Nonlinear diffusion equations I

JUAN LUIS VÁZQUEZ

Departamento de Matemáticas
Universidad Autónoma de Madrid
and Royal Academy of Sciences

CIME Summer Course

“Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions”

Cetraro, Italia July 4, 2016

Outline

Outline

Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?* Is diffusion more or less random walk ?
- *how to explain it with mathematics? is it pure or applied mathematics ?* what would Kolmogorov say?
- *How much of it can be explained with linear models, how much is essentially nonlinear?*
- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

The Laplacian Δ is the King of Differential Operators.

Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?* Is diffusion more or less random walk ?

- *how to explain it with mathematics? is it pure or applied mathematics ?* what would Kolmogorov say?

- *How much of it can be explained with linear models, how much is essentially nonlinear?*

- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

The Laplacian Δ is the King of Differential Operators.

Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?* Is diffusion more or less random walk ?
- *how to explain it with mathematics? is it pure or applied mathematics ?* what would Kolmogorov say?

- *How much of it can be explained with linear models, how much is essentially nonlinear?*

- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

The Laplacian Δ is the King of Differential Operators.

Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

• *what is diffusion anyway?* Is diffusion more or less random walk ?

• *how to explain it with mathematics? is it pure or applied mathematics ?* what would Kolmogorov say?

• *How much of it can be explained with linear models, how much is essentially nonlinear?*

• *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

The Laplacian Δ is the King of Differential Operators.

Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?* Is diffusion more or less random walk ?

- *how to explain it with mathematics? is it pure or applied mathematics ?* what would Kolmogorov say?

- *How much of it can be explained with linear models, how much is essentially nonlinear?*

- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

The Laplacian Δ is the King of Differential Operators.

Diffusion in Wikipedia

- Diffusion. The spreading of any quantity that can be described by the diffusion equation or a random walk model (e.g. concentration, heat, momentum, ideas, price) can be called diffusion.
- Some of the most important examples are listed below.
  ** Atomic diffusion*
  ** Brownian motion, for example of a single particle in a solvent*
  ** Collective diffusion, the diffusion of a large number of (possibly interacting) particles * Effusion of a gas through small holes.*
  ** Electron diffusion, resulting in electric current*
  ** Facilitated diffusion, present in some organisms.*
  ** Gaseous diffusion, used for isotope separation*
  ** Heat flow * Ito- diffusion * Knudsen diffusion*
  ** Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field*
  ** Osmosis is the diffusion of water through a cell membrane. * Photon diffusion*
  ** Reverse diffusion * Self-diffusion * Surface diffusion*

Diffusion in Wikipedia

- Diffusion. The spreading of any quantity that can be described by the diffusion equation or a random walk model (e.g. concentration, heat, momentum, ideas, price) can be called diffusion.
- Some of the most important examples are listed below.
  ** Atomic diffusion*
  ** Brownian motion, for example of a single particle in a solvent*
  ** Collective diffusion, the diffusion of a large number of (possibly interacting) particles * Effusion of a gas through small holes.*
  ** Electron diffusion, resulting in electric current*
  ** Facilitated diffusion, present in some organisms.*
  ** Gaseous diffusion, used for isotope separation*
  ** Heat flow * Ito- diffusion * Knudsen diffusion*
  ** Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field*
  ** Osmosis is the diffusion of water through a cell membrane. * Photon diffusion*
  ** Reverse diffusion * Self-diffusion * Surface diffusion*

The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.

  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on / from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t)X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.
  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on / from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t)X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.
  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on / from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.
  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on / from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

  where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

  This is the famous linear eigenvalue problem, Spectral Theory.

The heat equation semigroup and Gauss

- When heat propagates in free space the natural problem is the initial value problem

$$(1) \qquad u_t = \Delta u, \qquad u(x,0) = f(x)$$

which is solved by convolution with the evoution version of the Gaussian function

$$(2) \qquad G(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right).$$

Note that G has all nice analytical properties for $t > 0$, but note that $G(x,0) = \delta(x)$, a Dirac mass. G works as a kernel (Green, Gauss).

- The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot, t)$ form a linear continuous semigroup of contractions in all L^p spaces $1 \le p \le \infty$.
(This is pure Functional Analysis, XXth century)
- **Asymptotic behaviour as $t \to \infty$, convergence to the Gaussian.** Under very mild conditions on u_0 it is proved that

$$(3) \qquad \lim_{t\to\infty} t^{n/2}(u(x,t) - M\,G(x,t)) = 0$$

uniformly in the whole space, if $M = \int u_0(x)\,dx$. For convergence in L^p less is needed. This is the famous Central Limit Theorem in its continuous form (Probability).

The heat equation semigroup and Gauss

- When heat propagates in free space the natural problem is the initial value problem

  (1) $$u_t = \Delta u, \qquad u(x,0) = f(x)$$

  which is solved by convolution with the evoution version of the Gaussian function

  (2) $$G(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right).$$

  Note that G has all nice analytical properties for $t > 0$, but note that $G(x,0) = \delta(x)$, a Dirac mass. G works as a kernel (Green, Gauss).

- The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot, t)$ form a linear continuous semigroup of contractions in all L^p spaces $1 \le p \le \infty$.
  (This is pure Functional Analysis, XXth century)

- **Asymptotic behaviour as $t \to \infty$, convergence to the Gaussian.** Under very mild conditions on u_0 it is proved that

  (3) $$\lim_{t\to\infty} t^{n/2}(u(x,t) - M\, G(x,t)) = 0$$

  uniformly in the whole space, if $M = \int u_0(x)\, dx$. For convergence in L^p less is needed. This is the famous Central Limit Theorem in its continuous form (Probability).

The heat equation semigroup and Gauss

- When heat propagates in free space the natural problem is the initial value problem

  (1) $$u_t = \Delta u, \qquad u(x,0) = f(x)$$

  which is solved by convolution with the evoution version of the Gaussian function

  (2) $$G(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right).$$

  Note that G has all nice analytical properties for $t > 0$, but note that $G(x,0) = \delta(x)$, a Dirac mass. G works as a kernel (Green, Gauss).

- The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot, t)$ form a linear continuous semigroup of contractions in all L^p spaces $1 \le p \le \infty$.
  (This is pure Functional Analysis, XXth century)

- **Asymptotic behaviour as $t \to \infty$, convergence to the Gaussian.** Under very mild conditions on u_0 it is proved that

  (3) $$\lim_{t\to\infty} t^{n/2}(u(x,t) - M\, G(x,t)) = 0$$

  uniformly in the whole space, if $M = \int u_0(x)\, dx$. For convergence in L^p less is needed. This is the famous Central Limit Theorem in its continuous form (Probability).

The heat equation semigroup and Gauss

- When heat propagates in free space the natural problem is the initial value problem

  (1) $$u_t = \Delta u, \qquad u(x,0) = f(x)$$

  which is solved by convolution with the evoution version of the Gaussian function

  (2) $$G(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right).$$

  Note that G has all nice analytical properties for $t > 0$, but note that $G(x,0) = \delta(x)$, a Dirac mass. G works as a kernel (Green, Gauss).
- The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot,t)$ form a linear continuous semigroup of contractions in all L^p spaces $1 \le p \le \infty$.
  (This is pure Functional Analysis, XXth century)
- **Asymptotic behaviour as $t \to \infty$, convergence to the Gaussian.** Under very mild conditions on u_0 it is proved that

  (3) $$\lim_{t\to\infty} t^{n/2}(u(x,t) - M\, G(x,t)) = 0$$

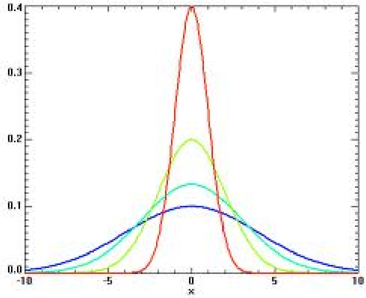
  uniformly in the whole space, if $M = \int u_0(x)\,dx$. For convergence in L^p less is needed. This is the famous Central Limit Theorem in its continuous form (Probability).

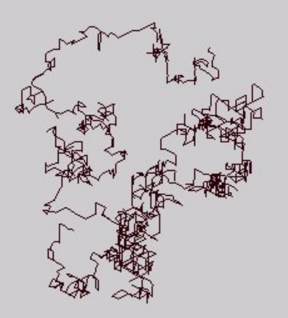
The personalities

J. Fourier and K. F. Gauss

Heat equation graphs

- The comparison of ordered dissipation vs underlying chaos



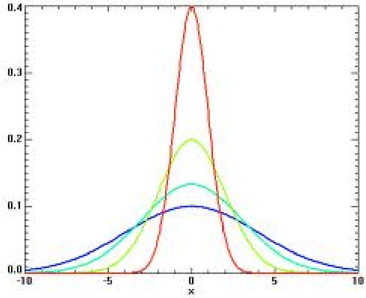


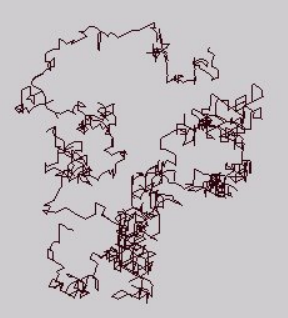
Left, the evolution to a nice Gaussian

Right, a sample of random walk, origin of Brownian motion

Heat equation graphs

- The comparison of ordered dissipation vs underlying chaos



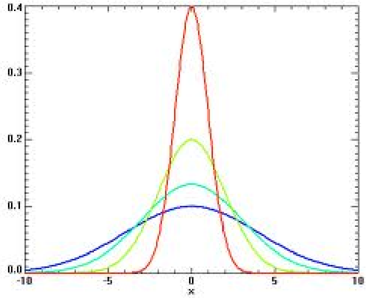


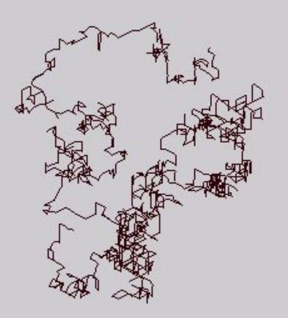
Left, the evolution to a nice Gaussian

Right, a sample of random walk, origin of Brownian motion

Heat equation graphs

- The comparison of ordered dissipation vs underlying chaos



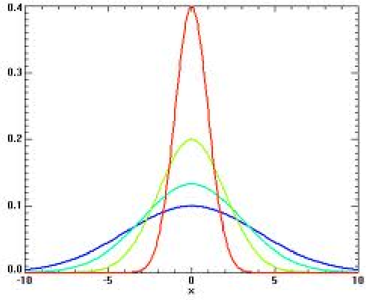


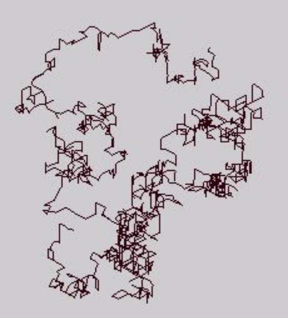
Left, the evolution to a nice Gaussian

Right, a sample of random walk, origin of Brownian motion

Heat equation graphs

- The comparison of ordered dissipation vs underlying chaos





Left, the evolution to a nice Gaussian

Right, a sample of random walk, origin of Brownian motion

Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$\boxed{u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f}$$

Main inventions in Parabolic Theory:
(1) a_{ij}, b_i, c, f regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; C^α spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski; Kolmogorov, Wiener, Levy; Ito, Skorokhod, ...

$$\boxed{dX = bdt + \sigma dW}$$

Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$\boxed{u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f}$$

Main inventions in Parabolic Theory:
(1) a_{ij}, b_i, c, f regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; C^α spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski; Kolmogorov, Wiener, Levy; Ito, Skorokhod, ...

$$\boxed{dX = bdt + \sigma dW}$$

Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$\boxed{u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f}$$

Main inventions in Parabolic Theory:
(1) a_{ij}, b_i, c, f regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; C^α spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski; Kolmogorov, Wiener, Levy; Ito, Skorokhod, ...

$$\boxed{dX = bdt + \sigma dW}$$

Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$\boxed{u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f}$$

Main inventions in Parabolic Theory:
(1) a_{ij}, b_i, c, f regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; C^α spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski; Kolmogorov, Wiener, Levy; Ito, Skorokhod, ...

$$\boxed{dX = bdt + \sigma dW}$$

Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$\boxed{u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)}$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$\boxed{u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)}$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$\boxed{u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)}$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$\boxed{u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)}$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$\boxed{u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)}$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

The Nonlinear Diffusion Models

The four classical 'sisters' of the 1980's

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE: \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC: \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0, \ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0, \ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation →*(hidden free boundary)*

$$u_t = \Delta u^m, \quad m > 1.$$

- The p-Laplacian Equation, $u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

  Recent interest in $p = 1$ (images) or $p = \infty$ (geometry and transport)

The Nonlinear Diffusion Models

The four classical 'sisters' of the 1980's

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0,\ \Delta u = 0 \quad \text{in} \quad \Omega(t); \qquad u = 0,\ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation →*(hidden free boundary)*

$$u_t = \Delta u^m, \qquad m > 1.$$

- The p-Laplacian Equation, $u_t = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$.

  Recent interest in $p = 1$ (images) or $p = \infty$ (geometry and transport)

The Nonlinear Diffusion Models

The four classical 'sisters' of the 1980's

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE: \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC: \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0,\ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0,\ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation $\to$*(hidden free boundary)*

$$u_t = \Delta u^m, \quad m > 1.$$

- The p-Laplacian Equation, $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

  Recent interest in $p = 1$ (images) or $p = \infty$ (geometry and transport)

The Nonlinear Diffusion Models

The four classical 'sisters' of the 1980's

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0, \ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0, \ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation →*(hidden free boundary)*

$$u_t = \Delta u^m, \quad m > 1.$$

- The p-Laplacian Equation, $u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

  Recent interest in $p = 1$ (images) or $p = \infty$ (geometry and transport)

The Nonlinear Diffusion Models

The four classical 'sisters' of the 1980's

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0, \ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0, \ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation →*(hidden free boundary)*

$$u_t = \Delta u^m, \quad m > 1.$$

- The p-Laplacian Equation, $u_t = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$.

  Recent interest in $p = 1$ (images) or $p = \infty$ (geometry and transport)

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

  $$\boxed{u_t = \Delta u + u^p}$$

  Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
  Problem: *what is the influence of diffusion / migration?*

- General scalar model

  $$\boxed{u_t = \mathcal{A}(u) + f(u)}$$

- The system model: $\overrightarrow{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$\boxed{u_t = \Delta u + u^p}$$

Main feature: If $p > 1$ the norm $\|u(\cdot,t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$\boxed{u_t = \mathcal{A}(u) + f(u)}$$

- The system model: $\vec{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$\boxed{u_t = \Delta u + u^p}$$

Main feature: If $p > 1$ the norm $\|u(\cdot,t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$\boxed{u_t = \mathcal{A}(u) + f(u)}$$

- The system model: $\vec{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$\boxed{u_t = \Delta u + u^p}$$

Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$\boxed{u_t = \mathcal{A}(u) + f(u)}$$

- The system model: $\overrightarrow{\mathbf{u}} = (u_1, \cdots, u_m) \rightarrow$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$\boxed{u_t = \Delta u + u^p}$$

  Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
  Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$\boxed{u_t = \mathcal{A}(u) + f(u)}$$

- The system model: $\vec{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

An opinion by John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

- *Continuity of solutions of elliptic and parabolic equations*, paper published in Amer. J. Math, 80, no 4 (1958), 931-954.
- **De Giorgi, Ennio**, *Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari.* Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), 25–43.
- **J. Moser**, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations.* Comm. Pure Appl. Math. **13** (1960) 457–468.
  A Harnack inequality for parabolic differential equations, Comm. Pure and Appl. Math. **17** (1964), 101–134.

An opinion by John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

- *Continuity of solutions of elliptic and parabolic equations*, paper published in Amer. J. Math, 80, no 4 (1958), 931-954.
- **De Giorgi, Ennio**, *Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari.* Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), 25–43.
- **J. Moser**, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations.* Comm. Pure Appl. Math. **13** (1960) 457–468.
  A Harnack inequality for parabolic differential equations, Comm. Pure and Appl. Math. **17** (1964), 101–134.

