Mean field games: additional notes

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1 One shot games: a simple example of the passage to the limit $N \to \infty$, from [4]

Let Q be a compact subset of $\mathbb{R}^d$.
Let V be an operator which continuously maps the set of probability measures on Q (endowed with the weak * topology) to a bounded subset of $Lip(Q)$, the Lipschitz functions on Q.
Consider the following Nash equilibrium with N players: the player indexed by i is to choose a probability measure $\bar\pi_i^N$ on Q such that, for all probability measure μ

$$J_i^N(\bar\pi_1^N, \dots, \bar\pi_{i-1}^N, \mu, \bar\pi_{i+1}^N, \dots, \bar\pi_N^N) \geq J_i^N(\bar\pi_1^N, \dots, \bar\pi_N^N),$$

where

$$J_i^N(\pi_1^N, \dots, \pi_N^N) = \int_{Q^N} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](x_i)\prod_{j=1}^N d\pi_j^N(x_j)$$

Nash theorem says that there exists a *symmetric* Nash equilibrium, i.e. a probability measure $\bar\pi^N$ such that $\bar\pi_i^N = \bar\pi^N$ $i = 1, \dots, N$ defines a Nash equilibrium.

Passage to the limit as $N \to \infty$ We have that, for any probability measure μ on Q

$$\int_{Q^N} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](x_i)\prod_{j=1}^N d\bar\pi^N(x_j) \leq \int_{Q^N} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](x_i)\prod_{j\neq i}^N d\bar\pi^N(x_j)d\mu(x_i). \tag{1}$$

Assume that for a subsequence still called $(\bar\pi^N)$, $\bar\pi^N \to \bar m$ weak $*$.
Suppose first that there exists a constant C such that

$$\|V[m] - V[m']\|_{C(Q)} \leq C d_1^{MK}(m, m'),$$

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for all probability measures m and m'.
We have that

$$\begin{aligned}
&\left\| \int_{Q^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]\prod_{j\neq i} d\bar\pi^N(x_j) - \int_{Q^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]\prod_{j\neq i} d\bar m(x_j)\right\|_{L^\infty(Q)}\\
&\le \sum_{k\neq i}\left\| \int_{Q^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]\prod_{j\neq i, j<k} d\bar\pi^N(x_j)\prod_{j\neq i, j>k} d\bar m(x_j)(d\bar\pi^N(x_k)-d\bar m(x_k))\right\|_{L^\infty(Q)}\\
&= \sum_{k\neq i}\left\| \int_{Q^{N-1}} \left(V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]-V\left[\frac{1}{N-2}\sum_{j\neq i, j\neq k}\delta_{x_j}\right]\right)\prod_{j\neq i, j<k} d\bar\pi^N(x_j)\prod_{j\neq i, j>k} d\bar m(x_j)(d\bar\pi^N(x_k)-d\bar m(x_k))\right|\\
&\le C\sum_{k\neq i}\left\| V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]-V\left[\frac{1}{N-2}\sum_{j\neq i, j\neq k}\delta_{x_j}\right]\right\|_{L^\infty(Q)} d_1^{MK}(\bar\pi^N,\bar m)
\end{aligned}$$

But

$$\left\| V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]-V\left[\frac{1}{N-2}\sum_{j\neq i, j\neq k}\delta_{x_j}\right]\right\|_{L^\infty(Q)} \le \frac{\mathrm{diam(Q)}}{N-1}$$

. Thus

$$\lim_{N\to\infty}\max_{1\le i\le N}\left\| \int_{Q^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]\left(\prod_{j\neq i} d\bar\pi^N(x_j) - \prod_{j\neq i} d\bar m(x_j)\right)\right\|_{L^\infty(Q)} = 0 \tag{2}$$

