorphically moving the boundary $b\mathcal{M}$ of the Mandelbrot set . We remind that $b\mathcal{M}$ is the bifurcation locus of $Per_1(0)$ which is a complex line contained in U_1 and can be identified to the family of quadratic polynomials.
We will see simultaneoulsly that the bifurcation current is uniformly laminar in the region U_1. Let us first recall some basic facts about holomorphic motions.

Definition 4.3.1 *Let M be a complex manifold and $E \subset M$ be any subset. A* holomorphic motion *of E in M is a map*

$$\sigma : E \times \Delta \ni (z, u) \mapsto \sigma(z, u) =: \sigma_u(z) \in M$$

which satisfies the following properties:

i) $\sigma_0 = Id|_E$

ii) $E \ni z \mapsto \sigma_u(z) \in M$ *is one-to-one for every* $u \in \Delta$

iii) $\Delta \ni u \mapsto \sigma_u(z) \in M$ *is holomorphic for every* $z \in E$.

When the family of holomorphic discs in M enjoys good compactness properties, any holomorphic motion extends to the closure. In particular, when M is the Riemann sphere $\widehat{\mathbf{C}}$, the Picard-Montel theorem combined with Hurwitz lemma easily leads to the following famous extension statement. This result is usually called λ-lemma since the "time" parameter is denoted λ rather that t (see Lemm a 3.1.3).

The main result of this subsection is the following. The prove we present here is due to Bassanelli and Berteloot (see [BB2]).

Theorem 4.3.2 *Let Ω_{hyp} be the union of all hyperbolic components of the Mandelbrot set $\mathcal{M}$ and $\heartsuit$ the main cardioid. Let B_{if_1} be the bifurcation locus in U_1 and $T_{bif}|_{U_1}$ be the associated bifurcation current. Let μ_1 be the harmonic measure of $\mathcal{M}$. There exists a continuous holomorphic motion*

$$\sigma : \big((\Omega_{hyp} \setminus \heartsuit) \cup b\mathcal{M}\big) \times \Delta \to U_1$$

such that

$$\sigma\,(b\mathcal{M} \times \Delta) = B_{if_1} \;\; \text{and} \;\; T_{bif}|_{U_1} = \int_{Per_1(0)} [\sigma(z, \Delta)]\, \mu_1.$$

In particular, B_{if_1} is a lamination with μ_1 as transverse measure. Moreover, the map σ is holomorphic on $\big(\Omega_{hyp} \setminus \heartsuit\big) \times \Delta$ and preserves the curves $Per_n(w)$ for $n \geq 2$ and $|w| \leq 1$.

Proof.

First step: Holomorphic motion of $\big(\Omega_{hyp} \setminus \heartsuit\big)$.

The curve $Per_1(0)$ is actually the complex line $\lambda_1 = 2$. We will write $(2, \lambda_2) =: z$ the points of this line.

Let us consider $U_n := \{\lambda \in \mathbf{C}^2 / f_\lambda$ has an attracting cycle of period $n\}$ and $\Omega_n := U_n \cap Per_1(0)$. We recall that $\Omega_{hyp} = \heartsuit \cup \bigcup_{n \geq 2} \Omega_n$.

Let us also set $U_{n,1} := U_n \cap U_1$. By the Fatou-Shishikura inequality, a quadratic rational map has at most two non-repelling cycles. Thus $U_{n,1} \cap U_{m,1} = \emptyset$ when $n \neq m$ and there exists a well defined holomorphic map

$$\psi_n : U_{n,1} \to \Delta \times \Delta$$

which associates to every $\lambda \in U_{n,1}$ the pair $(w_n(\lambda), w_1(\lambda))$ where $w_n(\lambda)$ is the multiplier of the attracting n-cycle of λ and $w_1(\lambda)$ the multiplier of its attracting fixed point.

The key point is the following transversality statement due to Douady and Hubbard (see also [BB1]):

Lemma 4.3.3 *the map ψ induces a biholomorphism*

$$\psi_{n,j} : U_{n,1,j} \to \Delta \times \Delta$$

on each connected component $U_{n,1,j}$ of $U_{n,1}$.

The connected components $\Omega_{n,j}$ of Ω_n coincides with $U_{n,1,j} \cap Per_1(0)$ and one clearly obtains a holomorphic motion $\sigma : \big(\Omega_{hyp} \setminus \heartsuit\big) \times \Delta \to U_1$ by setting:

$$\sigma(z,t) := (\psi_{n,j})^{-1}\big(w_n(z), t\big)$$

for any $z \in \Omega_{n,j}$.

<u>Second step:</u> extension of σ to $b\mathcal{M}$.

The key point here is that $\sigma(z,t) =: (\alpha(z,t), \beta(z,t))$ belongs to the complex line $Per_1(t)$ which, according to Proposition 2.2.9, is given by the equation

$$(t^2+1)\lambda_1 - t\lambda_2 - (t^3+1) = 0.$$

Thus $\sigma(z,t)$ is completely determined by $\beta(z,t)$:

$$\alpha_\lambda(t) = \frac{1}{1+t^2}\left(t\beta_\lambda(t) + t^3 + 2\right), \ \ \forall t \in \Delta. \tag{4.3.1}$$

We will now identify $Per_1(0)$ with the deleted Riemann sphere $\widehat{\mathbf{C}} \setminus \{\infty\}$ and set $\beta(\infty, t) = \infty$ for all $t \in \Delta$. Then, the map $\beta : \big(\{\infty\} \cup \big(\Omega_{hyp} \setminus \heartsuit\big)\big) \times \Delta \to \widehat{\mathbf{C}}$ is clearly a holomorphic motion which, by Lemma 3.1.3, extends to the closure of $\big(\Omega_{hyp} \setminus \heartsuit\big)$. We thus obtain a continuous holomorphic motion

$$\beta : \big(\{\infty\} \cup \big(\Omega_{hyp} \setminus \heartsuit\big) \cup b\mathcal{M}\big) \times \Delta \to \widehat{\mathbf{C}}.$$

As, by construction, $\beta(z,t) \neq \infty$ when $z \neq \infty$, the identity 4.3.1 shows that $\sigma(z,t) = (\alpha(z,t), \beta(z,t))$ extends to a continuous holomorphic motion of $\big(\big(\Omega_{hyp} \setminus \heartsuit\big) \cup b\mathcal{M}\big)$.

Third step: laminarity properties.