An opinion by John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

- *Continuity of solutions of elliptic and parabolic equations*, paper published in Amer. J. Math, 80, no 4 (1958), 931-954.
- **De Giorgi, Ennio**, *Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari.* Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), 25–43.
- **J. Moser**, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations.* Comm. Pure Appl. Math. **13** (1960) 457–468.
  A Harnack inequality for parabolic differential equations, Comm. Pure and Appl. Math. **17** (1964), 101–134.

Outline

The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\boxed{\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot \big(D(\phi,\vec{r})\nabla\phi(\vec{r},t)\big)}$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\boxed{\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot \left(D(\phi,\vec{r})\nabla\phi(\vec{r},t)\right)}$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0$ $\Longrightarrow$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\boxed{\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot \left(D(\phi,\vec{r}) \nabla \phi(\vec{r},t) \right)}$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = m u^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\boxed{\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot \left(D(\phi,\vec{r})\nabla\phi(\vec{r},t)\right)}$$

  It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

  $c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:

  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")

  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\boxed{\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot \left(D(\phi,\vec{r})\nabla\phi(\vec{r},t)\right)}$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:

  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")

  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\boxed{\frac{\partial\phi(\vec{r},t)}{\partial t} = \nabla\cdot\left(D(\phi,\vec{r})\nabla\phi(\vec{r},t)\right)}$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla\cdot(c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = m u^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\boxed{\frac{\partial\phi(\vec{r},t)}{\partial t} = \nabla\cdot\left(D(\phi,\vec{r})\nabla\phi(\vec{r},t)\right)}$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla\cdot(c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:

  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ (“slow case”)

  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ (“fast case”)

The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\tfrac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\tfrac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\frac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

  and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

  This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

  where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
  This separates $m > 1$ PME - from - $m < 1$ FDE

The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\frac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\frac{1}{2} u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.

This separates $m > 1$ PME - from - $m < 1$ FDE

The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\frac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

  and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

  This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

  where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
  This separates $m > 1$ PME - from - $m < 1$ FDE

Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933) $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$

  Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

  To the right, put $p = p_o\,\rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

$$\rho_t = \operatorname{div}\left(\frac{k}{\mu}\rho\nabla p\right) = \operatorname{div}\left(\frac{k}{\mu}\rho\nabla(p_o\rho^\gamma)\right) = c\Delta\rho^{\gamma+1}.$$

- Underground water infiltration (Boussinesq, 1903) $m = 2$

Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$

  Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

  To the right, put $p = p_o\,\rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

$$\rho_t = \operatorname{div}\left(\frac{k}{\mu}\rho\nabla p\right) = \operatorname{div}\left(\frac{k}{\mu}\rho\nabla(p_o\rho^\gamma)\right) = c\Delta\rho^{\gamma+1}.$$

- Underground water infiltration (Boussinesq, 1903) $m = 2$

Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933) $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

To the right, put $p = p_o\,\rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

$$\rho_t = \mathrm{div}\,(\frac{k}{\mu}\rho\nabla p) = \mathrm{div}\,(\frac{k}{\mu}\rho\nabla(p_o\rho^\gamma)) = c\Delta\rho^{\gamma+1}.$$

- Underground water infiltration (Boussinesq, 1903) $m = 2$

Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  $m = 1 + \gamma \geq 2$
  $$\begin{cases} \rho_t + \mathrm{div}\,(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$
  Second line left is the Darcy law for flows in porous media (Darcy, 1856).
  Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.

  To the right, put $p = p_o\, \rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).
  $$\rho_t = \mathrm{div}\,(\frac{k}{\mu}\rho \nabla p) = \mathrm{div}\,(\frac{k}{\mu}\rho \nabla (p_o \rho^\gamma)) = c\Delta \rho^{\gamma+1}.$$
- Underground water infiltration (Boussinesq, 1903) $m = 2$

Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  $m = 1 + \gamma \geq 2$
  $$\begin{cases} \rho_t + \mathrm{div}\,(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$
  Second line left is the Darcy law for flows in porous media (Darcy, 1856).
  Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.

  To the right, put $p = p_o\,\rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).
  $$\rho_t = \mathrm{div}\,(\frac{k}{\mu}\rho\nabla p) = \mathrm{div}\,(\frac{k}{\mu}\rho\nabla(p_o\rho^\gamma)) = c\Delta\rho^{\gamma+1}.$$
- Underground water infiltration (Boussinesq, 1903) $m = 2$

Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, ¡ 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_{\Omega} c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T\cdot \mathbf{n}dS.$$

  Put $k(T) = k_o T^n$, apply Gauss law and you get

$$c\rho\frac{\partial T}{\partial t} = \mathrm{div}(k(T)\nabla T) = c_1\Delta T^{n+1}.$$

  $\rightarrow$ When k is not a power we get $\boxed{T_t = \Delta\Phi(T)}$ with $\Phi'(T) = k(T)$.
- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, *McKean, PL Lions and Toscani et al.*)
- Many more (boundary layers, geometry).

Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, ¡ 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_\Omega c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T\cdot \mathbf{n}dS.$$

  Put $k(T) = k_o T^n$, apply Gauss law and you get

$$c\rho\frac{\partial T}{\partial t} = \mathrm{div}(k(T)\nabla T) = c_1 \Delta T^{n+1}.$$

  $\rightarrow$ When k is not a power we get $\boxed{T_t = \Delta\Phi(T)}$ with $\Phi'(T) = k(T)$.
- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, *McKean, PL Lions and Toscani et al.*)
- Many more (boundary layers, geometry).

Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, ¡ 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_{\Omega} c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T \cdot \mathbf{n} dS.$$

  Put $k(T) = k_o T^n$, apply Gauss law and you get

$$c\rho\frac{\partial T}{\partial t} = \operatorname{div}(k(T)\nabla T) = c_1 \Delta T^{n+1}.$$

  $\rightarrow$ When k is not a power we get $\boxed{T_t = \Delta\Phi(T)}$ with $\Phi'(T) = k(T)$.
- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, *McKean, PL Lions and Toscani et al.*)
- Many more (boundary layers, geometry).

Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, ¡ 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_{\Omega} c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T\cdot \mathbf{n}dS.$$

  Put $k(T) = k_o T^n$, apply Gauss law and you get

$$c\rho\frac{\partial T}{\partial t} = \mathrm{div}(k(T)\nabla T) = c_1\Delta T^{n+1}.$$

  $\rightarrow$ When k is not a power we get $\boxed{T_t = \Delta\Phi(T)}$ with $\Phi'(T) = k(T)$.
- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, *McKean, PL Lions and Toscani et al.*)
- Many more (boundary layers, geometry).

Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, ¡ 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_\Omega c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T\cdot\mathbf{n}dS.$$

  Put $k(T) = k_o T^n$, apply Gauss law and you get

$$c\rho\frac{\partial T}{\partial t} = \mathrm{div}(k(T)\nabla T) = c_1\Delta T^{n+1}.$$

  $\rightarrow$ When k is not a power we get $\boxed{T_t = \Delta\Phi(T)}$ with $\Phi'(T) = k(T)$.

- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, *McKean, PL Lions and Toscani et al.*)
- Many more (boundary layers, geometry).

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* C^k *for some* k?
- Regularity and movement of interfaces: C^k *for some* k?.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? C^k for some k?*
- Regularity and movement of interfaces: *C^k for some k?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? C^k for some k?*
- Regularity and movement of interfaces: *C^k for some k?.*
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* C^k *for some* k*?*
- Regularity and movement of interfaces: C^k *for some* k*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* C^k *for some* k*?*
- Regularity and movement of interfaces: C^k *for some* k*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* C^k *for some* k*?*
- Regularity and movement of interfaces: C^k *for some* k*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* C^k *for some* k*?*
- Regularity and movement of interfaces: C^k *for some* k*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* C^k *for some* k*?*
- Regularity and movement of interfaces: C^k *for some* k*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Planning of the Theory

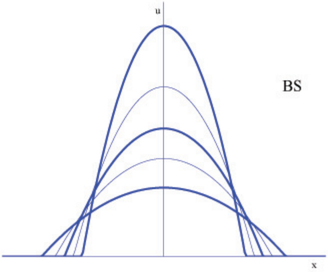
These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? C^k for some k?*
- Regularity and movement of interfaces: C^k *for some k?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
  They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$.
- Explicit formulas (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

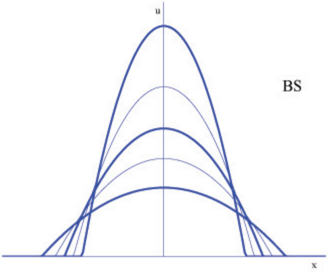
Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
  They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$.
- Explicit formulas (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

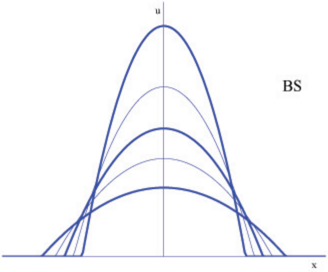
Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
  They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$.
- Explicit formulas (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

My Russian friend

Grisha I. Barenblatt

My Russian friend

Grisha I. Barenblatt

Concept of solution

There are many concepts of generalized solution of the PME:

- Classical solution: only in non-degenerate situations, $u > 0$.
- Limit solution: physical, but depends on the approximation (?).
- Weak solution Test against smooth functions and eliminate derivatives on the unknown function; it is the mainstream; (Oleinik, 1958)

$$\int\int (u\,\eta_t - \nabla u^m \cdot \nabla \eta)\, dxdt + \int u_0(x)\,\eta(x,0)\, dx = 0.$$

Very weak

$$\int\int (u\,\eta_t + u^m\,\Delta\eta)\, dxdt + \int u_0(x)\,\eta(x,0)\, dx = 0.$$

More on concepts of solution

Solutions are not always, not only weak:

- Strong solution. More regular than weak but not classical: weak derivatives are L^p functions. *Big benefit: usual calculus is possible.*
- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems Ambrosio, Savarè, Nochetto

More on concepts of solution

Solutions are not always, not only weak:

- Strong solution. More regular than weak but not classical: weak derivatives are L^p functions. *Big benefit: usual calculus is possible.*
- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems   Ambrosio, Savarè, Nochetto

More on concepts of solution

Solutions are not always, not only weak:

- Strong solution. More regular than weak but not classical: weak derivatives are L^p functions. *Big benefit: usual calculus is possible.*
- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems Ambrosio, Savarè, Nochetto

More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968). Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions (Perthame,...).
- Proper solutions (Galaktionov-Vazquez,...).

More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968). Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions (Perthame,...).
- Proper solutions (Galaktionov-Vazquez,...).

More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968). Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions (Perthame,...).
- Proper solutions (Galaktionov-Vazquez,...).

More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968). Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions (Perthame,...).
- Proper solutions (Galaktionov-Vazquez,...).

Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.

- (Caffarelli-Friedman, 1982) C^α regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is C^α continuous.
- If there is an interface Γ, it is also C^α continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* C^1
  Aronson-Caffarelli-V. Regularity stops here in $n = 1$

Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.

- (Caffarelli-Friedman, 1982) C^α regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is C^α continuous.
- If there is an interface Γ, it is also C^α continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* C^1 Aronson-Caffarelli-V. Regularity stops here in $n = 1$

Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.
- (Caffarelli-Friedman, 1982) C^α regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is C^α continuous.
- If there is an interface Γ, it is also C^α continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* C^1
  Aronson-Caffarelli-V. Regularity stops here in $n = 1$

Regularity results

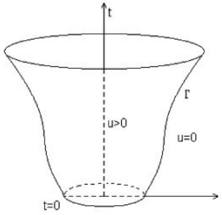
- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.
- (Caffarelli-Friedman, 1982) C^α regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is C^α continuous.
- If there is an interface Γ, it is also C^α continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* C^1
  Aronson-Caffarelli-V. Regularity stops here in $n = 1$

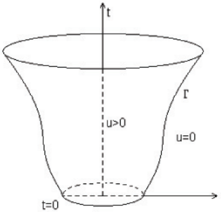
Free Boundaries in several dimensions



A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If u_0 has compact support, then after some time T the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If u_0 is transversal then FB is C^∞ after T. Pressure is "laterally" C^∞. *when it moves, it is always a broken profile* .
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii r_1 and r_2. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that $\beta < 1$!

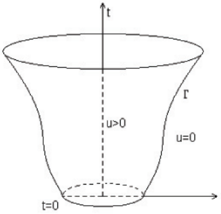
Free Boundaries in several dimensions



A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If u_0 has compact support, then after some time T the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If u_0 is transversal then FB is C^∞ after T. Pressure is “laterally” C^∞. *when it moves, it is always a broken profile .*
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii r_1 and r_2. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that $\beta < 1$.

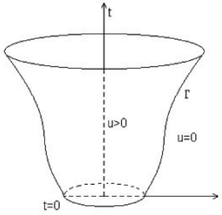
Free Boundaries in several dimensions



A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If u_0 has compact support, then after some time T the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If u_0 is transversal then FB is C^∞ after T. Pressure is "laterally" C^∞. *when it moves, it is always a broken profile .*
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii r_1 and r_2. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that $\beta < 1$.

Free Boundaries in several dimensions



A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If u_0 has compact support, then after some time T the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If u_0 is transversal then FB is C^∞ after T. Pressure is "laterally" C^∞. *when it moves, it is always a broken profile .*
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii r_1 and r_2. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that $\beta < 1$.

Parabolic to Elliptic

- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems Ambrosio, Savarè, Nochetto

- Separation of variables. Put $u(x,t) = F(x)G(t)$. Then PME gives

$$F(x)G'(t) = G^m(t)\Delta F^m(x),$$

so that $G'(t) = -G^m(t)$, i.e., $G(t) = (m-1)t^{-1/(m-1)}$ if $m > 1$ and

$$-\Delta F^m(x) = F(x), \qquad -\Delta v(x) = v^p(x),\ p = 1/m.$$

This is more interesting for $m < 1$, specially for $m = (n-2)/(n-2)$.

Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2+n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

Asymptotic behaviour I

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^\alpha |u(x,t) - B(x,t)| \to 0}$$

For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^\alpha |u(x,t) - B(x,t)| \to 0}$$

For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

For every $p \ge 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \ge 0$, compact support and $f = 0$.

Calculations of entropy rates

- We rescale the function as $u(x,t) = r(t)^n \rho(y\, r(t), s)$ where $r(t)$ is the Barenblatt radius at $t+1$, and “new time” is $s = \log(1+t)$. Equation becomes
$$\rho_s = \mathrm{div}\,(\rho(\nabla \rho^{m-1} + \frac{c}{2}\nabla y^2)).$$
- Then define the entropy
$$E(u)(t) = \int (\frac{1}{m}\rho^m + \frac{c}{2}\rho y^2)\, dy$$
The minimum of entropy is identified as the Barenblatt profile.
- Calculate
$$\frac{dE}{ds} = -\int \rho|\nabla \rho^{m-1} + cy|^2\, dy = -D$$
Moreover,
$$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$
We conclude exponential decay of D in new time s, which is potential in real time t. It follows that E decays to a minimum $E_\infty > 0$ and we prove that this is the level of the Barenblatt solution.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.

About estimates and scaling

- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.

About asymptotic behaviour. (Following Lyapunov and Boltzmann)

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space.* Journal of Evolution Equations **3** (2003), 67–118.

On Nonlinear Diffusion

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry.* Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

- J. L. Vázquez, ”The Porous Medium Equation. Mathematical Theory”, Oxford Univ. Press, 2007, xxii+624 pages.

About estimates and scaling

- J. L. Vázquez, “Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type”, Oxford Univ. Press, 2006, 234 pages.

About asymptotic behaviour. (Following Lyapunov and Boltzmann)

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space.* Journal of Evolution Equations **3** (2003), 67–118.

On Nonlinear Diffusion

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry.* Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

- J. L. Vázquez, ”The Porous Medium Equation. Mathematical Theory”, Oxford Univ. Press, 2007, xxii+624 pages.

About estimates and scaling

- J. L. Vázquez, “Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type”, Oxford Univ. Press, 2006, 234 pages.

About asymptotic behaviour. (Following Lyapunov and Boltzmann)

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space.* Journal of Evolution Equations **3** (2003), 67–118.

On Nonlinear Diffusion

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry.* Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

- J. L. Vázquez, ”The Porous Medium Equation. Mathematical Theory”, Oxford Univ. Press, 2007, xxii+624 pages.

