Lecture No. 5

- **Problems under considerations:**

(S)
$$\begin{cases} H(x, Du) = 0 & \text{in } \Omega, \\ \gamma \cdot Du = g & \text{on } \partial\Omega, \end{cases}$$

(E)
$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \Omega \times (0, \infty), \\ \gamma \cdot Du = g & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

(∗)
$$\begin{cases} H(x, Du) = c & \text{in } \Omega, \\ \gamma \cdot Du = g & \text{on } \partial\Omega, \end{cases}$$

5 Optimal controls, extremal curves

Theorem 3.1

Let u be a solution of (E) and let $(x,t) \in Q := \bar{\Omega} \times (0,\infty)$. There exists a $(\eta, v, l) \in \mathrm{SP}(x)$ such that

$$u(x,t) = \int_0^t [L(\eta(s), -v(s)) + l(s) g(\eta(s))] \, \mathrm{d}s + u_0(\eta(t)).$$

- $\forall A \geq 0$, $\exists C_A > 0$ **such that**

$$L(x,\xi) \geq A|\xi| - C_A \quad \forall (x,\xi) \in \bar{\Omega} \times \mathbb{R}^N.$$

$$\because \qquad L(x,\xi) \geq \xi \cdot (A\xi/|\xi|) - H(x, A\xi/|\xi|)$$
$$\geq A|\xi| - \max_{\bar{\Omega} \times B_A(0)} H.$$

- **Note:** $\forall (\eta, v, l) \in \mathrm{SP}(x)$,

$$L(\eta(s), -v(s)) + l(s) g(\eta(s)) \geq L(\ldots) - Cl(s)$$
$$\geq L(\ldots) - Cv(s) \geq (A - C)|v(s)| - C_A.$$

- Let

$$\begin{cases} (\eta_k, v_k, l_k) \in \mathrm{SP}(x), \\ \displaystyle\int_0^t [L(\eta_k, -v_k) + l_k g(\eta_k)]\,\mathrm{d}s \le M < \infty \\ \text{(a minimizing sequence, for instance.)} \end{cases}$$

- $\forall A \ge 0$, $\exists C_A > 0$ such that

$$L(\eta_k, -v_k) + l_k g(\eta_k) \ge A|v_k| - C_A.$$

- Let $E \subset [0, t]$. Then

$$\begin{aligned} &\int_E [L(\eta_k, -v_k) + l_k g(\eta_k)]\,\mathrm{d}s \\ &\le \int_E [L(\eta_k, -v_k) + l_k g(\eta_k) + C_0]\,\mathrm{d}s \\ &\le \int_0^t [L(\eta_k, -v_k) + l_k g(\eta_k) + C_0]\,\mathrm{d}s \\ &\le M + C_0 t. \end{aligned}$$

$$\therefore \qquad M + C_0 t \geq \int_E [L(\eta_k, -v_k) + l_k g(\eta_k)] \,\mathrm{d}s$$
$$\geq \int_E (A|v_k| - C_A) \,\mathrm{d}s.$$
$$\therefore \qquad \int_E |v_k| \,\mathrm{d}s \leq A^{-1}(M + C_0 t) + A^{-1} C_A |E|.$$

- $\{v_k\}$ is uniformly integrable. Since $|\dot\eta_k| \vee l_k \leq C|v_k|$ a.e., $\{(\dot\eta_k, l_k)\}$ is also unif. integrable. Thus, as $k \to \infty$ along a subsequence,

$$(\dot\eta_k, v_k, l_k) \to (\zeta, v, l) \quad \text{weakly in } L^1([0,t], \mathbb{R}^{2N+1})$$

(Dunford-Pettis theorem), and

$$\eta_k(s) \to \eta(s) := x + \int_0^s \zeta(\tau)\,\mathrm{d}\tau \quad \text{uniformly on } [0,\, t].$$