Let us show that $T_{\rm bif}|_{U_1} = \int_{Per_1(0)}[\sigma(z,\Delta)]\ \mu_1$. According to the approximation formula given by Theorem 4.2.1 applied on $Per_1(0)$ we have:

$$\mu_1 = \lim_m 2^{-m} \sum_{z\in Per_1(0)\cap Per_m(0)} \delta_{\sigma(z,0)}. \qquad (4.3.2)$$

Let us set $T := \int_{Per_1(0)}[\sigma(z,\Delta)]\ \mu_1$. We have to check that $T = T_{\rm bif}|_{U_1}$. Let ϕ be a $(1,1)$-test form in U_1. As the holomorphic motion σ is continuous, the function $z \mapsto \langle[\sigma(z,\Delta)],\phi\rangle$ is continuous as well. Then, using 4.3.2 one gets

$$\langle T,\phi\rangle = \lim_m 2^{-m} \sum_{z\in Per_n(0)\cap Per_m(0)} \langle[\sigma(z,\Delta)],\phi\rangle = \lim_m 2^{-m}\langle[Per_m(0)],\phi\rangle \qquad (4.3.3)$$

where the last equality uses the fact that, according to Proposition 2.2.10, the curves $Per_m(0)$ have no multiplicity in U_1. Now the conclusion follows by using 4.3.3 and the approximation formula of Theorem 4.2.1 in U_1.

By construction, the map σ is holomorphic on $\big(\Omega_{hyp}\setminus\heartsuit\big)\times\Delta$ and preserves the curves $Per_n(w)$ for $n\geq 2$ and $|w|<1$. This extends to $|w|=1$ by continuity.

Using the continuity of σ, one easily sees that $\sigma\big(b\mathcal{M},\Delta\big)$ is closed in U_1 and therefore contains the support of $\int_{Per_1(0)}[\sigma(z,\Delta)]\ \mu_1$. By the above formula we thus have

$$B{\rm if}_1 = Supp\big(T_{\rm bif}|_{U_1}\big) \subset \sigma\big(b\mathcal{M},\Delta\big).$$

The opposite inclusion easily follows from the construction of σ: any point in $\sigma\big(b\mathcal{M},\Delta\big)$ is a limit of $z_m\in Per_m(0)$ where $m\to+\infty$. □

COROLLARY: Closure of Bif in P2???.

Instead of using the basic λ-Lemma we could have use its far advanced generalization due to Slodkowski and get a motion on the full line $Per_1(0)$.

Theorem 4.3.4 (Slodkowski λ-lemma) *Let $E\subset\widehat{\mathbf{C}}$ be a subset of the Riemann sphere and $\sigma: E\times\Delta\ni(z,t)\mapsto\sigma(z,t)\in\widehat{\mathbf{C}}$ be a holomorphic motion. Then σ extends to a holomorphic motion $\tilde\sigma$ of $\widehat{\mathbf{C}}$. Moreover $\tilde\sigma$ is continuous on $\overline{E}\times\Delta$ and $z\mapsto\tilde\sigma(z,t)$ is a K-quasi-conformal homeomorphism for $K:=\frac{1+|t|}{1-|t|}$.*

Our reference for quasi-conformal maps is the book [H] where one can also find a nice proof of Slodkowski theorem due to Chirka and Rosay.

Using Slodkowski Theorem one may obtain further informations on the motion given by Theorem 4.3.2. We refer to our paper [BB2] for a proof.

Theorem 4.3.5 *Let $\sigma : \big((\Omega_{hyp} \setminus \heartsuit) \cup b\mathcal{M}\big) \times \Delta \to U_1$ be the holomorphic motion given by Theorem 4.3.2. Then σ extends to a continuous holomorphic motion $\tilde{\sigma}$: $Per_1(0) \times \Delta \longrightarrow U_1$ which is onto. All stable components in U_1 are of the form $\tilde{\sigma}(\omega \times \Delta)$ for some component ω in $Per_1(0)$. Moreover, the map $\lambda \mapsto \tilde{\sigma}(\lambda, t)$ is a quasi-conformal homeomorphism for each t and $\tilde{\sigma}$ is one-to-one on $\big(Per_1(0) \setminus \overline{\heartsuit}\big) \times \Delta$ where $\heartsuit$ is the main cardioid.*

Theorem 4.3.5 shows that non-hyperbolic components exist in U_1 if and only if such components exist within the quadratic polynomial family $Per_1(0)$. Let us underline that, in relation with Fatou's problem on the density of hyperbolic rational maps, it is conjectured that such components do not exist.

4.3.2 Further laminarity statements for $T_{\mathbf{bif}}$

The following result is an analogue of Theorem 4.3.2 in the regions

$$U_n = \{\lambda \in Mod_2 / f_\lambda \text{ has an attracting cycle of period } n\}.$$

It shows, in particular, that the bifurcation current in Mod_2 is uniformly laminar in the regions U_n. It has been established by Bassanelli and Berteloot in [BB2].

Theorem 4.3.6 *Let Bif_n be the bifurcation locus in U_n and $T_{bif}|_{U_n}$ be the associated bifurcation current. Let Bif_n^c be the bifurcation locus in the central curve $Per_n(0)$ and μ_n^c be the associated bifurcation measure. Then, there exists a map*

$$\begin{array}{rcl} \sigma : Bif_n^c \times \Delta & \longrightarrow & Bif_n \\ (\lambda, t) & \longmapsto & \sigma(\lambda, t) \end{array}$$

such that:

1) $\sigma(Bif_n^c \times \Delta) = Bif_n$

2) σ is continuous, $\sigma(\lambda, \cdot)$ is one-to-one and holomorphic for each $\lambda \in Bif_n^c$

3) $p_n(\sigma(\lambda, t), t) = 0$; $\forall \lambda \in Bif_n^c, \forall t \in \Delta$

4) the discs $(\sigma(\lambda, \Delta))_{\lambda \in Bif_n^c}$ are mutually disjoint.

Moreover the bifurcation current in U_n is given by

$$T_{bif}|_{U_n} = \int_{Bif_n^c} [\sigma(\lambda, \Delta)]\, \mu_n^c$$

and, in particular, Bif_n is a lamination with μ_n^c as transverse measure.

The proof is similar to that of Theorem 4.3.6 but requires a special treatment for the extension problems since there is no λ-lemma available. The key is to use the fact that, by construction, the starting motion $\sigma : (\cup_{m\neq n}(U_m \cap U_n)) \times \Delta \longrightarrow U_n$ satisfies the following property:

$$p_n\left(\sigma(\lambda,t),t\right) = 0;\ \forall \lambda \in B\mathrm{if}_n^c, \forall t \in \Delta.$$

Such a motion is what we call a p_n-guided holomorphic motion. Using Zalcman rescaling lemma, one proves the following compactness property for guided holomorphic motions. We stress that here, an holomorphic motion $\mathcal{G}$ is seen as family of disjoints holomorphic discs σ and $\mathcal{G}_{t_0}$ is the set of points $\sigma(t_0)$.