About estimates and scaling

- J. L. Vázquez, “Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type”, Oxford Univ. Press, 2006, 234 pages.

About asymptotic behaviour. (Following Lyapunov and Boltzmann)

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space.* Journal of Evolution Equations **3** (2003), 67–118.

On Nonlinear Diffusion

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry.* Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.

About estimates and scaling

- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.

About asymptotic behaviour. (Following Lyapunov and Boltzmann)

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space*. Journal of Evolution Equations **3** (2003), 67–118.

On Nonlinear Diffusion

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry*. Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.

About estimates and scaling

- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.

About asymptotic behaviour. (Following Lyapunov and Boltzmann)

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space*. Journal of Evolution Equations **3** (2003), 67–118.

On Nonlinear Diffusion

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry*. Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

- J. L. Vázquez, ”The Porous Medium Equation. Mathematical Theory”, Oxford Univ. Press, 2007, xxii+624 pages.

About estimates and scaling

- J. L. Vázquez, “Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type”, Oxford Univ. Press, 2006, 234 pages.

About asymptotic behaviour. (Following Lyapunov and Boltzmann)

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space*. Journal of Evolution Equations **3** (2003), 67–118.

On Nonlinear Diffusion

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry*. Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

Some of our lines of research

- Very fast diffusion flows, logarithmic diffusion, gradient diffusion (p-Laplacian) flows
  A Blanchet, M. Bonforte, J Dolbeault, G Grillo; with R. Iagar, ... Related work by Carrillo, McCann, Markowcih, Toscani. Other direction: Fila and Winkler
- Flows on manifolds:
  paper with Bonforte and Grillo in JEE, 2008;
  paper with Lu, Lei, Villani in arXiv, june 2008 ...
- Flows in inhomogeneous media.
  papers with G. Reyes; with Reyes and Kamin, 2010.
  elliptic work by Brezis, Eidus, Kamin, Tesei,
  work by Karlsen et al for general variable coefficients
- Dynamic boundary conditions.
  papers with E. Vitillaro (M3AS, JDE);
- Nonlinear diffusion with fractional diffusion operators.
  Most works by Caffarelli, Silvestre, Cabré and others on stationary problems $\sim 2004-08$. Parabolic work by Caffarelli, Chan, Vazquez, Vasseur and Madrid Group. France: Serfaty and Sire.

Some of our lines of research

- Very fast diffusion flows, logarithmic diffusion, gradient diffusion (p-Laplacian) flows
  A Blanchet, M. Bonforte, J Dolbeault, G Grillo; with R. Iagar, ... Related work by Carrillo, McCann, Markowcih, Toscani. Other direction: Fila and Winkler
- Flows on manifolds:
  paper with Bonforte and Grillo in JEE, 2008;
  paper with Lu, Lei, Villani in arXiv, june 2008 ...
- Flows in inhomogeneous media.
  papers with G. Reyes; with Reyes and Kamin, 2010.
  elliptic work by Brezis, Eidus, Kamin, Tesei,
  work by Karlsen et al for general variable coefficients
- Dynamic boundary conditions.
  papers with E. Vitillaro (M3AS, JDE);
- Nonlinear diffusion with fractional diffusion operators.
  Most works by Caffarelli, Silvestre, Cabré and others on stationary problems $\sim 2004-08$. Parabolic work by Caffarelli, Chan, Vazquez, Vasseur and Madrid Group. France: Serfaty and Sire.

Some of our lines of research

- Very fast diffusion flows, logarithmic diffusion, gradient diffusion (p-Laplacian) flows
  A Blanchet, M. Bonforte, J Dolbeault, G Grillo; with R. Iagar, ... Related work by Carrillo, McCann, Markowcih, Toscani. Other direction: Fila and Winkler
- Flows on manifolds:
  paper with Bonforte and Grillo in JEE, 2008;
  paper with Lu, Lei, Villani in arXiv, june 2008 ...
- Flows in inhomogeneous media.
  papers with G. Reyes; with Reyes and Kamin, 2010.
  elliptic work by Brezis, Eidus, Kamin, Tesei,
  work by Karlsen et al for general variable coefficients
- Dynamic boundary conditions.
  papers with E. Vitillaro (M3AS, JDE);
- Nonlinear diffusion with fractional diffusion operators.
  Most works by Caffarelli, Silvestre, Cabré and others on stationary problems $\sim 2004-08$. Parabolic work by Caffarelli, Chan, Vazquez, Vasseur and Madrid Group. France: Serfaty and Sire.

Some of our lines of research

- Very fast diffusion flows, logarithmic diffusion, gradient diffusion (p-Laplacian) flows
  A Blanchet, M. Bonforte, J Dolbeault, G Grillo; with R. Iagar, ... Related work by Carrillo, McCann, Markowcih, Toscani. Other direction: Fila and Winkler
- Flows on manifolds:
  paper with Bonforte and Grillo in JEE, 2008;
  paper with Lu, Lei, Villani in arXiv, june 2008 ...
- Flows in inhomogeneous media.
  papers with G. Reyes; with Reyes and Kamin, 2010.
  elliptic work by Brezis, Eidus, Kamin, Tesei,
  work by Karlsen et al for general variable coefficients
- Dynamic boundary conditions.
  papers with E. Vitillaro (M3AS, JDE);
- Nonlinear diffusion with fractional diffusion operators.
  Most works by Caffarelli, Silvestre, Cabré and others on stationary problems $\sim 2004 - 08$. Parabolic work by Caffarelli, Chan, Vazquez, Vasseur and Madrid Group. France: Serfaty and Sire.

Some of our lines of research

- Very fast diffusion flows, logarithmic diffusion, gradient diffusion (p-Laplacian) flows
  A Blanchet, M. Bonforte, J Dolbeault, G Grillo; with R. Iagar, ... Related work by Carrillo, McCann, Markowcih, Toscani. Other direction: Fila and Winkler
- Flows on manifolds:
  paper with Bonforte and Grillo in JEE, 2008;
  paper with Lu, Lei, Villani in arXiv, june 2008 ...
- Flows in inhomogeneous media.
  papers with G. Reyes; with Reyes and Kamin, 2010.
  elliptic work by Brezis, Eidus, Kamin, Tesei,
  work by Karlsen et al for general variable coefficients
- Dynamic boundary conditions.
  papers with E. Vitillaro (M3AS, JDE);
- Nonlinear diffusion with fractional diffusion operators.
  Most works by Caffarelli, Silvestre, Cabré and others on stationary problems $\sim 2004-08$. Parabolic work by Caffarelli, Chan, Vazquez, Vasseur and Madrid Group. France: Serfaty and Sire.

Some of our lines of research

- Very fast diffusion flows, logarithmic diffusion, gradient diffusion (p-Laplacian) flows
  A Blanchet, M. Bonforte, J Dolbeault, G Grillo; with R. Iagar, ... Related work by Carrillo, McCann, Markowcih, Toscani. Other direction: Fila and Winkler
- Flows on manifolds:
  paper with Bonforte and Grillo in JEE, 2008;
  paper with Lu, Lei, Villani in arXiv, june 2008 ...
- Flows in inhomogeneous media.
  papers with G. Reyes; with Reyes and Kamin, 2010.
  elliptic work by Brezis, Eidus, Kamin, Tesei,
  work by Karlsen et al for general variable coefficients
- Dynamic boundary conditions.
  papers with E. Vitillaro (M3AS, JDE);
- Nonlinear diffusion with fractional diffusion operators.
  Most works by Caffarelli, Silvestre, Cabré and others on stationary problems $\sim 2004-08$. Parabolic work by Caffarelli, Chan, Vazquez, Vasseur and Madrid Group. France: Serfaty and Sire.

A theorem from 2016

This is the main result of a project with Herbert Koch and Clemens Kienzler from Bonn. We solve an open problem posed in 1987, and once the idea was found the project has taken a number of years.

Theorem

Let $\rho \ge 0$ be a solution of the PME posed for all $x \in \mathbb{R}^n$, $n \ge 1$, and $t > 0$, and let the initial data ρ_0 be nonnegative, bounded and compactly supported. Then, there exists a time T_r depending on ρ_0 such that for all $t > T_r$
(i) The free boundary is a C^∞ hypersurface close to a ball of radius $R(t) = c_1(n,m)\, M^{(m-1)\lambda} t^\lambda$, with $\lambda = 1/(n(m-1)+2)$ (the Barenblatt exponent);
(ii) The solution pressure $\rho^{m-1}(t,x)$ is a C^∞ function inside the support and up to the free boundary. Moreover, the free boundary function $t = h(x)$ is C^∞ in the complement of the ball of radius $R(T_r)$.
(iii) Moreover, if the initial mass is $M > 0$ and the initial function is supported in the ball $B_R(0)$, then we can write the upper estimate of the regularization time as $T_r = T(n,m)M^{m-1}R^{2+n(m-1)}$.

The full paper is now in the final touches.
We use delicate flatness conditions, scalings, heat semigroups and harmonic analysis.
We have eliminated the non-degeneracy condition on the initial data.
The estimates are uniform. The result cannot be improved in a number of directions.

A theorem from 2016

This is the main result of a project with Herbert Koch and Clemens Kienzler from Bonn. We solve an open problem posed in 1987, and once the idea was found the project has taken a number of years.

Theorem

Let $\rho \geq 0$ be a solution of the PME posed for all $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and let the initial data ρ_0 be nonnegative, bounded and compactly supported. Then, there exists a time T_r depending on ρ_0 such that for all $t > T_r$

(i) The free boundary is a C^∞ hypersurface close to a ball of radius $R(t) = c_1(n,m)\, M^{(m-1)\lambda} t^\lambda$, with $\lambda = 1/(n(m-1)+2)$ (the Barenblatt exponent);

(ii) The solution pressure $\rho^{m-1}(t,x)$ is a C^∞ function inside the support and up to the free boundary. Moreover, the free boundary function $t = h(x)$ is C^∞ in the complement of the ball of radius $R(T_r)$.

(iii) Moreover, if the initial mass is $M > 0$ and the initial function is supported in the ball $B_R(0)$, then we can write the upper estimate of the regularization time as $T_r = T(n,m) M^{m-1} R^{2+n(m-1)}$.

The full paper is now in the final touches.

We use delicate flatness conditions, scalings, heat semigroups and harmonic analysis.

We have eliminated the non-degeneracy condition on the initial data.

The estimates are uniform. The result cannot be improved in a number of directions.

A theorem from 2016

This is the main result of a project with Herbert Koch and Clemens Kienzler from Bonn. We solve an open problem posed in 1987, and once the idea was found the project has taken a number of years.

Theorem

Let $\rho \geq 0$ be a solution of the PME posed for all $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and let the initial data ρ_0 be nonnegative, bounded and compactly supported. Then, there exists a time T_r depending on ρ_0 such that for all $t > T_r$
(i) The free boundary is a C^∞ hypersurface close to a ball of radius $R(t) = c_1(n,m)\, M^{(m-1)\lambda} t^\lambda$, with $\lambda = 1/(n(m-1)+2)$ (the Barenblatt exponent);
(ii) The solution pressure $\rho^{m-1}(t,x)$ is a C^∞ function inside the support and up to the free boundary. Moreover, the free boundary function $t = h(x)$ is C^∞ in the complement of the ball of radius $R(T_r)$.
(iii) Moreover, if the initial mass is $M > 0$ and the initial function is supported in the ball $B_R(0)$, then we can write the upper estimate of the regularization time as $T_r = T(n,m)M^{m-1} R^{2+n(m-1)}$.

The full paper is now in the final touches.
We use delicate flatness conditions, scalings, heat semigroups and harmonic analysis.
We have eliminated the non-degeneracy condition on the initial data.
The estimates are uniform. The result cannot be improved in a number of directions.

A theorem from 2016

This is the main result of a project with Herbert Koch and Clemens Kienzler from Bonn. We solve an open problem posed in 1987, and once the idea was found the project has taken a number of years.

Theorem

Let $\rho \geq 0$ be a solution of the PME posed for all $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and let the initial data ρ_0 be nonnegative, bounded and compactly supported. Then, there exists a time T_r depending on ρ_0 such that for all $t > T_r$
(i) The free boundary is a C^∞ hypersurface close to a ball of radius $R(t) = c_1(n,m)\, M^{(m-1)\lambda} t^\lambda$, with $\lambda = 1/(n(m-1)+2)$ (the Barenblatt exponent);
(ii) The solution pressure $\rho^{m-1}(t,x)$ is a C^∞ function inside the support and up to the free boundary. Moreover, the free boundary function $t = h(x)$ is C^∞ in the complement of the ball of radius $R(T_r)$.
(iii) Moreover, if the initial mass is $M > 0$ and the initial function is supported in the ball $B_R(0)$, then we can write the upper estimate of the regularization time as $T_r = T(n,m) M^{m-1} R^{2+n(m-1)}$.

The full paper is now in the final touches.
We use delicate flatness conditions, scalings, heat semigroups and harmonic analysis.
We have eliminated the non-degeneracy condition on the initial data.
The estimates are uniform. The result cannot be improved in a number of directions.

A theorem from 2016

This is the main result of a project with Herbert Koch and Clemens Kienzler from Bonn. We solve an open problem posed in 1987, and once the idea was found the project has taken a number of years.

Theorem

Let $\rho \geq 0$ be a solution of the PME posed for all $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and let the initial data ρ_0 be nonnegative, bounded and compactly supported. Then, there exists a time T_r depending on ρ_0 such that for all $t > T_r$
(i) The free boundary is a C^∞ hypersurface close to a ball of radius $R(t) = c_1(n,m)\, M^{(m-1)\lambda} t^\lambda$, with $\lambda = 1/(n(m-1)+2)$ (the Barenblatt exponent);
(ii) The solution pressure $\rho^{m-1}(t,x)$ is a C^∞ function inside the support and up to the free boundary. Moreover, the free boundary function $t = h(x)$ is C^∞ in the complement of the ball of radius $R(T_r)$.
(iii) Moreover, if the initial mass is $M > 0$ and the initial function is supported in the ball $B_R(0)$, then we can write the upper estimate of the regularization time as $T_r = T(n,m) M^{m-1} R^{2+n(m-1)}$.

The full paper is now in the final touches.
We use delicate flatness conditions, scalings, heat semigroups and harmonic analysis.
We have eliminated the non-degeneracy condition on the initial data.
The estimates are uniform. The result cannot be improved in a number of directions.

A theorem from 2016

This is the main result of a project with Herbert Koch and Clemens Kienzler from Bonn. We solve an open problem posed in 1987, and once the idea was found the project has taken a number of years.

Theorem

Let $\rho \geq 0$ be a solution of the PME posed for all $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and let the initial data ρ_0 be nonnegative, bounded and compactly supported. Then, there exists a time T_r depending on ρ_0 such that for all $t > T_r$
(i) The free boundary is a C^∞ hypersurface close to a ball of radius $R(t) = c_1(n,m)\, M^{(m-1)\lambda} t^\lambda$, with $\lambda = 1/(n(m-1)+2)$ (the Barenblatt exponent);
(ii) The solution pressure $\rho^{m-1}(t,x)$ is a C^∞ function inside the support and up to the free boundary. Moreover, the free boundary function $t = h(x)$ is C^∞ in the complement of the ball of radius $R(T_r)$.
(iii) Moreover, if the initial mass is $M > 0$ and the initial function is supported in the ball $B_R(0)$, then we can write the upper estimate of the regularization time as $T_r = T(n,m) M^{m-1} R^{2+n(m-1)}$.

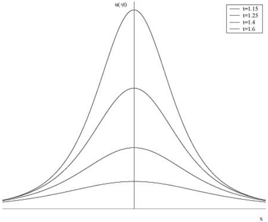
The full paper is now in the final touches.
We use delicate flatness conditions, scalings, heat semigroups and harmonic analysis.
We have eliminated the non-degeneracy condition on the initial data.
The estimates are uniform. The result cannot be improved in a number of directions.