Note that (2) holds with the more general assumptions on V made above, because we can approximate V by a sequence of operators V_n such that $\|V_n[m]-V_n[m']\|_{C(Q)} \le C_n d_1^{MK}(m,m')$: it is possible, since V maps the set of the probability measures on Q to a bounded subset of $Lip(Q)$, i.e. a compact of $C(Q)$.
Finally, a simple version of Hewitt and Savage theorem yields that

$$\lim_{N\to\infty}\max_{1\le i\le N}\left| \int_{Q^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](x)\prod_{j\neq i} d\bar m(x_j) - V[\bar m](x)\right|$$

holds pointwise. Since V maps the set of the probability measures on Q to a bounded subset of $Lip(Q)$, we also have

$$\lim_{N\to\infty}\max_{1\le i\le N}\left\| \int_{Q^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right]\prod_{j\neq i} d\bar m(x_j) - V[\bar m]\right\|_{L^\infty(Q)} = 0.$$

Therefore, we can pass to the limit in (1) and see that for any probability measure μ on Q,

$$\int_Q V[\bar m](x)d\bar m(x) \le \int_Q V[\bar m](x)d\mu(x).$$

2 Stochastic games with N players: Nash equilibria, from [11, 13]

Theorem 1 *Let $\mathbb{T}$ be the unit torus of $\mathbb{R}^d$. Let $L:\mathbb{T}\times\mathbb{R}^d$ be a smooth function such that $\lim_{|\alpha|\to\infty}\min_{x\in\mathbb{T}}\frac{L(x,\alpha)}{|\alpha|}=+\infty$. Call H the Hamiltonian*

$$H(x,p) = \sup_{\alpha\in\mathbb{R}^d} p\cdot\alpha - L(x,\alpha).$$

We assume that H is C^1 and that there exists a constant $\theta \in (0,1)$ such that for p large enough,

$$\inf_x \left(\frac{\partial H}{\partial x}(x,p) \cdot p + \frac{\theta}{d\nu} H^2(x,p) \right) > 0. \tag{3}$$

*Let V be an operator which continuously maps the set of probability measures on $\mathbb{T}$ (endowed with the weak * topology) to a bounded subset of $Lip(\mathbb{T})$, the Lipschitz functions on $\mathbb{T}$, and for example maps continously $C^{k,\alpha}(\mathbb{T})$ to $C^{k+1,\alpha}(\mathbb{T})$, for all $k \in \mathbb{N}$ and $0 \le \alpha < 1$.*

Recall that the trajectory of the player indexed by i is found by solving the SDE

$$dX_i^N(t) = \sigma dW_i(t) - \alpha_i^N(t)dt, \quad X_i^N(0) = x_i(0),$$

under $\mathbb{P}_\alpha$, where $(W_i(t))_{1\le i \le N}$ are independent Brownian motions in $\mathbb{R}^d$ and the control α_i^N is adapted to W_i.
The cost of the player indexed by i is

$$J_i^N(\alpha_1^N, \dots, \alpha_N^N) = \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}_\alpha \left(\int_0^T L(X_i^N(t), \alpha_i^N(t)dt + \int_0^T V\left[\frac{1}{N-1} \sum_{j\neq i} \delta_{X_j^N(t)} \right](X_i^N(t))dt \right).$$

The system of $2N$ PDEs in $\mathbb{T}$

$$-\nu\Delta u_i^N(x) + H(x, \nabla u_i^N) + \lambda_i^N = \int_{\mathbb{T}^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i} \delta_{x_j}\right](x) \prod_{j\neq i} m_j^N(x_j)dx_j, \tag{4}$$

$$-\nu\Delta m_i^N - \operatorname{div}\left(m_i^N \frac{\partial H}{\partial p}(x, \nabla u_i^N)\right) = 0, \tag{5}$$

$$\int_{\mathbb{T}} m_i^N dx = 1, \quad m_i^N \ge 0, \quad \int_{\mathbb{T}} u_i^N dx = 0 \tag{6}$$

has a solution such that $\forall 1 \le i \le N$, $(u_i^N, m_i^N, \lambda_i^N) \in C^2(\mathbb{T}) \times W^{1,q}(\mathbb{T}) \times \mathbb{R}$, for all $1 \le q < \infty$.