Theorem 4.3.7 *Let $p(\lambda, w)$ be a polynomial on $\mathbf{C}^2 \times \mathbf{C}$ such that the degree of $p(\cdot, w)$ does not depend on $w \in \Delta$. Let $\mathcal{G}$ be a p-guided holomorphic motion in $\mathbf{C}^2$ such that any component of the algebraic curve $\{p(\cdot, t) = 0\}$ contains at least three points of $\mathcal{G}_t$ for every $t \in \Delta$. Then, for any $\mathcal{F} \subset \mathcal{G}$ such that $\mathcal{F}_{t_0}$ is relatively compact in $\mathbf{C}^2$ for some $t_0 \in \Delta$, there exists a continuous p-guided holomorphic motion $\widehat{\mathcal{F}}$ in $\mathbf{C}^2$ such that $\mathcal{F} \subset \widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}_{t_0} = \overline{\mathcal{F}_{t_0}}$.*

The above Theorem plays the role of the λ-lemma in the proof of Theorem 4.3.6. We refer to the paper [BB2] for details.

We will now end this section by quoting another laminarity result for T_{bif} which is due to Dujardin (see [Du])

Theorem 4.3.8 *Within the polynomial family of degree 3, the bifurcation current is laminar outside the connectedness locus.*

We refer to section 2.3 for the discussion of the connectedness locus within polynomial families.

Chapter 5

The bifurcation measure

5.1 A Monge-Ampère mass related with strong bifurcations

5.1.1 Basic properties

Since the bifurcation current T_{bif} of any holomorphic family $(f_\lambda)_{\lambda\in M}$ of degree d rational maps has a continuous potential L (see Definition 3.2.6 and Theorem 3.2.9), one may define the powers $(T_{\text{bif}})^k := T_{\text{bif}} \wedge T_{\text{bif}} \wedge \cdots \wedge T_{\text{bif}}$ for any $k \leq m := dimM$. We recall that for any closed positive current T, the product $dd^cL \wedge T$ is defined by $dd^cL \wedge T := dd^c(LT)$. In particular, $(T_{\text{bif}})^m$ is a positive measure on M which is equal to the Monge-Ampère mass of the Lyapunov function L.

Definition 5.1.1 *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of degree d rational maps parametrized by a complex manifold M of dimension m. The* bifurcation measure *μ_{bif} of the family is the positive measure on M defined by*

$$\mu_{bif} = \frac{1}{m!}(T_{bif})^m = \frac{1}{m!}\,(dd^cL)^m$$

where T_{bif} is the bifurcation current and L the Lyapunov function of the family.

The following proposition is a direct consequence of the definition and the fact that μ_{bif} has locally bounded potentials.

Proposition 5.1.2 *The support of μ_{bif} is contained in the bifurcation locus and μ_{bif} does not charge pluripolar sets.*

It is actually possible to define the bifurcation measure in the moduli space Mod_d of degree d rational maps and show that this measure has strictly positive and finite mass (see [BB1] Proposition 6.6). Although all the results we will present here are true in Mod_d, we will restrict ourself to the technically simpler situation of holomorphic families. The example we have in mind are the polynomial families and the moduli space Mod_2 which, in some sense, can be treated as a holomorphic family.

In arbitrary holomorphic families, the measure μ_{bif} can identically vanish. Moreover, it is usually quite involved to that $\mu_{\text{bif}} > 0$. Note however that this will follow from standard arguments in polynomial families. The following simple observation already shows that $\mu_{\text{bif}} > 0$ in M_2 (and more generally M_d), it has also its own interest.

Proposition 5.1.3 *In any holomorphic family, all rigid Lattès examples belong to the support of* μ_{bif}.

Proof. The parameters corresponding to rigid Lattès examples are isolated. By Theorem 1.3.5, these points correspond to strict minima of the Lyapunov function L and therefore the Monge-Ampère measure $(dd^cL)^m$ cannot vanish around them. □

We end this subsection by showing that the activity currents have no self-intersection. This is a useful geometric information which, in particular, shows that the activity of all critical points is a necessary condition for a parameter to be in the support of the bifurcation masure. It was first proved by Dujardin-Favre in the context of polynomial families (see [DF] Proposition 6.9), we present here a general argument due to Gauthier ([G]).

Theorem 5.1.4 *Let* $(f_\lambda)_{\lambda\in M}$ *be any holomorphic family of degree* d *rational maps with marked critical points. The activity currents* T_i *satisfy* $T_i \wedge T_i = 0$. *In particular, when* $m := dim\ M = 2d-2$ *(or* $m = d-1$ *for polynomial families) then*

$$\mu_{bif} = T_1 \wedge T_2 \wedge \cdots \wedge T_m$$

and Supp μ_{bif} *is contained in the intersection of the activity loci of the critical points.*

The proof is very close to that of a density statement which will be presented in the next section, it combines the following potential-theoretic lemma with a dynamical observation.

Lemma 5.1.5 *Let* u *be a continuous p.s.h function on some open subset* Ω *in* $\mathbf{C}^2$. *Let* Γ *be the union of all analytic subsets of* Ω *on which* u *is harmonic. If the support of* dd^cu *is contained in* $\overline{\Gamma}$ *then* $dd^cu \wedge dd^cu$ *vanishes on* Ω.

Proof. Let us set $\mu := dd^cu \wedge dd^cu$. Let B_r be an open ball of radius r whose closure is contained in Ω, we have to show that $\mu\left(B_{\frac{r}{2}}\right) = 0$.

Denote by h the solution of the Dirichlet-Monge-Ampère problem with data u on bB_r:

$$h = u \text{ on the boundary of } B_r$$
$$dd^c h \wedge dd^c h = 0 \text{ on } B_r \text{ (i.e. } h \text{ is maximal on } B_r).$$

The function h is $p.s.h$ and continuous on $\overline{B_r}$ ((see [BT]). As h is $p.s.h$ maximal and coincides with the $p.s.h$ function u on $b\overline{B_r}$, we have $u \leq h$ on $\overline{B_r}$. For any $\epsilon > 0$ we define

$$D_\epsilon := \{\lambda \in B_{\frac{r}{2}} \;/\; 0 \leq h(\lambda) - u(\lambda) \leq \epsilon\}.$$

We will see that our assumption implies that

$$Supp\, \mu \cap B_{\frac{r}{2}} \subset D_\epsilon \text{ for all } \; \epsilon > 0. \tag{5.1.1}$$

Indeed, if γ is a complex curve in Ω on which u is harmonic then, the maximum modulus principle, applied to $(h - u)$ on $\gamma \cap B_r$ implies that $h = u$ on γ. Then, as $(h - u)$ is continuous on B_r and $Supp\;\mu \subset Supp\; dd^c u \subset \overline{\Gamma}$, we get $Supp\, \mu \cap B_r \subset \{h = u\}$.

Now, a result due to Briend-Duval (see[BD] or [DS] Théorème A.10.2) says that

$$\mu\left(D_\epsilon\right) \leq C\epsilon \tag{5.1.2}$$

where C only depends on u and B_r. From 5.1.1 and 5.1.2 we deduce that $\mu\left(B_{\frac{r}{2}}\right) = 0$.
□

We may now end the proof of Theorem 5.1.4.