- Using the convexity of L,

$$\int_0^t [L(\eta, -v) + lg(\eta)] \,\mathrm{d}s \leq \liminf_{k\to\infty} \int_0^t [L(\eta_k, -v_k) + l_k g(\eta_k)] \,\mathrm{d}s.$$

Theorem 3.2

Let u be a solution of (S) and $x \in \bar{\Omega}$. There exists a $(\eta, v, l) \in \mathrm{SP}(x)$ such that for all $t > 0$,

$$u(x) = \int_0^t [L(\eta(s), -v(s)) + l(s)g(\eta(s))] \, \mathrm{d}s + u(\eta(t)).$$

- **u is a solution of**

$$(\mathrm{E}^{\#}) \qquad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \Omega \times (0, \infty), \\ \gamma \cdot Du = g & \text{on } \partial\Omega \times (0, \infty), \end{cases}$$

- By Theorem 3.1, there exists a sequence $(\eta_1, v_1, l_1) \in \mathrm{SP}(x), (\eta_2, v_2, l_2) \in \mathrm{SP}(\eta_1(1)), \ldots$ such that for all k,

$$u(\eta_k(0)) - u(\eta_k(1)) = \int_0^1 [L(\eta_k, -v_k) + l_k g(\eta_k)]\, \mathrm{d}s.$$

- Then set

$$\begin{aligned}(\eta, v, l)(s) &= (\eta_1, v_1, l_1)(s) && 0 \le s < 1,\\ &= (\eta_2, v_2, l_2)(s-1) && 1 \le s < 2,\\ &= (\eta_3, v_3, l_3)(s-2) && 2 \le s < 3,\\ &\ \ \vdots\end{aligned}$$

6 An observation

- Let $u \in C^1(\bar\Omega)$ be a classical solution of (S).
- Assume that $L(x,\cdot) \in C^1(\mathbb{R}^N)$ for all x.
- Fix $x \in \bar\Omega$ and let $(\eta, v, l) \in \mathrm{SP}(x)$ be an extremal curve,

so that

$$u(\eta(0)) - u(\eta(t)) = \int_0^t [L(\eta, -v) + lg(\eta)]\,\mathrm{d}s \quad \forall t > 0.$$

Differentiate this and set $p(t) := Du(\eta(t))$, to get

$$-p \cdot \dot\eta = L(\eta, -v) + lg(\eta).$$

We have:

$$\begin{cases} H(\eta, p) = 0, \\ l(\gamma \cdot p - g) = 0. \end{cases}$$

Hence

$$-v \cdot p = -(\dot\eta + l\gamma(\eta)) \cdot p = L(\eta, -v) = H(\eta, p) + L(\eta, -v).$$

Thus, by the convex duality,

$$\begin{cases} D_p H(\eta, p) = -v \quad (\implies v \in L^\infty), \\ D_\xi L(\eta, -v) = p. \end{cases}$$

- Let $0 < \delta \ll 1$. Observe that

$$\begin{aligned} L(\eta, -(1+\delta)v) &= L(\eta, -v) - \delta v \cdot D_\xi L(\eta, -v) + o(\delta) \\ &= L(\ldots) - \delta v \cdot p + o(\delta) \\ &= L(\ldots) + \delta L(\ldots) + o(\delta) \\ &= (1+\delta) L(\eta, -v) + o(\delta). \end{aligned}$$

7 Large time behavior of solutions of HJ equations

- Assume that $c = 0$. (E) has a bounded solution u.

$u \in \mathrm{BUC}(Q)$.

- We have the variational formula for u:

$$u(x,t) = \inf \int_0^t \big(L(\eta(s), -v(s)) + l(s)g(\eta(s))\big)ds + u_0(\eta(t)),$$

where the infimum is taken over all $(\eta, v, l) \in \mathrm{SP}(x)$.

- Set

$$u_\infty(x) = \liminf_{t\to\infty} u(x,t) = \liminf_{t\to\infty,\, y\to x} u(y,t).$$

This function u_∞ is a solution of (S).