Outline

FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$



The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

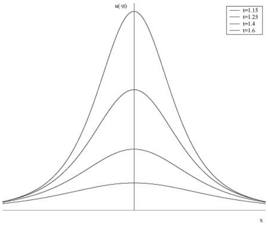
Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$



The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

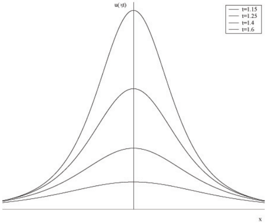
Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C+k\xi^2)^{1/(1-m)}}$$



The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

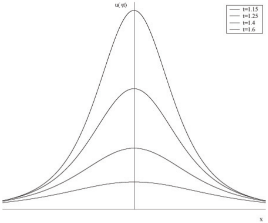
Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$



The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

Applied Motivation

Carleman model

Simple case of Diffusive limit of kinetic equations. Two types of particles in a one dimensional setting moving with speeds c and $-c$.
Densities are u and v respectively. Dynamics is

$$(4) \qquad \begin{cases} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{cases}$$

for some interaction kernel $k(u,v) \ge 0$. Typical case $k = (u+v)^\alpha c^2$.
Write now the equations for $\rho = u + v$ and $j = c(u-v)$ and pass to the limit $c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

$$(5) \qquad \frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right),$$

which is the FDE with $m = 1-\alpha$, cf. Lions Toscani, 1997. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

Carleman model

Simple case of Diffusive limit of kinetic equations. Two types of particles in a one dimensional setting moving with speeds c and $-c$.
Densities are u and v respectively. Dynamics is

$$\left\{\begin{array}{l} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{array}\right. \tag{4}$$

for some interaction kernel $k(u,v) \geq 0$. Typical case $k = (u+v)^\alpha c^2$.
Write now the equations for $\rho = u + v$ and $j = c(u-v)$ and pass to the limit $c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right), \tag{5}$$

which is the FDE with $m = 1-\alpha$, cf. Lions Toscani, 1997. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

Carleman model

Simple case of Diffusive limit of kinetic equations. Two types of particles in a one dimensional setting moving with speeds c and $-c$.
Densities are u and v respectively. Dynamics is

$$
\left\{ \begin{array}{l} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\[2mm] \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{array} \right. \tag{4}
$$

for some interaction kernel $k(u,v) \ge 0$. Typical case $k = (u+v)^\alpha c^2$.
Write now the equations for $\rho = u + v$ and $j = c(u-v)$ and pass to the limit $c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

$$
\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right), \tag{5}
$$

which is the FDE with $m = 1 - \alpha$, cf. Lions Toscani, 1997. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

Yamabe problem. Elliptic

Standard Yamabe problem . We have a Riemannian manifold (M, g_0) in space dimension $n \geq 3$, Question: of finding another metric g in the conformal class of g_0 having constant scalar curvature.
Write the conformal relation as

$$g = u^{4/(n-2)} g_0$$

locally on M for some positive smooth function u. The conformal factor is $u^{4/(n-2)}$. Denote by $R = R_g$ and R_0 the scalar curvatures of the metrics g, g_0 resp. Write Δ_0 for the Laplace-Beltrami operator of g_0, we have the formula $R = -u^{-N} Lu$ on M, with $N = (n+2)/(n-2)$ and

$$Lu := \kappa \Delta_0 u - R_0 u, \quad \kappa = \frac{4(n-1)}{n-2}.$$

The Yamabe problem becomes then

$$\text{(6)} \qquad \Delta_0 u - \left(\frac{n-2}{4(n-1)}\right) R_0 u + \left(\frac{n-2}{4(n-1)}\right) R_g u^{(n+2)/(n-2)} = 0.$$

The equation should determine u (hence, g) when g_0, R_0 and R_g are known. In the standard case we take $M = \mathbb{R}^n$ and g_0 the standard metric, so that Δ_0 is the standard Laplacian, $R_0 = 0$, we take $R_g = 1$ and then we get the well-known semilinear elliptic equation with critical exponent.

Yamabe problem. Evolution

Evolution Yamabe flow is defined as an evolution equation for a family of metrics. Used as a tool to construct metrics of constant scalar curvature within a given conformal class. Wwe look for a one-parameter family $g_t(x) = g(x,t)$ of metrics solution of the evolution problem

$$\partial_t g = -R\, g, \quad g(0) = g_0 \quad \text{on } M. \tag{7}$$

It is easy to show that this is equivalent to the equation

$$\partial_t(u^N) = Lu, \quad u(0) = 1 \quad \text{on } \ M.$$

after rescaling the time variable. Let now (M, g_0) be $\mathbb{R}^n$ with the standard flat metric, so that $R_0 = 0$. Put $u^N = v$, $m = 1/N = (n-2)/(n+2) \in (0,1)$. Then

$$\partial_t v = Lv^m, \tag{8}$$

which is a fast diffusion equation with exponent $m_y \in (0,1)$ given by

$$m_y = \frac{n-2}{n+2}, \quad 1 - m_y = \frac{4}{n+2}.$$

If we now try separate variables solutions of the form $v(x,t) = (T-t)^\alpha f(x)$, then necessarily $\alpha = 1/(1-m_y) = (n+2)/4$, and $F = f^m$ satisfies the semilinear elliptic equation with critical exponent that models the stationary version:

$$\Delta F + \frac{n+2}{4} F^{\frac{n+2}{n-2}} = 0. \tag{9}$$

Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \operatorname{div}(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric g given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

  where Ric is the Ricci tensor and R the scalar curvature.
  This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1\,dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
  Daskalopoulos, Del Pino
  Di Bendedetto, Diller
  Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \operatorname{div}(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric g given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

 where Ric is the Ricci tensor and R the scalar curvature.
 This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1\,dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
 Daskalopoulos, Del Pino
 Di Bendedetto, Diller
 Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \mathrm{div}\,(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric g given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

where Ric is the Ricci tensor and R the scalar curvature.
This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1 dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
Daskalopoulos, Del Pino
Di Bendedetto, Diller
Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \operatorname{div}(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric g given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

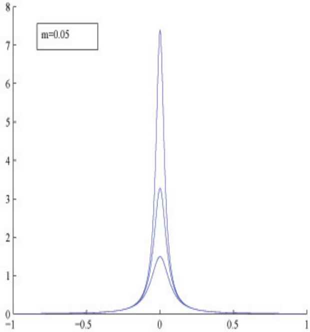
  where Ric is the Ricci tensor and R the scalar curvature.
  This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1 dx_2$.

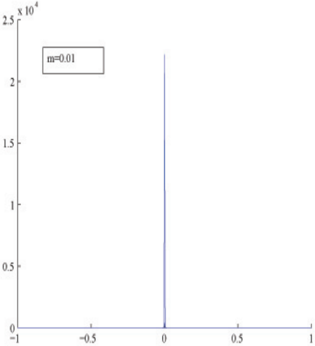
- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
  Daskalopoulos, Del Pino
  Di Bendedetto, Diller
  Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

Pictures

About fast diffusion in the limit





- Evolution of the ZKB solutions; dimension $n = 2$.
  exponent near $m = 0$

Functional Analysis Program

Main facts

- Existence of an evolution semigroup.

$$u_0 \mapsto S_t(u_0) = u(t)$$

  A key issue is the choice of functional space.
  $X = L^1(\mathbb{R}^n)$ (Brezis, Benilan, Crandall, 1971)
  $Y = L^1_{loc}(\mathbb{R}^n)$ (Herrero, Pierre 1985)
  $M =$ Locally bounded measures (Pierre, 1987; Dalhberg - Kenig 1988)
  $B =$ (possibly locally unbounded) Borel mesaures (Chasseigne-Vazquez ARMA 2002)

- Positivity. Nonnegative data produce positive solutions.

- "Smoothing effect": In *many cases* $L^p \rightarrow L^q$ with $q > p$. Then solutions are C^∞ smooth. In other cases, things go wrong (things=Functional Analysis)

- Theory for two signs is still poorly understood.
  Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa)

The good and bad range

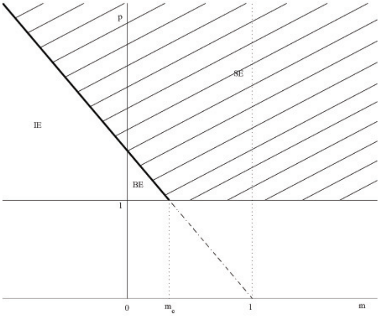


Figure 1. The (m, p) diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1 - m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n - 2)/(n + 2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

The good and bad range

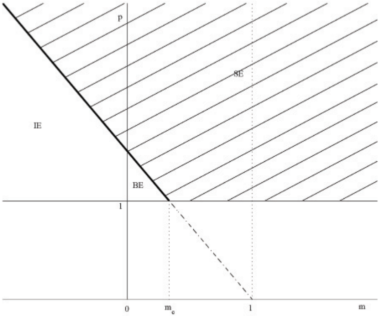


Figure 1. The (m, p) diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1-m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n-2)/(n+2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

The good and bad range

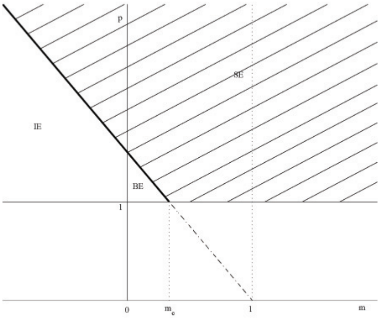


Figure 1. The (m, p) diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1 - m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n-2)/(n+2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

The good and bad range II

- Smoothing effect means that data in L^p imply that the weak solution is in L^∞ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in L^p imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

The good and bad range II

- Smoothing effect means that data in L^p imply that the weak solution is in L^∞ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in L^p imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

The good and bad range II

- Smoothing effect means that data in L^p imply that the weak solution is in L^∞ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in L^p imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

The good and bad range II

- Smoothing effect means that data in L^p imply that the weak solution is in L^∞ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in L^p imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

The good and bad range II

- Smoothing effect means that data in L^p imply that the weak solution is in L^∞ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in L^p imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

The good and bad range II

- Smoothing effect means that data in L^p imply that the weak solution is in L^∞ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in L^p imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

---

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

The good and bad range III

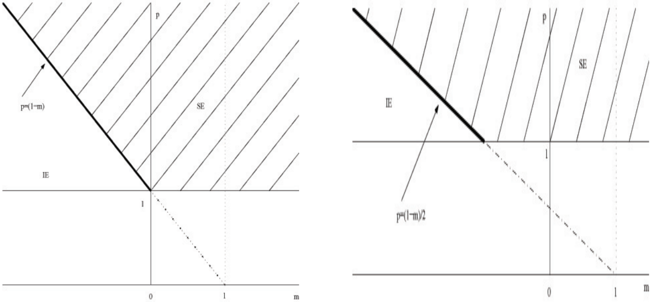


Figure 2. Left: (m, p) diagram for the PME/FDE in dimension $n = 2$
Right: (m, p) diagram for the PME/FDE in dimension $n = 1$

- There is existence and non-uniqueness if $n = 1$ and $-1 < m < 0$

The Fast Diffusion Equation is a long story.

Let us put the **End** here

Thank you

♠ ♠ ♠

Gracias, Grazie

The Fast Diffusion Equation is a long story.

Let us put the **End** here

Thank you

♠ ♠ ♠

Gracias, Grazie

The Fast Diffusion Equation is a long story.

Let us put the **End** here

Thank you

♠ ♠ ♠

Gracias, Grazie

The Fast Diffusion Equation is a long story.

Let us put the **End** here

Thank you

♠ ♠ ♠

Gracias, Grazie

The Fast Diffusion Equation is a long story.

Let us put the **End** here

Thank you

♠ ♠ ♠

Gracias, Grazie

The theory of nonlinear diffusion with fractional operators

JUAN LUIS VÁZQUEZ

Departamento de Matemáticas
Universidad Autónoma de Madrid
and Royal Academy of Sciences

CIME Summer Course

"Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions"

Cetraro, Italia July 4, 2016

Outline

Outline

1. **Linear and Nonlinear Diffusion**
   - Nonlinear equations
2. **Fractional diffusion**
3. **Nonlinear Fractional diffusion models**
   - Model I. A potential Fractional diffusion
   - Main estimates for this model

Outline

Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space. Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the stock market,...

These equations have occupied a large part of my research since 1980.

- The mathematical study of diffusion starts with the Heat Equation,

$$\boxed{u_t = \Delta u}$$

a linear example of immense influence in Science.

- The Heat Equation has produced a huge number of concepts, techniques and connections for pure and applied science, for analysts, probabilists, computational people and geometers, for physicists and engineers, and lately in finance and the social sciences.

- Today educated people talk about the Gaussian function, separation of variables, Fourier analysis, spectral decomposition, Dirichlet forms, Maximum Principles, Brownian motion, generation of semigroups, positive operators in Banach spaces, entropy dissipation, ...

Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space. Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the stock market,...

These equations have occupied a large part of my research since 1980.

- The mathematical study of diffusion starts with the Heat Equation,

$$\boxed{u_t = \Delta u}$$

a linear example of immense influence in Science.

- The Heat Equation has produced a huge number of concepts, techniques and connections for pure and applied science, for analysts, probabilists, computational people and geometers, for physicists and engineers, and lately in finance and the social sciences.

- Today educated people talk about the Gaussian function, separation of variables, Fourier analysis, spectral decomposition, Dirichlet forms, Maximum Principles, Brownian motion, generation of semigroups, positive operators in Banach spaces, entropy dissipation, ...

Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space. Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the stock market,...

These equations have occupied a large part of my research since 1980.

- The mathematical study of diffusion starts with the Heat Equation,

$$\boxed{u_t = \Delta u}$$

a linear example of immense influence in Science.

- The Heat Equation has produced a huge number of concepts, techniques and connections for pure and applied science, for analysts, probabilists, computational people and geometers, for physicists and engineers, and lately in finance and the social sciences.

- Today educated people talk about the Gaussian function, separation of variables, Fourier analysis, spectral decomposition, Dirichlet forms, Maximum Principles, Brownian motion, generation of semigroups, positive operators in Banach spaces, entropy dissipation, ...

Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space. Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the stock market,...

These equations have occupied a large part of my research since 1980.

- The mathematical study of diffusion starts with the Heat Equation,

$$\boxed{u_t = \Delta u}$$

a linear example of immense influence in Science.

- The Heat Equation has produced a huge number of concepts, techniques and connections for pure and applied science, for analysts, probabilists, computational people and geometers, for physicists and engineers, and lately in finance and the social sciences.
- Today educated people talk about the Gaussian function, separation of variables, Fourier analysis, spectral decomposition, Dirichlet forms, Maximum Principles, Brownian motion, generation of semigroups, positive operators in Banach spaces, entropy dissipation, ...

Nonlinear equations

- The heat example is generalized into the theory of linear parabolic equations, which is nowadays a basic topic in any advanced study of PDEs.
- However, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very not-linear. A general model of nonlinear diffusion takes the divergence form

$$\boxed{\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)}$$

  with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin et al.)
- In this generality the mathematical theory is too rich to admit a simple description. This includes the big areas of Nonlinear Diffusion and Reaction Diffusion, where I have been working.
- Reference works. Books by Ladyzhenskaya-Solonnikov-Uraltseva, Friedman, Smoller,... But they are only basic reference.

Nonlinear equations

- The heat example is generalized into the theory of linear parabolic equations, which is nowadays a basic topic in any advanced study of PDEs.
- However, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very not-linear. A general model of nonlinear diffusion takes the divergence form

$$\boxed{\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)}$$

  with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin et al.)
- In this generality the mathematical theory is too rich to admit a simple description. This includes the big areas of Nonlinear Diffusion and Reaction Diffusion, where I have been working.
- Reference works. Books by Ladyzhenskaya-Solonnikov-Uraltseva, Friedman, Smoller,... But they are only basic reference.

Nonlinear equations

- The heat example is generalized into the theory of linear parabolic equations, which is nowadays a basic topic in any advanced study of PDEs.
- However, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very not-linear. A general model of nonlinear diffusion takes the divergence form

$$\boxed{\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)}$$

  with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin et al.)
- In this generality the mathematical theory is too rich to admit a simple description. This includes the big areas of Nonlinear Diffusion and Reaction Diffusion, where I have been working.
- Reference works. Books by Ladyzhenskaya-Solonnikov-Uraltseva, Friedman, Smoller,... But they are only basic reference.

Nonlinear heat flows

- Many specific examples, now considered the "classical nonlinear diffusion models", have been investigated to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins
- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water), Hele-Shaw Problem (potential flow in a thin layer between solid plates), Porous Medium Equation: $u_t = \Delta(u^m)$, Evolution P-Laplacian Eqn: $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.