The feedback law

$$\bar{\alpha}_i^N(t) = \frac{\partial H}{\partial p}\left(X_i^N(t), \nabla u_i^N(X_i^N(t))\right), \quad i = 1, \dots, N \tag{7}$$

defines a Nash equilibrium. Moreover,

$$\lambda_i^N = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}_{\bar\alpha} \left(\int_0^T L(X_i^N(t), \bar\alpha_i^N(t)dt + \int_0^T V\left[\frac{1}{N-1} \sum_{j\neq i} \delta_{X_j^N(t)} \right](X_i^N(t))dt \right) \tag{8}$$

Proof. A) The existence of a solution for (4)- (6), with the regularity mentioned in the statement of the theorem is obtained by a Leray-Schauder fixed point method. The crucial step is an a priori estimate on $\|\nabla u_i^N\|_{L^\infty(\mathbb{T})}$, which is obtained by the classical Bernstein method. This is where the technical assumption (3) is used.

We start with the HJB equation (4). We first get a priori estimates on λ_i^N by looking at (4) at the extrema of u_i^N.
Take the gradient of (4): calling $w_i^N = \nabla u_i^N$, we have

$$-\nu\Delta w_i^N + (\nabla_x H)(x, w_i^N) + \sum_j \frac{\partial}{\partial p_j} H(x, w_i^N) \frac{\partial w_i^N}{\partial x_j} = G, \quad \text{in } \mathbb{T}, \tag{9}$$

where G is a function in L^∞. Taking the product of (9) with w_i^N, and calling $\psi_i^N = |w_i^N|^2$, we get that

$$\frac{1}{2}(-\nu\Delta\psi_i^N + 2\nu D^2u_i^N \cdot D^2u_i^N) + \frac{\partial}{\partial x}H(x, w_i^N)\cdot w_i^N + \frac{1}{2}\frac{\partial}{\partial p}H(x, w_i^N)\cdot\nabla\psi_i^N = G\cdot w_i^N. \tag{10}$$

But

$$D^2u_i^N \cdot D^2u_i^N \geq \sum_{k=1}^d |\partial_{kk}u_i^N|^2 \geq \frac{1}{d}|\Delta u_i^N|^2.$$

Hence,

$$\nu D^2u_i^N \cdot D^2u_i^N \geq \frac{1}{d\nu}|\lambda_i^N + H(\cdot, w_i^N) - F_i^N|^2.$$

where F_i^N is bounded by a constant.
From the bounds on F_i^N and on λ_i^N, we see that for any $0 < \eta < 1$, there exists a constant C_η such that

$$\nu D^2u_i^N \cdot D^2u_i^N \geq \frac{1-\eta}{d\nu}H^2(\cdot, w_i^N) - C_\eta.$$

Therefore, (10) implies that for a constant C,

$$-\nu\Delta\psi_i^N + \frac{\partial}{\partial p}H(x, w_i^N)\cdot\nabla\psi_i^N + \frac{2(1-\eta)}{d\nu}H^2(\cdot, w_i^N) + 2\frac{\partial}{\partial x}H(x, w_i^N)\cdot w_i^N \leq C(1+|w_i^N|).$$

We choose η such that $1-\eta > \theta$ where θ is the constant appearing in (3). At the maximum of ψ_i^N, we have

$$\frac{2(1-\eta-\theta)}{d\nu}H^2(x, w_i^N) \leq C(1+|w_i^N|).$$

On the other hand $H(x, w_i^N) \geq c(|w_i^N| - 1)$, which yields the desired estimate.