Theorem 3.3

In addition to the above hypotheses, assume that $p \mapsto H(x,p)$ are strictly convex for all $x \in \bar\Omega$. Then, as $t \to \infty$,

$$u(x,t) \to u_\infty(x) \ \text{ in } C(\bar\Omega).$$

- Fix $\varepsilon > 0$. Select $T = T(\varepsilon) > 0$ so that

$$\forall x \in \overline{\Omega}, \quad \exists \tau = \tau(x) \in [0,\, T], \quad u(x, \tau(x)) < \varepsilon + u_\infty(x).$$

- Fix $x \in \bar\Omega$ and choose an extremal curve $(\eta, v, l) \in \mathrm{SP}(x)$:

$$\int_0^t \big(L(\eta(s), -v(s)) + l(s)g(\eta(s))\big)ds$$
$$= u_\infty(x) - u_\infty(\eta(t)) \ \text{ for all } t > 0.$$

- Fix $t \gg T$. Set $\sigma := \tau(\eta(t)) \in [0,\, T]$ and

$$\theta := \frac{t}{t-\sigma} \ (\to 1 \text{ as } t \to \infty).$$

- **(Scaling)** Set

$$\eta_\theta(s) := \eta(\theta s).$$

Observe that

$$\begin{aligned} \dot\eta_\theta(s) &= \theta\dot\eta(\theta s) \\ &= \theta\big(v(\theta s) - l(\theta s)g(\eta(\theta s))\big). \end{aligned}$$

If we set

$$v_\theta(s) := \theta v(\theta s), \quad l_\theta(s) = \theta l(\theta s),$$

then $(\eta_\theta, v_\theta, l_\theta) \in \mathrm{SP}(x)$.

- Recall that $\theta(t-\sigma)=t$. Observe that

$$u(\eta_\theta(t-\sigma),\,\sigma)=u(\eta(t),\,\tau(\eta))<\varepsilon+u_\infty(\eta(t)),$$

$$\begin{aligned}u(x,t)&\leq\int_0^{t-\sigma}\big(L(\eta_\theta(s),-v_\theta(s))+l_\theta(s)g(\eta_\theta(s))\big)ds\\&\quad+u(\eta_\theta(t-\sigma),\sigma),\\&<\int_0^{t-\sigma}\big(L(\eta_\theta(s),-v_\theta(s))+l_\theta(s)g(\eta_\theta(s))\big)ds\\&\quad+\varepsilon+u_\infty(\eta(t)).\end{aligned}$$

$$
\begin{aligned}
&\int_0^{t-\sigma} L(\eta_\theta(s), -v_\theta(s))ds \\
&= \frac{1}{\theta}\int_0^{t/\theta} L(\eta(\theta s), -\theta v(\theta s))\theta ds \\
&= \frac{1}{\theta}\int_0^{t} L(\eta(s), -\theta v(s))ds \\
&\le \int_0^{t} L(\eta(s), -v(s))ds + t\, o(\theta - 1) \quad \text{by the strict convexity,}
\end{aligned}
$$

and

$$
\int_0^{t-\sigma} l_\theta(s) g(\eta_\theta(s))ds = \int_0^t l(s) g(\eta(s))ds.
$$

Note that as $t \to \infty$,

$$
\begin{aligned}
t\, o(\theta - 1) &= t\, o\left(\frac{t}{t-\tau} - 1\right) = t\, o\left(\frac{\tau}{t-\tau}\right) \\
&= t\frac{\tau}{t-\tau}\frac{o\left(\frac{\tau}{t-\tau}\right)}{\frac{\tau}{t-\tau}} \to 0.
\end{aligned}
$$