→ follow the talks by Manuel

♠

Nonlinear heat flows

- Many specific examples, now considered the “classical nonlinear diffusion models”, have been investigated to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins
- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water), Hele-Shaw Problem (potential flow in a thin layer between solid plates), Porous Medium Equation: $u_t = \Delta(u^m)$, Evolution P-Laplacian Eqn: $u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.

→ follow the talks by Manuel

Nonlinear heat flows

- Many specific examples, now considered the “classical nonlinear diffusion models”, have been investigated to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins
- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water), Hele-Shaw Problem (potential flow in a thin layer between solid plates), Porous Medium Equation: $u_t = \Delta(u^m)$, Evolution P-Laplacian Eqn: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.

→ follow the talks by Manuel

♠

Nonlinear heat flows

- Many specific examples, now considered the “classical nonlinear diffusion models”, have been investigated to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins
- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water), Hele-Shaw Problem (potential flow in a thin layer between solid plates), Porous Medium Equation: $u_t = \Delta(u^m)$, Evolution P-Laplacian Eqn: $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.

→ follow the talks by Manuel

♠

Outline

Recent Direction. Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit, the Brownian motion, by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

- The Basic evolution equation

$$\boxed{u_t + (-\Delta)^s u = 0}$$

- Intense work in Stochastic Processes for some decades, but not in Analysis of PDEs until 10 years ago, initiated around Prof. Caffarelli in Texas.

- A basic theory and survey for PDE people: M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1 [math.AP], to appear.

  Precedent : B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629–650.

Recent Direction. Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit, the Brownian motion, by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

- The Basic evolution equation

$$\boxed{u_t + (-\Delta)^s u = 0}$$

- Intense work in Stochastic Processes for some decades, but not in Analysis of PDEs until 10 years ago, initiated around Prof. Caffarelli in Texas.

- A basic theory and survey for PDE people: M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1 [math.AP], to appear.

  Precedent : B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629–650.

Recent Direction. Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit, the Brownian motion, by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

- The Basic evolution equation

$$\boxed{u_t + (-\Delta)^s u = 0}$$

- Intense work in Stochastic Processes for some decades, but not in Analysis of PDEs until 10 years ago, initiated around Prof. Caffarelli in Texas.

- A basic theory and survey for PDE people: M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1 [math.AP], to appear.

  Precedent : B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629–650.

The fractional Laplacian operator

- **Different formulas for fractional Laplacian operator**.
  We assume that the space variable $x \in \mathbb{R}^n$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi)$$

- Singular integral operator:

$$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy$$

  With this definition, it is the inverse of the Riesz integral operator $(-\Delta)^{-s} u$. This one has kernel $C_1 |x - y|^{n-2s}$, which is not integrable.

- Take the random walk for Lévy processes:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n$$

  where P_{ik} denotes the transition function which has a . tail (i.e, power decay with the distance $|i - k|$). In the limit you get an operator A as the infinitesimal generator of a Levy process: if X_t is the isotropic α-stable Lévy process we have

$$Au(x) = \lim_{h \to 0} \mathbb{E}(u(x) - u(x + X_h))$$

The fractional Laplacian operator

- **Different formulas for fractional Laplacian operator**.
  We assume that the space variable $x \in \mathbb{R}^n$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)$$

- Singular integral operator:

$$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy$$

  With this definition, it is the inverse of the Riesz integral operator $(-\Delta)^{-s}u$. This one has kernel $C_1|x-y|^{n-2s}$, which is not integrable.

- Take the random walk for Lévy processes:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n$$

  where P_{ik} denotes the transition function which has a . tail (i.e, power decay with the distance $|i-k|$). In the limit you get an operator A as the infinitesimal generator of a Levy process: if X_t is the isotropic α-stable Lévy process we have

$$Au(x) = \lim_{h \to 0} \mathbb{E}(u(x) - u(x + X_h))$$

The fractional Laplacian operator

- **Different formulas for fractional Laplacian operator.**
  We assume that the space variable $x \in \mathbb{R}^n$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:
  $$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)$$
- Singular integral operator:
  $$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy$$
  With this definition, it is the inverse of the Riesz integral operator $(-\Delta)^{-s}u$. This one has kernel $C_1|x-y|^{n-2s}$, which is not integrable.
- Take the random walk for Lévy processes:
  $$u_j^{n+1} = \sum_k P_{jk} u_k^n$$
  where P_{ik} denotes the transition function which has a . tail (i.e, power decay with the distance $|i-k|$). In the limit you get an operator A as the infinitesimal generator of a Levy process: if X_t is the isotropic α-stable Lévy process we have
  $$Au(x) = \lim_{h \to 0} \mathbb{E}(u(x) - u(x + X_h))$$

The fractional Laplacian operator II

- The α-harmonic extension: Find first the solution of the $(n+1)$ problem

$$\nabla \cdot (y^{1-\alpha}\nabla v) = 0 \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+; \quad v(x,0) = u(x), \quad x \in \mathbb{R}^n.$$

Then, putting $\alpha = 2s$ we have

$$(-\Delta)^s u(x) = -C_\alpha \lim_{y\to 0} y^{1-\alpha} \frac{\partial v}{\partial y}$$

When $s = 1/2$ i.e. $\alpha = 1$, the extended function v is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^n$. It was proposed in PDEs by Caffarelli and Silvestre.

Remark. In $\mathbb{R}^n$ all these versions are equivalent. In a bounded domain we have to re-examine all of them. Three main alternatives are studied in probability and PDEs, corresponding to different options about what happens to particles at the boundary or what is the domain of the functionals.

References. Books by Landkof (1966-72), Stein (1970), Davies (1996).

The fractional Laplacian operator II

- The α-harmonic extension: Find first the solution of the $(n+1)$ problem

$$\nabla \cdot (y^{1-\alpha} \nabla v) = 0 \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+; \quad v(x,0) = u(x), \quad x \in \mathbb{R}^n.$$

Then, putting $\alpha = 2s$ we have

$$(-\Delta)^s u(x) = -C_\alpha \lim_{y \to 0} y^{1-\alpha} \frac{\partial v}{\partial y}$$

When $s = 1/2$ i.e. $\alpha = 1$, the extended function v is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^n$. It was proposed in PDEs by Caffarelli and Silvestre.

Remark. In $\mathbb{R}^n$ all these versions are equivalent. In a bounded domain we have to re-examine all of them. Three main alternatives are studied in probability and PDEs, corresponding to different options about what happens to particles at the boundary or what is the domain of the functionals.

References. Books by Landkof (1966-72), Stein (1970), Davies (1996).

The fractional Laplacian operator II

- The α-harmonic extension: Find first the solution of the $(n+1)$ problem

$$\nabla\cdot(y^{1-\alpha}\nabla v)=0 \quad (x,y)\in\mathbb{R}^n\times\mathbb{R}_+; \quad v(x,0)=u(x), \quad x\in\mathbb{R}^n.$$

Then, putting $\alpha=2s$ we have

$$(-\Delta)^s u(x)=-C_\alpha \lim_{y\to 0} y^{1-\alpha}\frac{\partial v}{\partial y}$$

When $s=1/2$ i.e. $\alpha=1$, the extended function v is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x\in\mathbb{R}^n$. It was proposed in PDEs by Caffarelli and Silvestre.

Remark. In $\mathbb{R}^n$ all these versions are equivalent. In a bounded domain we have to re-examine all of them. Three main alternatives are studied in probability and PDEs, corresponding to different options about what happens to particles at the boundary or what is the domain of the functionals.

References. Books by Landkof (1966-72), Stein (1970), Davies (1996).

Outline

Nonlocal nonlinear diffusion model I

- The model arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \geq 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.

- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  See PME Book for these and other applications (around 20!).

Nonlocal nonlinear diffusion model I

- The model arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \ge 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.

- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  See PME Book for these and other applications (around 20!).

Nonlocal nonlinear diffusion model I

- The model arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \geq 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.
- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  See PME Book for these and other applications (around 20!).

Nonlocal nonlinear diffusion model I

- The model arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \ge 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.

- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  See PME Book for these and other applications (around 20!).

Nonlocal diffusion model. The problem

- The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel
$$k(x, y) = c|x-y|^{-(n-2s)}$$
(i.e., a Riesz operator). We have $(-\Delta)^s p = u$.
- The diffusion model with nonlocal effects is thus given by the system
$$\boxed{u_t = \nabla \cdot (u\,\nabla p), \quad p = \mathcal{K}(u).} \tag{1}$$
where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$. $u_t = \nabla \cdot (u\,\nabla(-\Delta)^{-s}u)$
- The problem is posed for $x \in \mathbb{R}^n$, $n \ge 1$, and $t > 0$, and we give initial conditions
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$
where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

Papers and surveys by us and others are available, see below

Nonlocal diffusion model. The problem

- The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel

$$k(x,y) = c|x-y|^{-(n-2s)}$$

(i.e., a Riesz operator). We have $(-\Delta)^s p = u$.

- The diffusion model with nonlocal effects is thus given by the system

$$\boxed{u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}(u).} \tag{1}$$

where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$. $u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u)$

- The problem is posed for $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

Papers and surveys by us and others are available, see below

Nonlocal diffusion model. The problem

- The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel
$$k(x,y) = c|x-y|^{-(n-2s)}$$
(i.e., a Riesz operator). We have $(-\Delta)^s p = u$.
- The diffusion model with nonlocal effects is thus given by the system
$$\boxed{u_t = \nabla\cdot(u\,\nabla p), \quad p = \mathcal{K}(u).} \tag{1}$$
where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$. $u_t = \nabla\cdot(u\,\nabla(-\Delta)^{-s}u)$
- The problem is posed for $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and we give initial conditions
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$
where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

Papers and surveys by us and others are available, see below

Nonlocal diffusion model. The problem

- The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel

$$k(x,y) = c|x-y|^{-(n-2s)}$$

(i.e., a Riesz operator). We have $(-\Delta)^s p = u$.

- The diffusion model with nonlocal effects is thus given by the system

$$\boxed{u_t = \nabla \cdot (u\,\nabla p), \quad p = \mathcal{K}(u).} \tag{1}$$

where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$. $u_t = \nabla \cdot (u\,\nabla(-\Delta)^{-s}u)$

- The problem is posed for $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

Papers and surveys by us and others are available, see below

Nonlocal diffusion model

- The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory as a large number works on elliptic equations, mainly of the linear or semilinear type (Caffarelli school; Bass, Kassmann, and others)
- There are many works on the subject. Here is a good reference to fractional elliptic work by a young Spanish author

  Xavier Ros-Otón. *Nonlocal elliptic equations in bounded domains: a survey*, Preprint in arXiv:1504.04099 [math.AP].

Nonlocal diffusion Model I. Applications

- Modeling dislocation dynamics as a continuum. This has been studied by P. Biler, G. Karch, and R. Monneau (2008), and then other collaborators, following old modeling by A. K. Head on *Dislocation group dynamics II. Similarity solutions of the continuum approximation.* (1972).
  This is a one-dimensional model. By integration in x they introduce viscosity solutions a la Crandall-Evans-Lions. Uniqueness holds.

- Equations of the more general form $u_t = \nabla \cdot (\sigma(u)\nabla \mathcal{L}u)$ have appeared recently in a number of applications in particle physics. Thus, Giacomin and Lebowitz (J. Stat. Phys. (1997)) consider a lattice gas with general short-range interactions and a Kac potential, and passing to the limit, the macroscopic density profile $\rho(r,t)$ satisfies the equation
  $$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho)\nabla \frac{\delta F(\rho)}{\delta \rho}\right]$$
  See also (GL2) and the review paper (GLP). The model is used to study phase segregation in (GLM, 2000).

- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

Nonlocal diffusion Model I. Applications

- Modeling dislocation dynamics as a continuum. This has been studied by P. Biler, G. Karch, and R. Monneau (2008), and then other collaborators, following old modeling by A. K. Head on *Dislocation group dynamics II. Similarity solutions of the continuum approximation.* (1972).
  This is a one-dimensional model. By integration in x they introduce viscosity solutions a la Crandall-Evans-Lions. Uniqueness holds.
- Equations of the more general form $u_t = \nabla \cdot (\sigma(u)\nabla \mathcal{L} u)$ have appeared recently in a number of applications in particle physics. Thus, Giacomin and Lebowitz (J. Stat. Phys. (1997)) consider a lattice gas with general short-range interactions and a Kac potential, and passing to the limit, the macroscopic density profile $\rho(r,t)$ satisfies the equation
  $$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho)\nabla \frac{\delta F(\rho)}{\delta \rho}\right]$$
  See also (GL2) and the review paper (GLP). The model is used to study phase segregation in (GLM, 2000).
- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

Nonlocal diffusion Model I. Applications

- Modeling dislocation dynamics as a continuum. This has been studied by P. Biler, G. Karch, and R. Monneau (2008), and then other collaborators, following old modeling by A. K. Head on *Dislocation group dynamics II. Similarity solutions of the continuum approximation.* (1972).
  This is a one-dimensional model. By integration in x they introduce viscosity solutions a la Crandall-Evans-Lions. Uniqueness holds.
- Equations of the more general form $u_t = \nabla \cdot (\sigma(u)\nabla \mathcal{L}u)$ have appeared recently in a number of applications in particle physics. Thus, Giacomin and Lebowitz (J. Stat. Phys. (1997)) consider a lattice gas with general short-range interactions and a Kac potential, and passing to the limit, the macroscopic density profile $\rho(r,t)$ satisfies the equation
  $$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho)\nabla \frac{\delta F(\rho)}{\delta \rho}\right]$$
  See also (GL2) and the review paper (GLP). The model is used to study phase segregation in (GLM, 2000).
- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

Extreme cases

- If we take $s = 0$, $\mathcal{K} =$ the identity operator, we get the standard porous medium equation, whose behaviour is well-known, see references later.
- In the other end of the s interval, when $s = 1$ and we take $\mathcal{K} = -\Delta$ we get

$$u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u. \tag{3}$$

In one dimension this leads to $u_t = u_x p_x - u^2$, $p_{xx} = -u$. In terms of $v = -p_x = \int u\,dx$ we have

$$v_t = u p_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + v v_x = 0$ which generates shocks in finite time but only if we allow for u to have two signs.

- HYDRODYNAMIC LIMIT. The case $s = 1$ in several dimensions is more interesting because it does not reduce to a simple Burgers equation.

$$u_t = \nabla \cdot (u\,\nabla p) = \nabla u \cdot \nabla p - u^2; \qquad , \qquad p = (-\Delta)^{-1} u\,,$$

Applications in superconductivity and superfluidity, see paper with Serfaty and below.

Extreme cases

- If we take $s = 0$, $\mathcal{K} =$ the identity operator, we get the standard porous medium equation, whose behaviour is well-known, see references later.
- In the other end of the s interval, when $s = 1$ and we take $\mathcal{K} = -\Delta$ we get

$$u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u. \tag{3}$$

In one dimension this leads to $u_t = u_x p_x - u^2, p_{xx} = -u$. In terms of $v = -p_x = \int u\,dx$ we have

$$v_t = up_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + vv_x = 0$ which generates shocks in finite time but only if we allow for u to have two signs.

- HYDRODYNAMIC LIMIT. The case $s = 1$ in several dimensions is more interesting because it does not reduce to a simple Burgers equation.

$$u_t = \nabla \cdot (u \nabla p) = \nabla u \cdot \nabla p - u^2; \qquad , \qquad p = (-\Delta)^{-1} u\,,$$

Applications in superconductivity and superfluidity, see paper with Serfaty and below.

Extreme cases

- If we take $s = 0$, $\mathcal{K} =$ the identity operator, we get the standard porous medium equation, whose behaviour is well-known, see references later.
- In the other end of the s interval, when $s = 1$ and we take $\mathcal{K} = -\Delta$ we get

$$u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u. \tag{3}$$

In one dimension this leads to $u_t = u_x p_x - u^2, p_{xx} = -u$. In terms of $v = -p_x = \int u\, dx$ we have

$$v_t = u p_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + v v_x = 0$ which generates shocks in finite time but only if we allow for u to have two signs.