B) Verification (first step) Consider the feedback law (7) and let us check (8): By Ito formula,

$$\begin{aligned} du_i^N(X_i^N(t)) = &\left(-\frac{\partial H}{\partial p}(X_i^N(t), \nabla u_i^N(X_i^N(t)))\cdot\nabla u_i^N(X_i^N(t)) + \nu\Delta u_i^N(X_i^N(t))\right)dt \\ &+ \sigma\nabla u_i^N(X_i^N(t))\cdot dW_i(t). \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{T}\mathbb{E}_{\bar\alpha}(u_i^N(X_i^N(t)) - u_i^N(X_i^N(0))) \\ =&\frac{1}{T}\mathbb{E}_{\bar\alpha}\left(\int_0^T\left(-\frac{\partial H}{\partial p}(X_i^N(t), \nabla u_i^N(X_i^N(t)))\cdot\nabla u_i^N(X_i^N(t)) + \nu\Delta u_i^N(X_i^N(t))\right)dt\right) \\ =&\lambda_i^N + \frac{1}{T}\mathbb{E}_{\bar\alpha}\left(\begin{array}{l}\displaystyle\int_0^T\left(-\frac{\partial H}{\partial p}(X_i^N(t), \nabla u_i^N(X_i^N(t)))\cdot\nabla u_i^N(X_i^N(t)) + H(X_i^N(t), \nabla u_i^N(X_i^N(t)))\right)dt \\ \displaystyle-\int_0^T\int_{\mathbb{T}^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](X_i^N(t))\prod_{j\neq i}m_j^N(x_j)dx_j dt\end{array}\right) \\ =&\lambda_i^N - \frac{1}{T}\mathbb{E}_{\bar\alpha}\left(\int_0^T L(X_i^N(t), \bar\alpha_i^N(t))dt + \int_0^T\int_{\mathbb{T}^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](X_i^N(t))\prod_{j\neq i}m_j^N(x_j)dx_j dt\right) \end{aligned}$$

But by ergodicity,

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}_{\bar\alpha}\left(\int_0^T\int_{\mathbb{T}^{N-1}}V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](X_i^N(t))\prod_{j\neq i}m_j^N(x_j)dx_jdt-\int_0^T V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_j^N}\right](X_i^N(t))dt\right)=0$$

Thus passing to the limit as $T\to\infty$,

$$\begin{aligned}\lambda_i^N &= \lim_{T\to\infty}\frac{1}{T}\mathbb{E}_{\bar\alpha}\left(\int_0^T L(X_i^N(t),\bar\alpha_i^N(t))dt+\int_0^T V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_j^N}\right](X_i^N(t))dt\right)\\ &= J_i^N(\bar\alpha_1^N,\dots,\bar\alpha_N^N).\end{aligned}$$

C) Verification (second step) Let us check that the feedback law (7) defines a Nash equilibrium. For $\alpha_j(x)=\bar\alpha_j(x)=\frac{\partial H}{\partial p}(x,\nabla u_j^N)$, $j\neq i$, and α_i an admissible control, we have that

$$\begin{aligned}&\frac{1}{T}\mathbb{E}_\alpha(u_i^N(X_i^N(t))-u_i^N(X_i^N(0)))\\ =&\frac{1}{T}\mathbb{E}_\alpha\left(\int_0^T\left(-\alpha_i(t)\cdot\nabla u_i^N(X_i^N(t))+\nu\Delta u_i^N(X_i^N(t))\right)dt\right)\\ =&\lambda_i^N+\frac{1}{T}\mathbb{E}_\alpha\left(\begin{array}{l}\int_0^T\left(-\alpha_i(t)\cdot\nabla u_i^N(X_i^N(t))+H(X_i^N(t),\nabla u_i^N(X_i^N(t)))\right)dt\\ -\int_0^T\int_{\mathbb{T}^{N-1}}V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](X_i^N(t))\prod_{j\neq i}m_j^N(x_j)dx_jdt\end{array}\right)\\ \geq&\lambda_i^N-\frac{1}{T}\mathbb{E}_\alpha\left(\int_0^T L(X_i^N(t),\alpha_i^N(t))dt+\int_0^T\int_{\mathbb{T}^{N-1}}V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](X_i^N(t))\prod_{j\neq i}m_j^N(x_j)dx_jdt\right)\end{aligned}$$