Proof. We only have to show that $T_i \wedge T_i = 0$, the remaining then follows from the identity $T_{\text{bif}} = \sum_i T_i$ (see Theorem 3.2.8).
The statement is local and we may therefore assume that $M = \mathbf{C}^k$. Moreover, an elementary slicing argument allows to reduce the dimension to $k = 2$. We apply the above lemma with $u = G_\lambda\left(\hat{c}_i(\lambda)\right)$ (see Lemma 3.2.7). We have to show that the support of $dd^c u = T_i$ is accumulated by curves on which the critical point c_i is passive. These curves are of the form $\{f_\lambda^n(c_i(\lambda) = c_i(\lambda)\}$ and their existence follows from Lemma 3.1.8. □.

An example, due to A.Douady, shows that the activity of all critical points is not sufficent for a parameter to be in the support of the bifurcation measure. We will present this example in the next subsection (see 5.1.7).

5.1.2 Some concrete families

We first discuss the case of the polynomial families introduced in subsection 2.1.2. We follow here the paper [DF] by Dujardin and Favre.

Proposition 5.1.6 *The bifurcation measure μ_{bif} of the degree d polynomial family $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$ is a probability measure supported on the connectedness locus $\mathcal{C}$. It coincides with the pluricomplex equilibrium measure of $\mathcal{C}$ and its support is the Shilov boundary of $\mathcal{C}$.*

Proof. Let us recall that the Green function of the polynomial $P_{c,a}$ is denoted $g_{c,a}$. The connectedness locus $\mathcal{C}$ is a compact subset of $\mathbf{C}^{d-1}$ which coincides with the intersection $\cap_{0\le i\le d-2}\mathcal{B}_i$ where $\mathcal{B}_i$ is the set of parameters for which the orbit of the critical point c_i is bounded (see Theorem 2.3.5). As the support of the activity current T_i is contained in $b\mathcal{B}_i$ we deduce that $Supp\ \mu_{\rm bif} \subset \mathcal{C}$ from Theorem 5.1.4. All the remaining follows from the fact that $\mu_{\rm bif} = (dd^c\mathcal{G})^{d-1}$ where $\mathcal{G}$ is the pluricomplex Green function of $\mathcal{C}$ with pole at infinity:

$$\mathcal{G} := sup\{u\ p.s.h\ /\ u - \ln^+ max\{|a|, |c_k|\} \le O(1), u \le 0 \text{ on } \mathcal{C}\}$$

an identity which we shall now prove.

Let us first establish that $\mu_{\rm bif} = (dd^cG)^{d-1}$ where $G := max\{g_0, g_1, \cdots, g_{d-2}\}$ and $g_i := g_{c,a}(c_i)$. We show by induction that $T_0 \wedge T_1 \wedge \cdots \wedge T_l = (dd^cG_l)^{l+1}$ for $0 \le l \le d-2$ where $G_l := max\{g_0, g_1, \cdots, g_l\}$. This follows from the following computation:

$$T_0 \wedge T_1 \wedge \cdots \wedge T_{l-1} \wedge T_l = dd^c\left(g_l(dd^cG_{l-1})^l\right) = dd^c\left(G_l(dd^cG_{l-1})^l\right) =$$
$$dd^c\left(G_{l-1}(dd^cG_{l-1})^{l-1} \wedge dd^cG_l\right) = dd^c\left(G_l(dd^cM_{l-1})^{l-1} \wedge dd^cG_l\right) = (dd^cG_l)^{l+1}$$

the second equality follows from $g_l = G_L$ on the support of $(dd^cG_{l-1})^l$ and the fourth from $G_l = G_{l-1}$ on the support of dd^cG_l, the last equality is obtained by repeating the same arguments $l-1$ times.

It remains to show that $G = \mathcal{G}$. The proof is standard and relies on the estimate given by Proposition 2.3.7 and the fact that G is maximal outside $\mathcal{C}$. We refer to the paper [DF], Proposition 6.14 for more details. □

We will now present the example mentionned at the end of last subsection.

Example 5.1.7 *In the holomorphic family of degree 3 polynomials*

$$\left((1+\alpha_1)z + (\tfrac{1}{2}+\alpha_2)z^2 + z^3\right)_{\alpha\in V_0}$$

where V_0 is a neighbourhood of the origin in $\mathbf{C}^2$ the critical points are both active at the origin $(0,0)$ but $(0,0) \notin Supp\ \mu_{bif}$.

This family is a deformation of the polynomial $P_0 := z + \frac{1}{2}z^2 + z^3$. If V_0 is small enough we have two marked critical points $c_1(\alpha)$ and $c_2(\alpha)$. The origin 0 is a parabolic fixed point for P_0 and we may assume that P_α has two fixed points counted with multiplicity near 0 for all $\alpha \in V_0$. As P is real, its critical points are complex conjugate and both of them are attracted by the parabolic fixed point at 0, moreover their orbits are not stationnary.

We first show that $(0,0) \notin Supp\, \mu_{\text{bif}}$. When the fixed points of P_α are distinct, we denote by $m_1(\alpha)$ and $m_2(\alpha)$ their multipliers. When this is the case, it turns out that either $|m_1(\alpha)| < 1$ or $|m_2(\alpha)| < 1$ and thus one of the fixed points attracts a critical point. This can be seen by using the holomophic index fixed point formula. By Theorem 5.1.4, this implies that $\alpha \notin Supp\, \mu_{\text{bif}}$. We thus see that μ_{bif} is supported on the subvariety of parameters α for which the fixed point is double. By Proposition 5.1.2 this implies that μ_{bif} vanishes near $(0,0)$.

Let us now see that both critical points are active. We may assume that the family is parametrized by a disc D in $\mathbf{C}$ such that P_α has two distinct fixed points when $\alpha \neq 0$. Assume to the contrary that a critical point $c(\alpha)$ is passive. Then, after taking a subsequence, the sequence $u_n(\alpha) := P_\alpha^n(c(\alpha))$ is uniformly converging to $u(\alpha)$. Since the polynomial P_0 is real, its critical points are complex conjugate and must therefore both be attracted by the parabolic fixed point 0. Moreover, their orbits are not stationnary. From this one easily deduce that the curve $(\alpha, u(\alpha))$ lies in the analytic set $Z := \{(\alpha, z) \in D \times \mathbf{P}^1 \ / \ P_\alpha(z) = z\}$. This is impossible since, for some α close to 0, the critical orbit should be attracted by a repelling fixed point.□

The situation in the moduli space Mod_2 is more complicated. We recall that Mod_2 can be identified to $\mathbf{C}^2$. Using the results which will be obtained in the last section of this chapter and the holomorphic motions constructed in section 4.3, it is possible to show that the support of the bifurcation locus is not bounded.

5.2 Density statements

We show that the remarkable parameters introduced in subsection 3.1.3 accumulate the support of the bifurcation measure.

5.2.1 Misiurewicz parameters

The results given in this subsection are essentially due to Dujardin and Favre. We present them in the setting of polynomial families and refer the reader to the original paper ([DF]) for a greater generality.