- HYDRODYNAMIC LIMIT. The case $s = 1$ in several dimensions is more interesting because it does not reduce to a simple Burgers equation.

$$u_t = \nabla \cdot (u \nabla p) = \nabla u \cdot \nabla p - u^2; \qquad , \qquad p = (-\Delta)^{-1} u\,,$$

Applications in superconductivity and superfluidity, see paper with Serfaty and below.

Our first project. Results

- Existence of weak energy solutions and property of finite propagation
  L. Caffarelli and J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Rational Mech. Anal. 2011; arXiv 2010.

- Existence of self-similar profiles, renormalized Fokker-Planck equation and entropy-based proof of stabilization
  L. Caffarelli and J. L. Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, appeared in Discrete Cont. Dynam. Systems, 2011; arXiv 2010.

- Regularity in three levels: $L^1 \to L^2$, $L^2 \to L^\infty$, and bounded implies C^α
  L. Caffarelli, F. Soria, and J. L. Vázquez, *Regularity of porous medium equation with fractional diffusion*, J. Eur. Math. Soc. (JEMS) **15** 5 (2013), 1701–1746. The very subtle case $s = 1/2$ is solved in a new paper L. Caffarelli, and J. L. Vázquez, appeared in ArXiv and as Newton Institute Preprint, 2014

Our first project. Results

- Existence of weak energy solutions and property of finite propagation
L. Caffarelli and J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Rational Mech. Anal. 2011; arXiv 2010.

- Existence of self-similar profiles, renormalized Fokker-Planck equation and entropy-based proof of stabilization
L. Caffarelli and J. L. Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, appeared in Discrete Cont. Dynam. Systems, 2011; arXiv 2010.

- Regularity in three levels: $L^1 \to L^2$, $L^2 \to L^\infty$, and bounded implies C^α
L. Caffarelli, F. Soria, and J. L. Vázquez, *Regularity of porous medium equation with fractional diffusion*, J. Eur. Math. Soc. (JEMS) **15** 5 (2013), 1701–1746. The very subtle case $s = 1/2$ is solved in a new paper L. Caffarelli, and J. L. Vázquez, appeared in ArXiv and as Newton Institute Preprint, 2014

Our first project. Results

- Existence of weak energy solutions and property of finite propagation
  L. Caffarelli and J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Rational Mech. Anal. 2011; arXiv 2010.

- Existence of self-similar profiles, renormalized Fokker-Planck equation and entropy-based proof of stabilization
  L. Caffarelli and J. L. Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, appeared in Discrete Cont. Dynam. Systems, 2011; arXiv 2010.

- Regularity in three levels: $L^1 \to L^2$, $L^2 \to L^\infty$, and bounded implies C^α
  L. Caffarelli, F. Soria, and J. L. Vázquez, *Regularity of porous medium equation with fractional diffusion*, J. Eur. Math. Soc. (JEMS) **15** 5 (2013), 1701–1746. The very subtle case $s = 1/2$ is solved in a new paper L. Caffarelli, and J. L. Vázquez, appeared in ArXiv and as Newton Institute Preprint, 2014

Our first project. Results

- Existence of weak energy solutions and property of finite propagation
L. Caffarelli and J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Rational Mech. Anal. 2011; arXiv 2010.

- Existence of self-similar profiles, renormalized Fokker-Planck equation and entropy-based proof of stabilization
L. Caffarelli and J. L. Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, appeared in Discrete Cont. Dynam. Systems, 2011; arXiv 2010.

- Regularity in three levels: $L^1 \to L^2$, $L^2 \to L^\infty$, and bounded implies C^α
L. Caffarelli, F. Soria, and J. L. Vázquez, *Regularity of porous medium equation with fractional diffusion*, J. Eur. Math. Soc. (JEMS) **15** 5 (2013), 1701–1746. The very subtle case $s = 1/2$ is solved in a new paper L. Caffarelli, and J. L. Vázquez, appeared in ArXiv and as Newton Institute Preprint, 2014

Our first project. Results

- Existence of weak energy solutions and property of finite propagation
L. Caffarelli and J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Rational Mech. Anal. 2011; arXiv 2010.

- Existence of self-similar profiles, renormalized Fokker-Planck equation and entropy-based proof of stabilization
L. Caffarelli and J. L. Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, appeared in Discrete Cont. Dynam. Systems, 2011; arXiv 2010.

- Regularity in three levels: $L^1 \to L^2$, $L^2 \to L^\infty$, and bounded implies C^α
L. Caffarelli, F. Soria, and J. L. Vázquez, *Regularity of porous medium equation with fractional diffusion*, J. Eur. Math. Soc. (JEMS) **15** 5 (2013), 1701–1746. The very subtle case $s = 1/2$ is solved in a new paper L. Caffarelli, and J. L. Vázquez, appeared in ArXiv and as Newton Institute Preprint, 2014

Our first project. Results

- Existence of weak energy solutions and property of finite propagation
L. Caffarelli and J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Rational Mech. Anal. 2011; arXiv 2010.

- Existence of self-similar profiles, renormalized Fokker-Planck equation and entropy-based proof of stabilization
L. Caffarelli and J. L. Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, appeared in Discrete Cont. Dynam. Systems, 2011; arXiv 2010.

- Regularity in three levels: $L^1 \to L^2$, $L^2 \to L^\infty$, and bounded implies C^α
L. Caffarelli, F. Soria, and J. L. Vázquez, *Regularity of porous medium equation with fractional diffusion*, J. Eur. Math. Soc. (JEMS) **15** 5 (2013), 1701–1746. The very subtle case $s = 1/2$ is solved in a new paper L. Caffarelli, and J. L. Vázquez, appeared in ArXiv and as Newton Institute Preprint, 2014

Our first project. Results

- Limit $s \to 1$ S. Serfaty, and J. L. Vazquez, *Hydrodynamic Limit of Nonlinear. Diffusion with Fractional Laplacian Operators*, Calc. Var. PDEs **526**, online; arXiv:1205.6322v1 [math.AP], may 2012.

  A presentation of this topic and results for the Proceedings from the Abel Symposium 2010.

- J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators.* in "Nonlinear partial differential equations: the Abel Symposium 2010", Holden, Helge & Karlsen, Kenneth H. eds., Springer, 2012. Pp. 271–298.

  Last reference is proving that the selfsimilar solutions of Barenblatt type (Caffareli-Vazquez, Biler-Karch-Monneau) are attractors with calculated rate in 1D

- *Exponential Convergence Towards Stationary States for the 1D Porous Medium Equation with Fractional Pressure*, by J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vázquez. JDE, 2015.
  Uses entropy analysis. Problem is open (and quite interesting in higher dimensions).

Our first project. Results

- Limit $s \to 1$ S. Serfaty, and J. L. Vazquez, *Hydrodynamic Limit of Nonlinear. Diffusion with Fractional Laplacian Operators*, Calc. Var. PDEs **526**, online; arXiv:1205.6322v1 [math.AP], may 2012.

  A presentation of this topic and results for the Proceedings from the Abel Symposium 2010.

- J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators.* in "Nonlinear partial differential equations: the Abel Symposium 2010", Holden, Helge & Karlsen, Kenneth H. eds., Springer, 2012. Pp. 271–298.

  Last reference is proving that the selfsimilar solutions of Barenblatt type (Caffareli-Vazquez, Biler-Karch-Monneau) are attractors with calculated rate in 1D

- *Exponential Convergence Towards Stationary States for the 1D Porous Medium Equation with Fractional Pressure*, by J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vázquez. JDE, 2015.
  Uses entropy analysis. Problem is open (and quite interesting in higher dimensions).

Our first project. Results

- Limit $s \to 1$ S. Serfaty, and J. L. Vazquez, *Hydrodynamic Limit of Nonlinear. Diffusion with Fractional Laplacian Operators*, Calc. Var. PDEs **526**, online; arXiv:1205.6322v1 [math.AP], may 2012.

  A presentation of this topic and results for the Proceedings from the Abel Symposium 2010.

- J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators*. in "Nonlinear partial differential equations: the Abel Symposium 2010", Holden, Helge & Karlsen, Kenneth H. eds., Springer, 2012. Pp. 271–298.

  Last reference is proving that the selfsimilar solutions of Barenblatt type (Caffareli-Vazquez, Biler-Karch-Monneau) are attractors with calculated rate in 1D

- *Exponential Convergence Towards Stationary States for the 1D Porous Medium Equation with Fractional Pressure*, by J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vázquez. JDE, 2015.
  Uses entropy analysis. Problem is open (and quite interesting in higher dimensions).

Main estimates for this model

We recall that the equation of M1 is $\partial_t u = \nabla \cdot (u \nabla K(u))$, posed in the whole space $\mathbb{R}^n$.

We consider $K = (-\Delta)^{-s}$ for some $0 < s < 1$ acting on Schwartz class functions defined in the whole space. It is a positive essentially self-adjoint operator. We let $H = K^{1/2} = (-\Delta)^{-s/2}$.

We do next formal calculations, assuming that $u \geq 0$ satisfies the required smoothness and integrability assumptions. This is to be justified later by approximation.

- Conservation of mass
$$\frac{d}{dt}\int u(x,t)\,dx = 0. \tag{4}$$
- First energy estimate:
$$\frac{d}{dt}\int u(x,t)\log u(x,t)\,dx = -\int |\nabla H u|^2\,dx. \tag{5}$$
- Second energy estimate
$$\frac{d}{dt}\int |Hu(x,t)|^2\,dx = -2\int u|\nabla K u|^2\,dx. \tag{6}$$

Main estimates

- Conservation of positivity: $u_0 \geq 0$ implies that $u(t) \geq 0$ for all times.
- L^∞ estimate. We prove that the L^∞ norm does not increase in time.

  Proof. At a point of maximum of u at time $t = t_0$, say $x = 0$, we have

  $$u_t = \nabla u \cdot \nabla P + u\,\Delta K(u).$$

  The first term is zero, and for the second we have $-\Delta K = L$ where $L = (-\Delta)_q$ with $q = 1 - s$ so that

  $$\Delta K u(0) = -Lu(0) = -\int \frac{u(0) - u(y)}{|y|^{n+2(1-s)}}\,dy \leq 0.$$

  This concludes the proof.
- We did not find a clean comparison theorem, a form of the usual maximum principle is not proved for Model 1. Good comparion works for Model 2 to be presented below, actually, it helps produce a very nice theory.
- Finite propagation is true for model M1. Infinite propagation is true for model M2.

  $$\partial_t u + (-\Delta)^s u^m = 0,$$

  the most recent member of the family, that we love so much.

Boundedness

- Solutions are bounded in terms of data in L^p, $1 \le p \le \infty$.
  For Model 1 Use (the de Giorgi or the Moser) iteration technique on the Caffarelli-Silvestre extension as in Caffarelli-Vasseur.
  Or use energy estimates based on the properties of the quadratic and bilinear forms associated to the fractional operator, and then the iteration technique

- **Theorem (for M1)** *Let u be a weak solution the IVP for the FPME with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive constant C such that for every $t > 0$*

  $$\sup_{x \in \mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^n)} \tag{7}$$

  with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$. The constant C depends only on n and s.
  This theorem allows to extend the theory to data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$, with global existence of bounded weak solutions.

Boundedness

- Solutions are bounded in terms of data in L^p, $1 \le p \le \infty$.
  For Model 1 Use (the de Giorgi or the Moser) iteration technique on the Caffarelli-Silvestre extension as in Caffarelli-Vasseur.
  Or use energy estimates based on the properties of the quadratic and bilinear forms associated to the fractional operator, and then the iteration technique

- **Theorem (for M1)** *Let u be a weak solution the IVP for the FPME with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive constant C such that for every $t > 0$*

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^n)} \tag{7}$$

  with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$. The constant C depends only on n and s.
  This theorem allows to extend the theory to data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$, with global existence of bounded weak solutions.

Boundedness

- Solutions are bounded in terms of data in L^p, $1 \le p \le \infty$.
  For Model 1 Use (the de Giorgi or the Moser) iteration technique on the Caffarelli-Silvestre extension as in Caffarelli-Vasseur.
  Or use energy estimates based on the properties of the quadratic and bilinear forms associated to the fractional operator, and then the iteration technique

- **Theorem (for M1)** *Let u be a weak solution the IVP for the FPME with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive constant C such that for every $t > 0$*

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^n)} \tag{7}$$

  with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$. The constant C depends only on n and s.
  This theorem allows to extend the theory to data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$, with global existence of bounded weak solutions.

Energy and bilinear forms

- **Energy solutions:** The basis of the boundedness analysis is a property that goes beyond the definition of weak solution. We will review the formulas with attention to the constants that appear since this is not done in [CSV]. The general energy property is as follows: for any F smooth and such that $f = F'$ is bounded and nonnegative, we have for every $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}\int F(u(t_2))\,dx - \int F(u(t_1))\,dx \quad &= -\int_{t_1}^{t_2}\int \nabla[f(u)]u\nabla p\,dx\,dt = \\ &-\int_{t_1}^{t_2}\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx\,dt\end{aligned}$$

where h is a function satisfying $h'(u) = uf'(u)$. We can write the last integral as a bilinear form

$$\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx = \mathcal{B}_s(h(u), u)$$

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v, w) = C_{n,s}\iint \nabla v(x)\frac{1}{|x-y|^{n-2s}}\nabla w(y)\,dx\,dy\,. \qquad (8)$$

Energy and bilinear forms

- **Energy solutions:** The basis of the boundedness analysis is a property that goes beyond the definition of weak solution. We will review the formulas with attention to the constants that appear since this is not done in [CSV]. The general energy property is as follows: for any F smooth and such that $f = F'$ is bounded and nonnegative, we have for every $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned} \int F(u(t_2))\,dx - \int F(u(t_1))\,dx \quad &= -\int_{t_1}^{t_2}\int \nabla[f(u)]u\nabla p\,dx\,dt = \\ &-\int_{t_1}^{t_2}\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx\,dt \end{aligned}$$

where h is a function satisfying $h'(u) = uf'(u)$. We can write the last integral as a bilinear form

$$\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx = \mathcal{B}_s(h(u),u)$$

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v,w) = C_{n,s}\iint \nabla v(x)\frac{1}{|x-y|^{n-2s}}\nabla w(y)\,dx\,dy\,. \tag{8}$$

Energy and bilinear forms II

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v,w) = C_{n,s} \iint \nabla v(x) \tfrac{1}{|x-y|^{n-2s}} \nabla w(y)\, dx\, dy = \iint \mathcal{N}_{-s}(x,y) \nabla v(x) \nabla w(y)\, dx\, dy$$

  where $\mathcal{N}_{-s}(x,y) = C_{n,s}|x-y|^{-(n-2s)}$ is the kernel of operator $(-\Delta)^{-s}$.

- After some integrations by parts we also have

$$\mathcal{B}_s(v,w) = C_{n,1-s} \iint (v(x) - v(y)) \frac{1}{|x-y|^{n+2(1-s)}} (w(x) - w(y))\, dx\, dy \tag{9}$$

  since $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$.

- It is known (Stein) that $\mathcal{B}_s(u,u)$ is an equivalent norm for the fractional Sobolev space $W^{1-s,2}(\mathbb{R}^n)$.
  We will need in the proofs that $C_{n,1-s} \sim K_n(1-s)$ as $s \to 1$, for some constant K_n depending only on n.

Energy and bilinear forms II

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v,w) = C_{n,s} \iint \nabla v(x) \frac{1}{|x-y|^{n-2s}} \nabla w(y)\, dx\, dy = \iint \mathcal{N}_{-s}(x,y) \nabla v(x) \nabla w(y)\, dx\, dy$$

where $\mathcal{N}_{-s}(x,y) = C_{n,s}|x-y|^{-(n-2s)}$ is the kernel of operator $(-\Delta)^{-s}$.

- After some integrations by parts we also have

$$\mathcal{B}_s(v,w) = C_{n,1-s} \iint (v(x)-v(y)) \frac{1}{|x-y|^{n+2(1-s)}} (w(x)-w(y))\, dx\, dy \tag{9}$$

since $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$.

- It is known (Stein) that $\mathcal{B}_s(u,u)$ is an equivalent norm for the fractional Sobolev space $W^{1-s,2}(\mathbb{R}^n)$.
We will need in the proofs that $C_{n,1-s} \sim K_n(1-s)$ as $s \to 1$, for some constant K_n depending only on n.