From ergodicity, we get that

$$J_i^N(\bar\alpha_1^N,\dots,\bar\alpha_{i-1}^N,\alpha_i,\bar\alpha_{i+1}^N,\dots,\bar\alpha_N^N)\geq\lambda_i^N=J_i^N(\bar\alpha_1^N,\dots,\bar\alpha_N^N).$$

■

3 An example of a system of PDEs arising in MFG with finite horizon : existence results.

from the lecture of P-L. Lions in Collège de France, 09-01-2009

Consider the system of PDEs arising for finite horizon MFG:

$$\frac{\partial u}{\partial t}-\nu\Delta u+H(x,\nabla u) = V[m],\quad \text{in } \mathbb{T}\times(0,T), \tag{11}$$

$$\frac{\partial m}{\partial t}+\nu\Delta m+\operatorname{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) = 0,\quad \text{in } \mathbb{T}\times(0,T), \tag{12}$$

$$\int_{\mathbb{T}}m(x,t)dx=1,\quad m>0, \tag{13}$$

$$u(t=0)=u_0,\quad m(t=T) = m_0\,. \tag{14}$$

Hereafter, the following assumptions will be made:

Assumptions

1. $\nu > 0$
2. H is smooth and convex w.r.t. p
3. u_0 and m_0 are smooth functions, m_0 is a probability density on $\mathbb{T}$ and $m_0 > 0$

3.1 A priori estimates in some simple cases

3.1.1 The case when $\frac{\partial H}{\partial p}$ is bounded

In addition to the assumptions above, we assume furthermore that there exists a constant C such that $|\frac{\partial H}{\partial p}(x,p)| \le C$ for all x and p, and that

- either $V[m](x) = F(m(x))$ where F is a smooth function defined on $\mathbb{R}_+$
- or $V[m]$ is an operator which is continuous from $C^{k,\alpha}$ to $C^{k,\alpha}$, for all k and $0 \le \alpha < 1$.

First step From the assumptions on H, we see that $\frac{\partial H}{\partial p}(x, \nabla u) = A(x,t)$ where $A \in L^\infty$. We consider the Fokker Planck equation (12). It reads

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}(Am) = 0, \quad \text{in } \mathbb{T} \times (0,T), \tag{15}$$

Standard arguments based on Gärding's inequality ensure that there exists a unique weak solution in $L^2((0,T), H^1(\mathbb{T}))$ to (12) such that $m|_{t=T} = m_0$. The weak maximum principle yields that there exists a positive constant $\bar{m}$ which depends on m_0, T and $\|A\|_{L^\infty}$ such that

$$0 \le m \le \bar{m}, \text{a.e. in } \mathbb{T} \times [0,T].$$

Therefore $\|m\|_{L^\infty}$ is bounded by a constant which does not depend on u. This yields that

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}(B) = 0, \quad \text{in } \mathbb{T} \times (0,T), \tag{16}$$

where $B \in L^\infty$. Standard results on the heat equation tell us that $m \in L^p((0,T), W^{1,p}(\mathbb{T}))$ for all $1 \le p < \infty$, that $m \in C^{\alpha,\alpha/2}(\bar{\mathbb{T}} \times [0,T])$, for all $0 \le \alpha < 1$, and that $\|m\|_{L^p((0,T),W^{1,p}(\mathbb{T}))} + \|m\|_{C^{\alpha,\alpha/2}(\bar{\mathbb{T}}\times[0,T])}$ is bounded by a constant which does not depend on u.