Theorem 5.2.1 *In the degree d polynomial family $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^{d-1}}$ let us define a sequence of analytic sets by:*

$$W_{n_0,\cdots,n_l} := \cap_{j=0}^{l} \{P_{c,a}^{n_j}(c_j) = P_{c,a}^{k(n_j)}(c_j)\}$$

where $l \leq d-2$ and $k(n_j) < n_j$. Then

$$\lim_{n_{d-2}\to\infty} \cdots \lim_{n_0\to\infty} \frac{1}{d^{n_{d-2}} + \cdots + d^{n_0}} [W_{n_0,\cdots,n_{d-2}}] = \mu_{bif}$$

and $W_{n_0,\cdots,n_{d-2}}$ is discrete.

Proof. We treat the case $d = 3$ which is actually not very different from the general case.

Let us first observe that W_{n_0,n_1} has codimension at least two and is contained in the connectedness locus which is compact (see Theorem 2.3.5). Thus W_{n_0,n_1} is a discrete set.

Applying a version of Theorem 4.1.1 suitably adapted to the family W_{n_0} yields

$$\lim_{n_1\to\infty} d^{n_1}[W_{n_0,n_1}] = T_1 \wedge [W_{n_0}]$$

where T_1 is the activity current of the critical point c_1. By the same Theorem one has $\lim_{n_0\to\infty} d^{n_0}[W_{n_0}] = T_0$ where T_0 is the activity current of c_0 and this, since T_1 has continuous potentials, gives

$$\lim_{n_0\to\infty} T_1 \wedge d^{n_0}[W_{n_0}] = T_1 \wedge T_0.$$

The conclusion follows immediately since, according to Theorem 5.1.4, $\mu_{\text{bif}} = T_0 \wedge T_1$. □

An important consequence of the above result is that the support of the bifurcation measure is accumulated by Misiurewicz polynomials. An alternative proof of that fact will be given in the next subsection for arbitrary families. We refer to 3.1.16 for a definition of Misiurewicz parameters.

Corollary 5.2.2 *In polynomial families, the support of the bifurcation measure is contained in the closure of strongly Misiurewicz parameters: Supp $\mu_{bif} \subset \overline{M_{iss}}$.*

Proof. By the above Theorem

$$\lim_{n_{d-2}\to\infty} \cdots \lim_{n_0\to\infty} \frac{1}{d^{n_{d-2}} + \cdots + d^{n_0}} [\cap_0^{d-2}\{P_{c,a}^{n_j}(c_j) = P_{c,a}^{n_j-1}(c_j)\}] = \mu_{\text{bif}}\,. \quad (5.2.1)$$

Let us observe that

$$H_j := \{P_{c,a}^{n_j}(c_j) = P_{c,a}^{n_j-1}(c_j)\} = Preper_{n_j} + Fix_j$$

where, for parameters in $Preper_{n_j}$ the critical point c_j is strictly preperiodic to a (necessarily) repelling fixed point while c_j is fixed for parameters in Fix_j.

Now Theorem 4.1.1 may ne rewritten as

$$\lim_{n_j\to\infty} \frac{1}{d^{n_j}}[Preper_{n_j}] + \frac{\alpha_{n_j}}{d^{n_j}}[Fix_j] = T_j \tag{5.2.2}$$

but, as T_j cannot charge the hypersurface Fix_j, we must have $\frac{\alpha_{n_j}}{d^{n_j}} \to 0$. Thus $\lim_{n_j\to\infty} \frac{1}{d^{n_j}}[Preper_j] = T_j$ and 5.2.1 yields

$$\lim_{n_{d-2}\to\infty} \cdots \lim_{n_0\to\infty} \frac{1}{d^{n_{d-2}} + \cdots + d^{n_0}}[\cap_{j=0}^{d-2} Preper_{n_j}] = \mu_{\text{bif}}\,.$$

The conclusion follows immediately since $\cap_{j=0}^{d-2} Preper_{n_j} \subset M_{\text{iss}}$. □

5.2.2 Shishikura or hyperbolic parameters

We aim here to show that the support of the bifurcation measure in Mod_d is simultaneously accumulated by Shishikura and hyperbolic parameters (see subsection 3.1.3 for definitions):

$$Supp\ \mu_{\text{bif}} \subset \overline{S_{\text{hi}}} \cap \overline{H_{\text{yp}}}.$$

It is worth emphasize that both statements will be deduced in the same way from the following generalized version of Theorem 4.2.5.

Theorem 5.2.3 *Let μ_{bif} be the bifurcation measure of a holomorphic family $(f_\lambda)_{\lambda\in M}$ of rational maps. Let m denote the complex dimension of M. Let $0 < r \leq 1$. Then there exists increasing sequences of integers $k_2(n), ..., k_m(n)$ such that:*

$$\mu_{bif} = \lim_n \frac{d^{-(n+k_2(n)+\cdots+k_m(n))}}{m!(2\pi)^m} \int_{[0,2\pi]^m} [Per_n(re^{i\theta_1})] \wedge \bigwedge_{j=2}^{m} [Per_{k_j(n)}(re^{i\theta_j})]\ d\theta_1 \cdots d\theta_m.$$

Moreover, we may assume that $k_j(n) \neq k_i(n)$ when $i \neq j$.

We will derive that result from Theorem 4.2.5 by simple calculus arguments with currents.

Proof. For any fixed variety $Per_p(re^{i\theta_p})$, the set of $\theta \in [0, 2\pi]$ for which $Per_p(re^{i\theta_p})$ shares a non trivial component with $Per_m(re^{i\theta})$ for some $m \in \mathbf{N}^*$ is at most countable. This follows from Fatou's theorem on the finiteness of the set of non-repelling cycles. Thus, the wedge products $[Per_{n_1}(re^{i\theta_1})] \wedge \cdots \wedge [Per_{n_m}(re^{i\theta_m})]$ make sense for almost every $(\theta_1, \cdots, \theta_m) \in [0, 2\pi]^m$ and the integrals $\int_{[0,2\pi]^m} [Per_n(re^{i\theta_1})] \wedge \bigwedge_{j=2}^m [Per_{k_j(n)}(re^{i\theta_j})]\ d\theta_1 \cdots d\theta_m$ are well defined.

Next, we need the following formula which has been justified for $q = 1$ at the end of the proof of Theorem 4.2.5. The proof is similar for $q > 1$ and we shall omit it. Recall that $L_n^r(\lambda) := \frac{d^{-n}}{2\pi} \int_0^{2\pi} \ln |p_n(\lambda, re^{i\theta})|\ d\theta$.

$$dd^c L^r_{n_1} \wedge \cdots \wedge dd^c L^r_{n_q} = \frac{d^{-(n_1+\cdots+n_q)}}{(2\pi)^q} \int_{[0,2\pi]^q} \bigwedge_{k=1}^q [Per_{n_k}(re^{i\theta_k})] d\theta_1 \cdots d\theta_q.$$

To prove the convergence, we may replace M by $\mathbf{C}^m$ since the problem is local. The conclusion is obtained by using Theorem 4.2.5, the above formula and the next lemma inductively.