Energy and bilinear forms II

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v,w) = C_{n,s} \iint \nabla v(x) \frac{1}{|x-y|^{n-2s}} \nabla w(y)\, dx\, dy = \iint \mathcal{N}_{-s}(x,y) \nabla v(x) \nabla w(y)\, dx\, dy$$

where $\mathcal{N}_{-s}(x,y) = C_{n,s}|x-y|^{-(n-2s)}$ is the kernel of operator $(-\Delta)^{-s}$.

- After some integrations by parts we also have

$$\mathcal{B}_s(v,w) = C_{n,1-s} \iint (v(x)-v(y)) \frac{1}{|x-y|^{n+2(1-s)}} (w(x)-w(y))\, dx\, dy \tag{9}$$

since $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$.

- It is known (Stein) that $\mathcal{B}_s(u,u)$ is an equivalent norm for the fractional Sobolev space $W^{1-s,2}(\mathbb{R}^n)$.
  We will need in the proofs that $C_{n,1-s} \sim K_n(1-s)$ as $s \to 1$, for some constant K_n depending only on n.

Additional and Recent work, open problems

- The asymptotic behaviour as $t \to \infty$ is a very interesting topic developed in a paper with Luis Caffarelli. Rates of convergence are found for in dimension $n = 1$ but they are not available for $n > 1$, they are tied to some functional inequalities that are not known.
- The study of the free boundary is in progress, but it is still open for small $s > 0$.
- The equation is generalized into $u_t = \nabla \cdot (u^{m-1}\nabla(-\Delta)^{-s}u)$ with $m > 1$. Recent work with D. Stan and F. del Teso shows that finite propagation is true for $m \geq 2$ and propagation is infinite is $m < 2$. This is quite different from the standard porous medium case $s = 0$, where $m = 1$ is the dividing value.
- Gradient flow in Wasserstein metrics is work by S. Lisini, E. Mainini and A. Segatti, just appeared in arXiv, *A gradient flow approach to the porous medium equation with fractional pressure.* Thanks to the Pavia people!
  Previous work by J. A. Carrillo et al. in $n = 1$.

Additional and Recent work, open problems

- The asymptotic behaviour as $t \to \infty$ is a very interesting topic developed in a paper with Luis Caffarelli. Rates of convergence are found for in dimension $n = 1$ but they are not available for $n > 1$, they are tied to some functional inequalities that are not known.
- The study of the free boundary is in progress, but it is still open for small $s > 0$.
- The equation is generalized into $u_t = \nabla \cdot (u^{m-1}\nabla(-\Delta)^{-s}u)$ with $m > 1$. Recent work with D. Stan and F. del Teso shows that finite propagation is true for $m \geq 2$ and propagation is infinite is $m < 2$. This is quite different from the standard porous medium case $s = 0$, where $m = 1$ is the dividing value.
- Gradient flow in Wasserstein metrics is work by S. Lisini, E. Mainini and A. Segatti, just appeared in arXiv, *A gradient flow approach to the porous medium equation with fractional pressure.* Thanks to the Pavia people!
  Previous work by J. A. Carrillo et al. in $n = 1$.

- The questions of uniqueness and comparison are solved in dimension $n = 1$ thanks to the trick of integration in space used by Biler, Karch, and Monneau. New tools are needed to make progress in several dimensions.
- Recent uniqueness results. Paper by X H Zhou, W L Xiao, J C Chen, *Fractional porous medium and mean field equations in Besov spaces*, EJDE 2014. They obtain local in time strong solutions in Besov spaces. Thus, for initial data in $B^{\alpha}_{1,\infty}$ if $1/2 \le s < 1$ and $\alpha > n + 1$ and $n \ge 2$.
- The problem in a bounded domain with Dirichlet or Neumann data has not been studied.
- Good numerical studies are needed.

- The questions of uniqueness and comparison are solved in dimension $n = 1$ thanks to the trick of integration in space used by Biler, Karch, and Monneau. New tools are needed to make progress in several dimensions.
- Recent uniqueness results. Paper by X H Zhou, W L Xiao, J C Chen, *Fractional porous medium and mean field equations in Besov spaces*, EJDE 2014. They obtain local in time strong solutions in Besov spaces. Thus, for initial data in $B^{\alpha}_{1,\infty}$ if $1/2 \le s < 1$ and $\alpha > n+1$ and $n \ge 2$.
- The problem in a bounded domain with Dirichlet or Neumann data has not been studied.
- Good numerical studies are needed.

- Most of the recent effort by me and collaborators has been devoted to the parallel model
$$u_t + (-\Delta)^s u^m = 0$$
which does not have an entropy theory. However, it generates a semigroup of contractions in $L^1(\mathbb{R}^n)$. This needs a whole new lecture

- The theory in bounded domains is being developed just now, I mention my collaborators M. Bonforte and Y. Sire, and among other groups, E. Valdinoci and coauthors (mostly elliptic).

- *(Go to arXiv, the consolation of the learning masses)*

- Most of the recent effort by me and collaborators has been devoted to the parallel model
$$u_t + (-\Delta)^s u^m = 0$$
which does not have an entropy theory. However, it generates a semigroup of contractions in $L^1(\mathbb{R}^n)$. This needs a whole new lecture

- The theory in bounded domains is being developed just now, I mention my collaborators M. Bonforte and Y. Sire, and among other groups, E. Valdinoci and coauthors (mostly elliptic).

- *(Go to arXiv, the consolation of the learning masses)*

- Most of the recent effort by me and collaborators has been devoted to the parallel model
$$u_t + (-\Delta)^s u^m = 0$$
which does not have an entropy theory. However, it generates a semigroup of contractions in $L^1(\mathbb{R}^n)$. This needs a whole new lecture

- The theory in bounded domains is being developed just now, I mention my collaborators M. Bonforte and Y. Sire, and among other groups, E. Valdinoci and coauthors (mostly elliptic).

- *(Go to arXiv, the consolation of the learning masses)*

- Most of the recent effort by me and collaborators has been devoted to the parallel model
$$u_t + (-\Delta)^s u^m = 0$$
which does not have an entropy theory. However, it generates a semigroup of contractions in $L^1(\mathbb{R}^n)$. This needs a whole new lecture

- The theory in bounded domains is being developed just now, I mention my collaborators M. Bonforte and Y. Sire, and among other groups, E. Valdinoci and coauthors (mostly elliptic).

- *(Go to arXiv, the consolation of the learning masses)*

The End

Thank you, Gracias, Grazie

The End

Thank you, Gracias, Grazie

The End

Thank you, Gracias, Grazie

Entropies for Nonlinear diffusion equations

JUAN LUIS VÁZQUEZ

Departamento de Matemáticas
Universidad Autónoma de Madrid
and Royal Academy of Sciences

CIME Summer Course

"Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions"

Cetraro, Italia July 4, 2016

Outline

1. **Estimates for the Heat Equations**
   - Heat Equation Methods

2. **Traditional porous medium**
   - Asymptotic behaviour
   - The Fast Diffusion Problem in $\mathbb{R}^N$

3. **Nonlinear Fractional diffusion**
   - Introduction to Fractional diffusion
   - Asymptotic behavior for the nonlocal HE / PME
   - Renormalized estimates

Outline

1. **Estimates for the Heat Equations**
   - Heat Equation Methods
2. **Traditional porous medium**
   - Asymptotic behaviour
   - The Fast Diffusion Problem in $\mathbb{R}^N$
3. **Nonlinear Fractional diffusion**
   - Introduction to Fractional diffusion
   - Asymptotic behavior for the nonlocal HE / PME
   - Renormalized estimates

Outline

Energy estimates

- We are going to use energy functions of different types to study the evolution of dissipation equations.
- The basic equation is the classical heat equation, but the scope is quite general. Our aim is not to establish the convergence of general solutions to the fundamental solution (which is well done by other methods), but a bit more: to find the speed of convergence. After change of variables (renormalization) this reads as rate of convergence to equilibrium and relies on important functional inequalities.
- The methods will apply to more general linear parabolic equations that generate semigroups. The method works for equations evolving on manifolds as a base space.
- Since 2000 we have been studying nonlinear diffusion equations.
- Nonlinear models: porous medium equation, fast diffusion equation,

  p-Laplacian evolution equation, chemotaxis system, thin films, ...

  plus

  fractional heat equation and fractional porous medium equations (since 2007), ...

Energy estimates

- We are going to use energy functions of different types to study the evolution of dissipation equations.
- The basic equation is the classical heat equation, but the scope is quite general. Our aim is not to establish the convergence of general solutions to the fundamental solution (which is well done by other methods), but a bit more: to find the speed of convergence. After change of variables (renormalization) this reads as rate of convergence to equilibrium and relies on important functional inequalities.
- The methods will apply to more general linear parabolic equations that generate semigroups. The method works for equations evolving on manifolds as a base space.
- Since 2000 we have been studying nonlinear diffusion equations.
- Nonlinear models: porous medium equation, fast diffusion equation, p-Laplacian evolution equation, chemotaxis system, thin films, ...

  plus

  fractional heat equation and fractional porous medium equations (since 2007), ...

Energy estimates

- We are going to use energy functions of different types to study the evolution of dissipation equations.
- The basic equation is the classical heat equation, but the scope is quite general. Our aim is not to establish the convergence of general solutions to the fundamental solution (which is well done by other methods), but a bit more: to find the speed of convergence. After change of variables (renormalization) this reads as rate of convergence to equilibrium and relies on important functional inequalities.
- The methods will apply to more general linear parabolic equations that generate semigroups. The method works for equations evolving on manifolds as a base space.
- Since 2000 we have been studying nonlinear diffusion equations.
- Nonlinear models: porous medium equation, fast diffusion equation, p-Laplacian evolution equation, chemotaxis system, thin films, ...

  plus

  fractional heat equation and fractional porous medium equations (since 2007), ...

Energy estimates

- We are going to use energy functions of different types to study the evolution of dissipation equations.
- The basic equation is the classical heat equation, but the scope is quite general. Our aim is not to establish the convergence of general solutions to the fundamental solution (which is well done by other methods), but a bit more: to find the speed of convergence. After change of variables (renormalization) this reads as rate of convergence to equilibrium and relies on important functional inequalities.
- The methods will apply to more general linear parabolic equations that generate semigroups. The method works for equations evolving on manifolds as a base space.
- Since 2000 we have been studying nonlinear diffusion equations.
- Nonlinear models: porous medium equation, fast diffusion equation,

  p-Laplacian evolution equation, chemotaxis system, thin films, ...

  plus

  fractional heat equation and fractional porous medium equations (since 2007), ...

Energy estimates for the Heat Equation

- Take the classical Heat Equation posed in the whole space $\mathbb{R}^N$ for $\tau > 0$:

$$u_\tau = \frac{1}{2}\Delta_y u$$

  with notation $u = u(y,\tau)$ that is useful as we will see. We know the (self-similar) fundamental solution, that is an attractor of its basin

$$U(y,\tau) = C\,\tau^{-N/2} e^{-y^2/2\tau}.$$

- First step: the logarithmic time-space rescaling

$$u(y,\tau) = v(x,t)\,(1+\tau)^{-N/2}, \quad y = x(1+\tau)^{1/2}, \quad t = \log(1+\tau),$$

  that leads to the well-known Fokker-Plank equation for $v(x,t)$:

$$v_t = \frac{1}{2}\Delta_x v + \frac{1}{2}\nabla\cdot(x\,v)$$

- If we now pass to the quotient $w = v/G$, where $G = c\,e^{-x^2/2}$ is the stationary state (Gaussiann kernel), to get the Ornstein-Uhlenbeck version

$$w_t = \frac{1}{2}G^{-1}\,\nabla\cdot\left(\,G\nabla w\,\right),$$

  a symmetrically weighted heat equation. The equivalence of these three equations is a main tool in Linear Diffusion and Semigroup Theory.

Energy estimates for the Heat Equation

- Take the classical Heat Equation posed in the whole space $\mathbb{R}^N$ for $\tau > 0$:

$$u_\tau = \frac{1}{2}\Delta_y u$$

with notation $u = u(y,\tau)$ that is useful as we will see. We know the (self-similar) fundamental solution, that is an attractor of its basin

$$U(y,\tau) = C\,\tau^{-N/2} e^{-y^2/2\tau}.$$

- First step: the logarithmic time-space rescaling

$$u(y,\tau) = v(x,t)\,(1+\tau)^{-N/2}, \quad y = x(1+\tau)^{1/2}, \quad t = \log(1+\tau),$$

that leads to the well-known Fokker-Plank equation for $v(x,t)$:

$$v_t = \frac{1}{2}\Delta_x v + \frac{1}{2}\nabla\cdot(x\,v)$$

- If we now pass to the quotient $w = v/G$, where $G = c\,e^{-x^2/2}$ is the stationary state (Gaussiann kernel), to get the Ornstein-Uhlenbeck version

$$w_t = \frac{1}{2}G^{-1}\,\nabla\cdot\left(\,G\nabla w\,\right),$$

a symmetrically weighted heat equation. The equivalence of these three equations is a main tool in Linear Diffusion and Semigroup Theory.

Energy estimates for the Heat Equation

- Take the classical Heat Equation posed in the whole space $\mathbb{R}^N$ for $\tau > 0$:

$$u_\tau = \frac{1}{2}\Delta_y u$$

with notation $u = u(y, \tau)$ that is useful as we will see. We know the (self-similar) fundamental solution, that is an attractor of its basin

$$U(y, \tau) = C\,\tau^{-N/2} e^{-y^2/2\tau}.$$

- First step: the logarithmic time-space rescaling

$$u(y, \tau) = v(x, t)\,(1+\tau)^{-N/2}, \quad y = x(1+\tau)^{1/2}, \quad t = \log(1+\tau),$$

that leads to the well-known Fokker-Plank equation for $v(x, t)$:

$$v_t = \frac{1}{2}\Delta_x v + \frac{1}{2}\nabla\cdot(x\,v)$$

- If we now pass to the quotient $w = v/G$, where $G = c\,e^{-x^2/2}$ is the stationary state (Gaussiann kernel), to get the Ornstein-Uhlenbeck version

$$w_t = \frac{1}{2}G^{-1}\,\nabla\cdot\left(\,G\nabla w\,\right),$$

a symmetrically weighted heat equation. The equivalence of these three equations is a main tool in Linear Diffusion and Semigroup Theory.

Energy estimates for the Heat Equation II

- We may assume without lack of generality that $\int w\, d\mu = \int v\, dx = \int u\, dy = 1$. We now make a crucial estimate on the time decay of the energy for the OUE:

$$\mathcal{F}(w(t)) = \int_{\mathbb{R}^N} |w-1|^2\, G\, dx, \quad \frac{d\mathcal{F}(w(t))}{dt} = -\int_{\mathbb{R}^N} |\nabla w|^2\, G\, dx = -\mathcal{D}(w(t)).$$

- We can now use a result from abstract functional analysis: the Gaussian Poincaré inequality with measure $d\mu = G(x)\, dx$:

$$\int_{\mathbb{R}^N} w^2 d\mu - \left(\int_{\mathbb{R}^N} w\, d\mu)^2\right) \le \int_{\mathbb{R}^N} |\nabla w|^2\, d\mu$$

- Then, the Left Hand Side is just $\mathcal{F}$ and the inequality implies after integration that:

$$\int_{\mathbb{R}^N} |w-1|^2\, \mathrm{d}\mu \le \mathrm{e}^{-t} \int_{\mathbb{R}^d} |w_0-1|^2\, \mathrm{d}\mu \quad \forall\, t \ge 0$$

- Then by regularity theory, we get $\|w-1\|_\infty \le \mathcal{K}\mathrm{e}^{-t}$, that means, once we go back to the original variables that:

$$(1-Ce^{-t})G(x,t) \le \; v(x,t) \; \le (1+Ce^{-t})\, G(x) \qquad (e^{-t} = \frac{1}{1+\tau}).$$

These are the well known Heat Kernel Estimates of solutions to the HE.