Hence, as $t \to \infty$,

$$\begin{aligned} u(x,t) &< \int_0^t \big(L(\eta(s), -v(s)) + l(s)g(\eta(s))\big)ds \\ &\quad + \varepsilon + u_\infty(x) + o(1) \\ &= u_\infty(x) - u_\infty(\eta(t)) + \varepsilon + u_\infty(\eta(t)) + o(1) \\ &= u_\infty(x) + \varepsilon + o(1), \end{aligned}$$

and

$$\limsup_{t\to\infty} u(x,t) \leq u_\infty(x).$$

Thus,

$$\lim_{t\to\infty} u(x,t) = u_\infty(x).$$

8 REMARKS

- **Strict convexity assumption ($c = 0$):**

There exists a continuous function $\omega : [0, \infty) \to [0, \infty)$ satisfying $\omega(r) > 0$ for $r > 0$ such that for any $x \in \bar{\Omega}$ and $p, q, \xi \in \mathbb{R}^n$, if $H(x, p) = 0$ and $\xi \in D_p^- H(x, p)$, then

$$H(x, p + q) \geq \xi \cdot q + \omega((\xi \cdot q)_+).$$

This is equivalent to: there exists a continuous function $\omega_1 : [0, \infty) \to [0, \infty)$ satisfying $\lim_{r \to 0+} \omega_1(r) = 0$ such that for any $x \in \bar{\Omega}$ and $p, \xi \in \mathbb{R}^n$, if $H(x, p) = 0$ and $\xi \in D_p^- H(x, p)$, then for $0 < \delta \ll 1$,

$$L(x, (1 + \delta)\xi) \leq (1 + \delta)L(x, \xi) + \delta\omega_1(\delta).$$

Lemma 3.4

Let $\phi \in C(\bar{\Omega})$ be a subsolution of (S) and $\eta \in \mathrm{AC}(I, \mathbb{R}^n)$, where I is an interval, be such that $\eta(s) \in \bar{\Omega}$ for $s \in I$. Set

$$I_\partial = \{s \in I \,:\, \eta(s) \in \partial\Omega\}.$$

Then there exists a function $p \in L^\infty(I, \mathbb{R}^n)$ such that

$$\frac{d}{ds}\phi(\eta(s)) = p(t) \cdot \dot{\eta}(t) \quad \text{a.e. } t \in I,$$
$$H(\eta(t), p(t)) \leq 0 \quad \text{a.e. } t \in I,$$
$$\gamma(\eta(t)) \cdot p(t) \leq g(\eta(t)) \quad \text{a.e. } t \in I_\partial.$$

Heuristically, $p(t) \approx D\phi(\eta(t))$.

Roughly speaking, for the proof, we approximate ϕ by a "classical" subsolution ψ of (SP) by using sup-convolution of ϕ. The desired properties for ψ to satisfy:

$$H(x, D\psi(x)) \lessapprox 0 \quad \text{a.e. in a neighborhood of } \bar{\Omega},$$
$$\frac{\partial\psi}{\partial\gamma} \lessapprox g \quad \text{a.e. in a neighborhood of } \partial\Omega.$$

Inequalities in the above lemma turn out to be equalities if ϕ is a solution of (S) and (η, v, l) is extremal. Then

$$H(\eta(s), p(s)) = 0, \quad -v(s) \in D_p^- H(\eta(s), -v(s)),$$

and so, as $\delta \to 0+$,

$$L(\eta(s), -(1+\delta)v(s)) = (1+\delta)L(\eta(s), -v(s)) + o(\delta).$$

- **Formulas for the limit function u_∞: we assume that $c = 0$.**

Theorem 3.5

We have:

$$\begin{aligned} u_\infty(x) &= \inf\{\phi(x) \,:\, \phi \in \mathcal{S}(\bar{\Omega}),\ \phi \geq u_0^-\}, \\ &= \inf\{d(x,y) + u_0^-(y) \,:\, y \in \mathcal{A}\}. \end{aligned}$$

and

$$\begin{aligned} u_0^-(x) &= \sup\{\psi(x) \,:\, \psi \in \mathcal{S}^-(\bar{\Omega}),\ \psi \leq u_0\}, \\ &= \inf\{d(x,y) + u_0(y) \,:\, y \in \bar{\Omega}\}. \end{aligned}$$