Lemma 5.2.4 *If $S_n \to (dd^c L)^p$ for some sequence $(S_n)_n$ of closed, positive (p,p)-currents on M then $dd^c L^r_{k(n)} \wedge S_n \to (dd^c L)^{p+1}$ for some increasing sequence of integers $k(n)$.*

Let us briefly justify lemma 5.2.4. Let us denote by s_n the trace measure of S_n, as M has been identified with $\mathbf{C}^m$ this measure is given by $s_n := S_n \wedge (dd^c|z|^2)^{m-p}$. Since S_n is positive, s_n is positive as well. Let us consider the sequence $(u_k)_k$ defined by $u_k := L^r_k - L$. We now that $(u_k)_k$ converges pointwise to 0 (see remark 4.2.6) and is locally uniformly bounded (the function L is continuous). The positive current S_n may be considered as a (p,p) form whose coefficients are measures which are dominated by the trace measure s_n. Thus, by the dominated convergence theorem, $(L^r_k - L)S_n = u_k S_n$ tends to 0 as $k \to \infty$ and n is fixed. On the other hand, LS_n converges to LS because L is continuous. It follows that some subsequence $L^r_{k(n)} S_n$ converges to LS. □

Corollary 5.2.5 *In the moduli space Mod_d the support of the bifurcation measure μ_{bif} is contained in $\overline{S_{hi}} \cap \overline{H_{yp}}$.*

Proof. Use Remark 2.1.3 to work with families and then apply Theorem 5.2.3. For $0 < r < 1$ one gets $Supp\ \mu_{\text{bif}} \subset \overline{H_{\text{yp}}}$ and for $r = 1$ $Supp\ \mu_{\text{bif}} \subset \overline{S_{\text{hi}}}$ □

Let us end this subsection with a few remarks.

We first notice that the above arguments yields a rather simple proof of the existence of Shishikura maps, the original proof uses quasi-conformal surgery.

Also, combining the above Corollary with Proposition 5.1.3 one sees that rigid Lattès maps are accumulated by Shishikura's maps or by hyperbolic maps. This last information apparently answers a question raised by Michel Herman.

Theorem 5.2.3 remains true, with the same proof, if one replace the integrals by

$$\int_{[0,2\pi]^m} [Per_n(r_1 e^{i\theta_1})] \wedge \bigwedge_{j=2}^m [Per_{k_j(n)}(r_j e^{i\theta_j})]\ d\theta_1 \cdots d\theta_m$$

where $0 < r_j \leq 1$ for $1 \leq j \leq m$. As a consequence, if $\alpha + \nu = m$ and $\mathcal{P}_{\alpha,\nu}$ is the set of parameters λ such that f_λ has ν distinct attracting cycles and ν distinct neutral cycles Then $Supp\ \mu_{\text{bif}}$ is contained in the closure of $\mathcal{P}_{\alpha,\nu}$.

5.2.3 Shishikura parameters with chosen multipliers

As Theorem 4.2.9 shows, polynomial with a neutral cycle of a given multiplier are dense in the support of the bifurcation current. We believe that a similar property is still true for the bifurcation measure which means that Shishikura parameters with arbitrarily fixed multipliers should be dense in the support μ_{bif} . The following result goes in this direction.

Theorem 5.2.6 *Denote by $p(f)$ (resp. $s(f)$, $c(f)$) the number of distinct parabolic (resp. Siegel, Cremer) cycles of $f \in Mod_d$. Then*

$$Supp\, \mu_{bif} \subset \overline{\{f \in Mod_d \ / \ p(f) = p, s(f) = s \text{ and } c(f) = c\}}$$

for any triple of integers p, s and c such that $p + s + c = 2d - 2$.

The proof is essentially based on Lemma 5.1.5 and Mañé-Sad-Sullivan theorem. More precisely we will use the following

Lemma 5.2.7 *Let E be a dense subset of $[0, 2\pi]$. Then for any holomorphic family of degree d rational map $(f_\lambda)_M$ the set*

$$\cup_n \cup_{\theta \in E} Per_n(e^{i\theta})$$

is dense in the bifurcation locus.

Proof. Use Mañé-Sad-Sullivan theorem or Theorem 4.2.5 with $r = 1$. □

Let us now prove Theorem 5.2.6. We restrict ourself to Mod_2. The general case requires to use a slicing argument, we refer to [BB1] for details.

Proof. Let E_1 and E_2 be two dense subsets of $[0, 2\pi]$. Let λ_0 be a point in the support of μ_{bif} and U_0 be an arbitrarily small neighbourhood of λ_0. By Lemma 5.2.7, the support of the bifurcation current dd^cL is accumulated by holomorphic discs contained in $\cup_n \cup_{\theta \in E_1} Per_n(e^{i\theta})$. Among such discs, let us consider those which go through U_0 and pick one disc Γ_1 on which the Lyapunov function L is not harmonic. Such a disc exists since otherwise, according to lemma 5.1.5, the measure μ_{bif} would vanish on U_0. The bifurcation locus of $(f_\lambda)_{\Gamma_1}$ is not empty and thus, to get a Shishikura parameter in U_0 with multipliers $e^{i\theta_1}$ and $e^{i\theta_2}$ where $\theta_j \in E_j$, it suffices to apply again Lemma 5.2.7 with the dense set E_2 to the family $(f_\lambda)_{\Gamma_1}$. □

5.3 The support of the bifurcation measure

In this section we will show in which sense the support of the bifurcation measure in Mod_d can be consider as a strong-bifurcation locus.

5.3.1 A transversality result

Transversality statements play a very important role for understanding the structure of parameter spaces. We have alrady encounter such results like for instance Lemma 4.3.3 or the Fatou-Shishikura inequality. We refer to the fundamental work of Epstein [Eps2] for a general and synthetic treatement of transversality problems in homorphic dynamics.

All the results presented here are true in Rat_d or in the moduli spaces Mod_d. For simplicity we shall restrict ourself to the moduli space Mod_2 which will be treated as a holomorphic family $(f_\lambda)_{\lambda\in\mathbf{C}^2}$ (see Theorem 2.2.8). We shall also assume, to simplify the exposition, to have two marked critical points $c_j(\lambda)$ $j = 1, 2$.

Assume that $f_0 \in Mod_2$ is strongly Misiurewicz. This means that there exists two repelling cycles

$$\mathcal{C}_j(0) := \{z_j(0), \cdots, f_0^{n_j-1}(z_j(0))\}$$

and an integer $k_0 \geq 1$ such that

$$f_0^{k_0}(c_j(0)) = z_j(0) \text{ but } c_j(0) \notin \mathcal{C}_j(0)$$

for $j = 1, 2$.