Second step We see that $H(x,p) = H(x,0) + \left(\int_0^1 \frac{\partial H}{\partial p}(x,tp)dt\right) \cdot p$, so $H(x,p) = h(x) + g(x,p)\cdot p$, where h and g are bounded uniformly w.r.t. x and p, h is C^1. Hence, the HJB equation reads

$$\frac{\partial u}{\partial t} - \nu \Delta u + g \cdot \nabla u = F(m) - h \quad \text{in } \mathbb{T} \times (0,T), \tag{17}$$

From the information obtained on m, regularity results for (17) yield that $u \in C^{\theta,\theta/2}(\bar{\mathbb{T}} \times [0,T])$ for all $\theta \in [1,2)$ and that $\|u\|_{C^{\theta,\theta/2}(\bar{\mathbb{T}}\times[0,T])}$ is bounded by a constant.

Third step In the Fokker-Planck equation (15), $A \in C^{\eta,\eta}$ for all $0 \le \eta < 1$. This implies that $m \in C^{\theta,\theta/2}(\bar{\mathbb{T}} \times [0,T])$ for all $\theta \in [1,2)$ and that $\|m\|_{C^{\theta,\theta/2}(\bar{\mathbb{T}}\times[0,T])}$ is bounded by a constant.

Fourth step Iterating the argument above, we see that we can keep gaining regularity alternatively on u and m. Thus we get that u and m are smooth, and uniform bounds on $\|u\|_{C^p} + \|m\|_{C^p}$ for all p.

3.1.2 The case when V is a smoothing operator

We now assume that V maps continuously the set of probability measures on $\mathbb{T}$ to a bounded subset of $Lip(\mathbb{T})$, the Lipschitz functions on $\mathbb{T}$, and for example maps continously $C^{k,\alpha}(\mathbb{T})$ to $C^{k+1,\alpha}(\mathbb{T})$, for all $k \in \mathbb{N}$ and $0 \le \alpha < 1$. We also assume that $\frac{\partial H}{\partial x}(x,p) \cdot p \ge -C(1+|p|^2)$

First step We start with the HJB equation (11) and we apply a Bernstein type argument. Take the gradient of (11): calling $w = \nabla u$, we have

$$\frac{\partial w}{\partial t} - \nu \Delta w + (\nabla x H)(x,w) + \sum_i \frac{\partial}{\partial p_i} H(x,w) \nabla w_i = G, \quad \text{in } \mathbb{T} \times (0,T), \tag{18}$$

where G is a function in L^∞. Taking the product of (18) with w, and calling $\psi = |w|^2$, we get that

$$\frac{1}{2}\left(\frac{\partial \psi}{\partial t} - \nu \Delta \psi + 2\nu D^2 u \cdot D^2 u\right) + (\nabla x H)(x,w) \cdot w + \frac{1}{2}\frac{\partial}{\partial p} H(x,w) \cdot \nabla \psi = G \cdot w, \quad \text{in } \mathbb{T} \times (0,T), \tag{19}$$

This implies that

$$\frac{1}{2}\left(\frac{\partial \psi}{\partial t} - \nu \Delta \psi + \frac{\partial}{\partial p} H(x,w) \cdot \nabla \psi\right) \le C(1+\psi).$$

A standard comparison argument with a solution of the ODE $\phi' = 2C(1+\phi)$ yields a uniform bound on ψ. Hence $\|\nabla u\|_{L^\infty}$ is bounded by a constant.

Second step Since $\|\nabla u\|_{L^\infty}$ is bounded by a constant, we can redo the four steps of § 3.1.1, and obtain C^p estimates for u and m as in § 3.1.1, for all $p > 0$.

3.2 Existence

3.2.1 case 1

We assume that

- there exists a constant C such that $|\frac{\partial H}{\partial p}(x,p)| \le C$ for all x and p
- V is a smoothing operator which maps continuously $C^{k,\alpha}(\mathbb{T})$ to a bounded set of $C^{k+1,\alpha}(\mathbb{T})$, for all $k \ge 0$ and $\alpha \in [0,1)$.

We are going to apply Shauder fixed point theorem to a mapping χ defined in $C^{\alpha,\alpha/2}(\mathbb{T} \times [0,T])$ for some $\alpha \in (0,1)$: consider first the mapping $\Theta : C^{\alpha,\alpha/2}(\mathbb{T} \times [0,T]) \to C(\mathbb{T} \times [0,T])$, $m \mapsto u$ where u solves (11) and $u|_{t=0} = u_0$. Existence and uniqueness for this problem are well known. Moreover, from the estimates above, for every $0 < \alpha < 1$, $\|u\|_{C^{2+\alpha/2,1+\alpha/2}(\mathbb{T}\times[0,T])}$ is bounded by a constant independent of m. Then consider the mapping ζ: $C^{2+\alpha/2,1+\alpha/2}(\mathbb{T} \times [0,T]) \to C(\mathbb{T} \times [0,T])$ $u \mapsto \tilde{m}$ where $\tilde{m}$ solves the Fokker-Planck equation (12) and $m|_{t=T} = m_0$. Well known results on Fokker-Planck equation tell us that m belong to $C^{2,1}(\mathbb{T} \times [0,T])$, with uniform bounds in this space.
Therefore, we can apply Leray-Schauder fixed point theorem to $\chi = \zeta \circ \theta$. There exists a strong solution of the MFG system. From the a priori estimates found in § 3.1, this solution is smooth.

3.2.2 case 2

We assume that

- V maps probability measures on $\mathbb{T}$ to a bounded subset of $Lip(\mathbb{T})$
- V continuously maps $C^{k,\alpha}(\mathbb{T})$ to $C^{k+1,\alpha}(\mathbb{T})$, for all $k \in \mathbb{N}$ and $0 \le \alpha < 1$.
- $\frac{\partial H}{\partial x}(x,p) \cdot p \ge -C(1+|p|^2)$.

We are going to approximate the problem in order to use the results proved for case 1.
Let δ be a small parameter. First, we approximate the Hamiltonian H by a H_δ such that

- H_δ is smooth, convex w.r.t. p and such that $\frac{\partial H_\delta}{\partial p}$ is bounded
- $\forall x, p$, $\lim_{\delta \to 0} H_\delta(x,p) = H(x,p)$

Let β_δ be a smooth, odd, and increasing function from $\mathbb{R}$ to $[-2/\delta, 2/\delta]$ coinciding with $x \mapsto x$ in $[-1/\delta, 1/\delta]$ and taking the value $2/\delta$ for $x \ge 3/\delta$. The operator $V_\delta[m](x) = \beta_\delta(V[m](x))$ has the properties required in case 1.
Then if we consider first the equation

$$\frac{\partial u}{\partial t} - \nu \Delta u + H_\delta(x, \nabla u) = V_\delta[m] \tag{20}$$

instead of (11), we are in Case 1 and we know that the MFG system (20), (12)- (14) has a solution (u_δ, m_δ). Moreover, looking carefully at the Bernstein estimate in § 3.1.2, we see that u_δ and m_δ are bounded uniformly w.r.t. δ in every $C^k(\mathbb{T}\times[0,T])$. Hence, up to a subsequence, we can pass to the limit as δ tend to $+\infty$. There exists a smooth solution of the MFG system.

3.2.3 case 3

We assume that

- $\frac{\partial H}{\partial p}$ is bounded,
- $V[m](x) = F(m(x))$ where F is a smooth function defined on $\mathbb{R}_+$

Here again, we are going to modify the operators in order to fall into case 1. Let ρ_ϵ be a non negative mollifier and consider first the equation

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = \beta_\epsilon(F(m \star \rho_\epsilon)), \quad \text{in } \mathbb{T} \times (0,T) \tag{21}$$

instead of (11). The MFG system (20), (12)- (14) has a solution (u_ϵ, m_ϵ). Moreover, since $\frac{\partial H}{\partial p}$ is bounded, the a priori estimates in § 3.1.1 tell us that (u_ϵ, m_ϵ) are bounded all C^k uniformly w.r.t. ϵ. We can extract subsequences which converge in say C^2 and pass to the limit. We obtain the existence of a smooth solution of the MFG system.

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