By the implicit function theorem, we may follow the cycles $\mathcal{C}_j$ on a small ball $B(0, r)$ centered at the origin. Writting $\mathcal{C}_j(\lambda) := \{z_j(\lambda), \cdots, f_\lambda^{n_j-1}(z_j(\lambda))\}$ the cycles corresponding to the parameter $\lambda \in B(0, r)$, we may define an important tool for studying the parameter space near f_0.

Definition 5.3.1 *The map $\chi : B(0, r) \to \mathbf{C}^2$ defined by*

$$\lambda \mapsto \left(f_\lambda^{k_0}(c_j(\lambda)) - z_j(\lambda)\right)_{j=1,2}$$

is called activity map *near the Misiurewicz parameter f_0.*

As we shall see, thanks to the next result, the activity map will allow to transfer informations from the dynamical space of f_0 to the parameter space.

Theorem 5.3.2 *The activity map χ near a strongly Misiurewicz parameter f_0 is locally invertible.*

This Theorem was proved by Buff and Epstein (see [BE]) in the general setting of Rat_d. In that case one has to assume that f_0 is not a flexible Lattès map (such maps do not exist in degree two). The proof of Buff and Epstein uses quadratic differentials thechniques. We shall prove here a weaker statement which is due to Gauthier ([G]) and is sufficent for the applications we have in mind.

Theorem 5.3.3 *The activity map χ near a strongly Misiurewicz parameter f_0 is locally proper.*

The proof of that result is based on more classical arguments going back to Sullivan (see also [vS] or [A]). The key point relies on the folowing Lemma.

Lemma 5.3.4 *All holomorphic curve contained in M_{iss} consists of flexible Lattès maps.*

Proof. Assume to the contrary that $(f_\lambda)_{\lambda\in D}$ is a holomorphic family parametrized by a one-dimensional disc which is consisting of strongly Misiurewicz parameters and such that the f_λ are not flexible Lattès maps.

Then the Julia set of f_λ coincides with $\mathbf{P}^1$ for all $\lambda \in D$ and, according to a Theorem of Mañé-Sad-Sullivan (see [MSS] Theorem B), there exists a quasiconformal holomorphic motion $\Phi : D \times \mathbf{P}^1 \to \mathbf{P}^1$ which conjugates f_λ to f_0 on $\mathbf{P}^1$. Let us denote by μ^λ the Beltrami form satisfying

$$\tfrac{\partial \Phi_\lambda}{\partial \bar{z}} = \mu^\lambda \tfrac{\partial \Phi_\lambda}{\partial z}.$$

There exists $\lambda_1 \in D\setminus\{0\}$ for which the support of μ^{λ_1} has strictly positive Lebesgue measure. Indeed, if this would not be the case, f_λ would be holomorphically conjugated to f_0 for all $\lambda \in D$. Then the Julia set of f_{λ_1} carries an invariant line field and thus f_{λ_1} is a flexible Lattès map (see [BM] Theorem VII. 22 and [Mc] corollary 3.18). This is a contradiction. □

We may now easily prove Theorem 5.3.3

Proof. Let us first establish that at least one critical point must be activ. Assume to the contrary that both critical points are passiv around f_0. Then, according to Lemma 3.1.17, f_λ is strongly Misiurewicz for all $\lambda \in B(0,r)$ after maybe reducing r. Cutting $B(0,r)$ by a disc D passing through the origin we obtain, by lemma 5.3.4, a disc of flexible Lattès maps. Since this is impossible in Mod_2 (and by assumption in other cases) we have reached a contradiction and proved that at least one critical point, say c_1, is activ at f_0.

The activity of c_1 means that $\chi_1^{-1}(0)$ has codimension one. The conclusion is obtained by repeating the argument on the hypersurface $\chi_1^{-1}(0)$. □

5.3.2 The bifurcation measure and strong-bifurcation loci

We want to establish that the inclusion $Supp\ \mu_{\text{bif}} \subset \overline{S_{\text{hi}}} \cap \overline{M_{\text{iss}}}$ obtained in subsection 5.2.2 is actually an equality. This is the reason for which we shall consider the support of the bifurcation measure as a strong-bifurcation locus. This is essentially a consequence of the following result due to Buff and Epstein [BE].

Theorem 5.3.5 *In the moduli space Mod_d the set of strongly Misiurewicz parameters is contained in the support of the bifurcation measure: $M_{iss} \subset Supp\ \mu_{bif}$.*

Corollary 5.3.6 *In the moduli space Mod_d one has $Supp\ \mu_{bif} = \overline{S_{hi}} = \overline{M_{iss}}$.*

Proof. To simplify the presentation we will work in the degree 3 polynomial family $\left(P_{c,a}\right)_{(c,a)\in\mathbf{C}^2}$. As usual we write λ the parameter (c,a) and $c_1(\lambda)$, $c_2(\lambda)$ the marked critical points, the fact that in this setting $c_1 = 0$ does not play any role here.

Assume that p_0 is a strongly Misiuerewicz polynomial. By definition, there exists an integer k_0 such that $p_0^{k_0}(c_j(0)) =: z_j(0)$ is a repelling periodic point for $j = 1, 2$. To get lighter notations we whall assume that the $z_j(0)$ are fixed repelling points.

We denote by $z_j(\lambda)$ the repellling fixed points which are obtained by holomorphically moving $z_j(0)$ on some neighbourhood of the origin and by $w_j(\lambda)$ the corresponding multipliers. Observe that $|w_j(\lambda)| \geq a > 1$ on a sufficently small neighbourhood of 0.

The activity map χ (see definition 5.3.1) may be written: $\chi = (\chi_1, \chi_2)$ where

$$\chi_j(\lambda) = p_\lambda^{k_0}\left(c_j(\lambda)\right) - z_j(\lambda).$$

We will use here Theorem 5.3.2 and assume that χ is locally invertible at the origin. It is possible adapt the proof for using the weaker transversality statement given by Theorem 5.3.3). For this one uses the fact that the sets obtained by rescaling the ramifiction locus of χ are not charged by the measure μ_{bif} and, thanks to some Besicovitch covering argument, reduces the problem to some estimate similar to those which we will now perform in the invertible case. We refer to te papers [BE] and [G] for details.

Let us denote by $D^2(0, \epsilon)$ the bidisc centered at the origin and of multiradius (ϵ, ϵ) in $\mathbf{C}^2$. For ϵ small enough we may define a sequence of rescaling

$$\delta_n : D^2(0, \epsilon) \to \Omega_n$$

by setting $\delta_n(x) := \chi^{-1}\left(\frac{x_1}{w_1(0)}, \frac{x_2}{w_2(0)}\right)$. To prove the Theorem, it suffices to show that $\mu_{\mathrm{bif}}\left(\Omega_n\right) > 0$ for all n. The crucial point of the proof is revealed by the following computations.

$$2\mu_{\mathrm{bif}}\left(\Omega_n\right) = \int_{\Omega_n} T_{\mathrm{bif}}^2 = \int_{D^2(0,\epsilon)} \delta_n^\star\left(T_1 + T_2\right)^2 \geq \int_{D^2(0,\epsilon)} \delta_n^\star\left(T_1 \wedge T_2\right) =$$
$$= \int_{D^2(0,\epsilon)} \delta_n^\star\left[dd^c g_\lambda(c_1(\lambda)) \wedge dd^c g_\lambda(c_2(\lambda))\right]$$

using the homogeneity property of the Green function

$$g_\lambda\left(c_j(\lambda)\right) = 3^{-(k_0+n)} g_\lambda\left(p_\lambda^{k_0+n}(c_j(\lambda))\right)$$

one thus gets

$$2\cdot 3^{(k_0+n)}\mu_{\text{bif}}\left(\Omega_n\right) \geq \int_{D^2(0,\epsilon)} \delta_n^\star\left[dd^c g_\lambda \circ p_\lambda^{k_0+n}(c_1(\lambda)) \wedge dd^c g_\lambda \circ p_\lambda^{k_0+n}(c_2(\lambda))\right] =$$

$$= \int_{D^2(0,\epsilon)} dd^c g_{\delta_n(x)} \circ p_{\delta_n(x)}^{k_0+n}(c_1(\delta_n(x))) \wedge dd^c g_{\delta_n(x)} \circ p_{\delta_n(x)}^{k_0+n}(c_2(\delta_n(x))).$$

Let us express the quantities $p_{\delta_n(x)}^{k_0+n}(c_j(\delta_n(x))$ by using the activity map χ. By definition we have $p_\lambda^{k_0+n}(c_j(\lambda)) = p_\lambda^n\left(z_j(\lambda)+\chi_j(\lambda)\right)$ and thus

$$p_{\delta_n(x)}^{k_0+n}(c_j(\delta_n(x))) = p_{\delta_n(x)}^n\left(z_j(\delta_n(x)) + \frac{x_j}{w_j(0)^n}\right).$$

To conclude, we momentarily admit the following

Claim: $p_{\delta_n(x)}^n\left(z_j(\delta_n(x)) + \frac{x_j}{w_j(0)^n}\right)$ is uniformly converging to some local biholomorphism $\psi_j : \mathbf{C}_{,0} \to \mathbf{P}^1{}_{,z_j(0)}$.

As the Green function $g_\lambda(z)$ is continuous in (λ, z), the Claim implies that $g_{\delta_n(x)} \circ p_{\delta_n(x)}^{k_0+n}(c_2(\delta_n(x)))$ uniformly converges towards $g_0\left(\psi_0(x_j)\right)$ and our estimate yields

$$\liminf_n \mu_{\text{bif}}\left(\Omega_n\right) \geq \int_{D^2(0,\epsilon)} dd^c g_0\left(\psi_1(x_1)\right) \wedge dd^c g_0\left(\psi_2(x_2)\right) =$$

$$= \left(\int_{\psi_1(D(0,\epsilon))} dd^c g_0\right)\left(\int_{\psi_2(D(0,\epsilon))} dd^c g_0\right) > 0$$

where the positivity of the last term follows from the fact that the repelling fixed points $z_j(0) = \psi_j(0)$ belong to the Julia set of p_0.

It remains to justify the Claim. Let us write $w_j(\lambda)$ on the form $w_j(0)\left(1+\epsilon_j(\lambda)\right)$. As $\|\delta_n(x)\| \leq C\frac{1}{a^n}$ one sees that $\left(1+\epsilon_j(\delta_n(x))\right)^n$ is uniformly converging to 1. Then

$$p_{\delta_n(x)}^n\left(z_j(\delta_n(x)) + \frac{x_j}{w_j(0)^n}\right) = p_{\delta_n(x)}^n\left[z_j(\delta_n(x)) + \frac{x_j}{w_j(\delta_n(x))^n}\left(1+\epsilon_j(\delta_n(x))\right)^n\right]$$

behaves like $p_{\delta_n(x)}^n\left[z_j(\delta_n(x)) + \frac{x_j}{w_j(\delta_n(x))^n}\right]$.

Now let us linearize p_λ near the repelling fixed points $z_j(\lambda)$. The linearization holomorphically depends on the parameter λ and one gets local biholomorphisms $\psi_{j,\lambda}$ such that $\psi_{j,\lambda}(0) = z_j(\lambda)$ and

$$p_\lambda \circ \psi_{j,\lambda}(z) = \psi_{j,\lambda}(w(\lambda)z) \text{ on } B(0, \tfrac{\epsilon}{|w(\lambda)|}).$$

Then the local biholomorphism ψ_j of the Claim is simply $\psi_{j,0}$. □

Remark 5.3.7 *By Theorem 5.1.4, $(T_1+T_2)^2 = 2T_1 \wedge T_2$ and therefore the estimate in the proof of Theorem 5.3.5 becomes an equality:*

$$2 \cdot 3^{(k_0+n)} \mu_{bif}(\Omega_n) = \int_{D^2(0,\epsilon)} dd^c g_{\delta_n(x)} \circ p_{\delta_n(x)}^{k_0+n}(c_1(\delta_n(x))) \wedge dd^c g_{\delta_n(x)} \circ p_{\delta_n(x)}^{k_0+n}(c_2(\delta_n(x))).$$

This allows to estimate the pointwise Hausdorff dimension of μ_{bif} at (strongly) Misiurewicz parameters.

5.3.3 Hausdorff dimension estimates

Here, we will simply mention some further results obtained by T. Gauthier in his thesis (see [G]).

Using the transversality map χ associated to Misiurewicz parameters (see Theorem 5.3.3) it is possible to construct a "transfer map" which copies some pieces of the dynamical plane into the parameter space. These tecniques have also been used by Shishikura for proving that the boundary of the Mandelbrot set is of Hausdorff dimension 2 and by Tan Lei for proving that the bifurcation locus in any polynomial families has also maximal Hausdorff dimension.

This allows to relate the Hausdorff dimension of the strong -bifurcation locus with the hyperbolic dimension of Misiurewicz parameters. Using Misiurewicz parameters whose critical orbits are captured by hyperbolic sets with high Hausdorff dimension then gives the following

Theorem 5.3.8 *The strong-bifurcation locus has full Hausdorff dimension.*

Combining this with Theorem 5.2.6 then yields the

Corollary 5.3.9 *Denote by $p(f)$ (resp. $s(f)$, $c(f)$) the number of distinct parabolic (resp. Siegel, Cremer) cycles of $f \in Rat_d$. Let p, s and c be three integers such that $p+s+c = 2d-2$. Then the set*

$$\overline{\{f \in Rat_d \ / \ p(f) = p, s(f) = s \text{ and } c(f) = c\}}$$

is homogeneous and has maximal Hausdorff dimension $2(2d-2)$.

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