Energy estimates for the Heat Equation II

- We may assume without lack of generality that $\int w\,d\mu = \int v\,dx = \int u\,dy = 1$. We now make a crucial estimate on the time decay of the energy for the OUE:

$$\mathcal{F}(w(t)) = \int_{\mathbb{R}^N} |w-1|^2\, G\,dx, \quad \frac{d\mathcal{F}(w(t))}{dt} = -\int_{\mathbb{R}^N} |\nabla w|^2\, G\,dx = -\mathcal{D}(w(t)).$$

- We can now use a result from abstract functional analysis: the Gaussian Poincaré inequality with measure $d\mu = G(x)\,dx$:

$$\int_{\mathbb{R}^N} w^2 d\mu - \left(\int_{\mathbb{R}^N} w\,d\mu)^2\right) \le \int_{\mathbb{R}^N} |\nabla w|^2\,d\mu$$

- Then, the Left Hand Side is just $\mathcal{F}$ and the inequality implies after integration that:

$$\int_{\mathbb{R}^N} |w-1|^2\,\mathrm{d}\mu \le \mathrm{e}^{-t} \int_{\mathbb{R}^d} |w_0-1|^2\,\mathrm{d}\mu \quad \forall\, t \ge 0$$

- Then by regularity theory, we get $\|w-1\|_\infty \le \mathcal{K}\mathrm{e}^{-t}$, that means, once we go back to the original variables that:

$$(1-Ce^{-t})G(x,t) \le \; v(x,t) \; \le (1+Ce^{-t})\,G(x) \qquad (e^{-t} = \frac{1}{1+\tau}).$$

These are the well known Heat Kernel Estimates of solutions to the HE.

Energy estimates for the Heat Equation II

- We may assume without lack of generality that $\int w\,d\mu = \int v\,dx = \int u\,dy = 1$. We now make a crucial estimate on the time decay of the energy for the OUE:

$$\mathcal{F}(w(t)) = \int_{\mathbb{R}^N} |w-1|^2\, G\,dx, \qquad \frac{d\mathcal{F}(w(t))}{dt} = -\int_{\mathbb{R}^N} |\nabla w|^2\, G\,dx = -\mathcal{D}(w(t)).$$

- We can now use a result from abstract functional analysis: the Gaussian Poincaré inequality with measure $d\mu = G(x)\,dx$:

$$\int_{\mathbb{R}^N} w^2 d\mu - \Big(\Big(\int_{\mathbb{R}^N} w\,d\mu\Big)^2\Big) \le \int_{\mathbb{R}^N} |\nabla w|^2\, d\mu$$

- Then, the Left Hand Side is just $\mathcal{F}$ and the inequality implies after integration that:

$$\int_{\mathbb{R}^N} |w-1|^2\, \mathrm{d}\mu \le \mathrm{e}^{-t} \int_{\mathbb{R}^d} |w_0 - 1|^2\, \mathrm{d}\mu \quad \forall\, t \ge 0$$

- Then by regularity theory, we get $\|w-1\|_\infty \le \mathcal{K}\mathrm{e}^{-t}$, that means, once we go back to the original variables that:

$$(1 - Ce^{-t})G(x,t) \le \ v(x,t)\ \le (1 + Ce^{-t})\,G(x) \qquad (e^{-t} = \frac{1}{1+\tau}).$$

These are the well known Heat Kernel Estimates of solutions to the HE.

Energy estimates for the Heat Equation II

- We may assume without lack of generality that $\int w\,d\mu = \int v\,dx = \int u\,dy = 1$. We now make a crucial estimate on the time decay of the energy for the OUE:

$$\mathcal{F}(w(t)) = \int_{\mathbb{R}^N} |w-1|^2\, G\,dx, \quad \frac{d\mathcal{F}(w(t))}{dt} = -\int_{\mathbb{R}^N} |\nabla w|^2\, G\,dx = -\mathcal{D}(w(t)).$$

- We can now use a result from abstract functional analysis: the Gaussian Poincaré inequality with measure $d\mu = G(x)\,dx$:

$$\int_{\mathbb{R}^N} w^2 d\mu - \left(\int_{\mathbb{R}^N} w\,d\mu)^2\right) \le \int_{\mathbb{R}^N} |\nabla w|^2\, d\mu$$

- Then, the Left Hand Side is just $\mathcal{F}$ and the inequality implies after integration that:

$$\int_{\mathbb{R}^N} |w-1|^2\,\mathrm{d}\mu \le \mathrm{e}^{-t} \int_{\mathbb{R}^d} |w_0-1|^2\,\mathrm{d}\mu \quad \forall\, t \ge 0$$

- Then by regularity theory, we get $\|w-1\|_\infty \le \mathcal{K}\mathrm{e}^{-t}$, that means, once we go back to the original variables that:

$$(1-Ce^{-t})G(x,t) \le \ v(x,t)\ \le (1+Ce^{-t})\,G(x) \qquad (e^{-t} = \frac{1}{1+\tau}).$$

These are the well known Heat Kernel Estimates of solutions to the HE.

Entropy estimates for the Heat Equation III

- There is another approach that starts the analysis from Boltzmann's ideas on entropy dissipation. We start from the Fokker-Planck equation $v_t = \Delta v + \nabla \cdot (xv)$ and consider the functional called **entropy**

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} v \log(v/G)\, dx = \int_{\mathbb{R}^N} v \log(v) dx + \frac{1}{2}\int_{\mathbb{R}^N} x^2 v\, dx + C\,.$$

- Differentiating along the flow (i.e., for a solution) leads to

$$\frac{d\mathcal{E}(v)}{dt} = -\mathcal{I}(v), \quad \mathcal{I}(u) = \int_{\mathbb{R}^N} v\left|\frac{\nabla v}{v} + x\right|^2 dx = \int_{\mathbb{R}^N} v\,|\nabla \log(v/G)|^2\, dx\,.$$

- Put now $v = Gf^2$ to find that

$$\mathcal{E}(v) = 2\int_{\mathbb{R}^N} f^2 \log(f)\, dx, \quad \mathcal{I}(v) = 4\int_{\mathbb{R}^N} |\nabla f|^2\, dx$$

- The famous logarithmic Sobolev inequality [Gross 75] says than that (for all functions, not only solutions)

$$\mathcal{E} \le \frac{1}{2}\mathcal{I}$$

and we obtain the decay $\mathcal{E}(t) \le \mathcal{E}(0)\, e^{-2t}$. Translate that into a good norm.

Entropy estimates for the Heat Equation III

- There is another approach that starts the analysis from Boltzmann's ideas on entropy dissipation. We start from the Fokker-Planck equation $v_t = \Delta v + \nabla \cdot (xv)$ and consider the functional called **entropy**

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} v \, \log(v/G)\, dx = \int_{\mathbb{R}^N} v \, \log(v) dx + \frac{1}{2}\int_{\mathbb{R}^N} x^2 v \, dx + C\,.$$

- Differentiating along the flow (i.e., for a solution) leads to

$$\frac{d\mathcal{E}(v)}{dt} = -\mathcal{I}(v), \quad \mathcal{I}(u) = \int_{\mathbb{R}^N} v \left|\frac{\nabla v}{v} + x\right|^2 dx = \int_{\mathbb{R}^N} v \, |\nabla \log(v/G)|^2 \, dx\,.$$

- Put now $v = Gf^2$ to find that

$$\mathcal{E}(v) = 2\int_{\mathbb{R}^N} f^2 \log(f)\, dx, \quad \mathcal{I}(v) = 4\int_{\mathbb{R}^N} |\nabla f|^2 \, dx$$

- The famous logarithmic Sobolev inequality [Gross 75] says than that (for all functions, not only solutions)

$$\mathcal{E} \le \frac{1}{2}\mathcal{I}$$

and we obtain the decay $\mathcal{E}(t) \le \mathcal{E}(0)\, e^{-2t}$. Translate that into a good norm.

Entropy estimates for the Heat Equation III

- There is another approach that starts the analysis from Boltzmann's ideas on entropy dissipation. We start from the Fokker-Planck equation $v_t = \Delta v + \nabla \cdot (xv)$ and consider the functional called **entropy**

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} v \, \log(v/G)\, dx = \int_{\mathbb{R}^N} v \, \log(v) dx + \frac{1}{2} \int_{\mathbb{R}^N} x^2 v \, dx + C\,.$$

- Differentiating along the flow (i.e., for a solution) leads to

$$\frac{d\mathcal{E}(v)}{dt} = -\mathcal{I}(v), \quad \mathcal{I}(u) = \int_{\mathbb{R}^N} v \left|\frac{\nabla v}{v} + x\right|^2 dx = \int_{\mathbb{R}^N} v \left|\nabla \log(v/G)\right|^2 dx\,.$$

- Put now $v = Gf^2$ to find that

$$\mathcal{E}(v) = 2\int_{\mathbb{R}^N} f^2 \, \log(f)\, dx, \quad \mathcal{I}(v) = 4\int_{\mathbb{R}^N} |\nabla f|^2 \, dx$$

- The famous logarithmic Sobolev inequality [Gross 75] says than that (for all functions, not only solutions)

$$\mathcal{E} \le \frac{1}{2}\mathcal{I}$$

and we obtain the decay $\mathcal{E}(t) \le \mathcal{E}(0)\, e^{-2t}$. Translate that into a good norm.

Entropy estimates for the Heat Equation III

- There is another approach that starts the analysis from Boltzmann's ideas on entropy dissipation. We start from the Fokker-Planck equation $v_t = \Delta v + \nabla\cdot(xv)$ and consider the functional called **entropy**

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} v\,\log(v/G)\,dx = \int_{\mathbb{R}^N} v\,\log(v)dx + \frac{1}{2}\int_{\mathbb{R}^N} x^2 v\,dx + C\,.$$

- Differentiating along the flow (i.e., for a solution) leads to

$$\frac{d\mathcal{E}(v)}{dt} = -\mathcal{I}(v), \quad \mathcal{I}(u) = \int_{\mathbb{R}^N} v\,|\frac{\nabla v}{v} + x|^2\,dx = \int_{\mathbb{R}^N} v\,|\nabla\log(v/G)|^2\,dx\,.$$

- Put now $v = Gf^2$ to find that

$$\mathcal{E}(v) = 2\int_{\mathbb{R}^N} f^2\,\log(f)\,dx, \quad \mathcal{I}(v) = 4\int_{\mathbb{R}^N} |\nabla f|^2\,dx$$

- The famous logarithmic Sobolev inequality [Gross 75] says than that (for all functions, not only solutions)

$$\mathcal{E} \le \frac{1}{2}\mathcal{I}$$

and we obtain the decay $\mathcal{E}(t) \le \mathcal{E}(0)\,e^{-2t}$. Translate that into a good norm.

About entropy in physics

- Entropy has been introduced as a state function in thermodynamics by R. Clausius in 1865, in the framework of the second law of thermodynamics, in order to interpret the results of S. Carnot.
- A statistical physics approach: Boltzmann's formula (1877) defines the entropy of a system in terms of a counting of the micro-states of a physical system. Boltzmann's equation: $\partial_t f + v \cdot \nabla_x f = Q(f,f)$. It describes the evolution of a gas of particles having binary collisions at the kinetic level; $f(t,x,v)$ is a time dependent distribution function (probability density) defined on the phase space $\mathbb{R}^N \times \mathbb{R}^N$.
- The Boltzmann entropy: $H[f] := \iint f \log(f) dxdv$ measures irreversibility: The famous H-Theorem (1872) says that

$$\frac{d}{dt}H[f] = \iint Q(f,f) log(f) dxdv \le 0\,.$$

Other notions of entropy: The Shannon entropy in information theory, entropy in probability theory (with reference to an arbitrary measure).
Other approaches: Carathéodory (1908), Lieb-Yngvason (1997).

Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

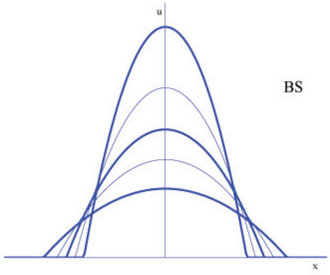
with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

Barenblatt profiles and Asymptotics

- These profiles are the alternative to the Gaussian profiles that have star role for the HE.

  They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$. us
- Explicit formulas (1950, 52) are self-similar:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

Free boundary at distance $|x| = ct^{\beta}$

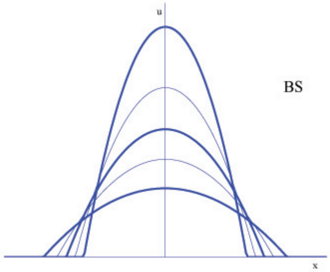
Scaling law; anomalous diffusion versus Brownian motion.

Barenblatt profiles and Asymptotics

- These profiles are the alternative to the Gaussian profiles that have star role for the HE.

  They are source solutions. Source *means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$. us
- Explicit formulas (1950, 52) are self-similar:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion.

Asymptotic behaviour I

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^N$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^N)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^\alpha |u(x,t) - B(x,t)| \to 0}$$

Let $f = 0$ (or small at infinity in L^p). For every $p \ge 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: *α and $\beta = \alpha/n = 1/(2 + n(m-1))$ are the zooming exponents as in $B(x,t)$.*

- Starting result by FK takes $u_0 \ge 0$, compact support and $f = 0$. Proof is done by rescaling method. Needs a good uniqueness theorem.

Asymptotic behaviour I

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^N$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^N)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

Let $f = 0$ (or small at infinity in L^p). For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$. Proof is done by rescaling method. Needs a good uniqueness theorem.

Asymptotic behaviour I

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^N$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^N)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

Let $f = 0$ (or small at infinity in L^p). For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$. Proof is done by rescaling method. Needs a good uniqueness theorem.

Asymptotic behaviour I

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^N$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^N)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

Let $f = 0$ (or small at infinity in L^p). For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$. Proof is done by rescaling method. Needs a good uniqueness theorem.

Calculations of the entropy rates

- This is next step of information after plain convergence.
  We rescale the function as $u(x,t) = r(t)^n\,\rho(y\,r(t), s)$
  where $r(t)$ is the Barenblatt radius at $t+1$, and "new time" is $s = \log(1+t)$.
  Equation becomes
  $$\rho_s = \mathrm{div}\,(\rho(\nabla\rho^{m-1} + \frac{c}{2}\nabla y^2)).$$
- Then define a new entropy (not Boltzmann entropy, but a new type called Rényi entropy)
  $$E(u)(t) = \int (\frac{1}{m}\rho^m + \frac{c}{2}\rho y^2)\,dy$$
  The minimum of entropy is identified as the Barenblatt profile.
- Calculate
  $$\frac{dE}{ds} = -\int \rho|\nabla\rho^{m-1} + cy|^2\,dy = -D$$
  Moreover, a calculation known as Bakry-Emery gives
  $$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$
- *We conclude exponential decay of D in new time s, i.e., a power rate in real time t. It follows that E decays to a minimum $E_\infty > 0$ and we then prove that this is the level of the Barenblatt solution, which attains the functional minimum.*

Calculations of the entropy rates

- This is next step of information after plain convergence.
  We rescale the function as $u(x,t) = r(t)^n \rho(y\, r(t), s)$
  where $r(t)$ is the Barenblatt radius at $t+1$, and "new time" is $s = \log(1+t)$.
  Equation becomes
  $$\rho_s = \mathrm{div}\,(\rho(\nabla\rho^{m-1} + \frac{c}{2}\nabla y^2)).$$
- Then define a new entropy (not Boltzmann entropy, but a new type called Rényi entropy)
  $$E(u)(t) = \int (\frac{1}{m}\rho^m + \frac{c}{2}\rho y^2)\, dy$$
  The minimum of entropy is identified as the Barenblatt profile.
- Calculate
  $$\frac{dE}{ds} = -\int \rho|\nabla\rho^{m-1} + cy|^2\, dy = -D$$
  Moreover, a calculation known as Bakry-Emery gives
  $$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$
- *We conclude exponential decay of D in new time s, i.e., a power rate in real time t. It follows that E decays to a minimum $E_\infty > 0$ and we then prove that this is the level of the Barenblatt solution, which attains the functional minimum.*

Calculations of the entropy rates

- This is next step of information after plain convergence.
  We rescale the function as $u(x,t) = r(t)^n \rho(y\, r(t), s)$
  where $r(t)$ is the Barenblatt radius at $t+1$, and "new time" is $s = \log(1+t)$.
  Equation becomes
$$\rho_s = \mathrm{div}\,(\rho(\nabla \rho^{m-1} + \frac{c}{2}\nabla y^2)).$$
- Then define a new entropy (not Boltzmann entropy, but a new type called Rényi entropy)
$$E(u)(t) = \int (\frac{1}{m}\rho^m + \frac{c}{2}\rho y^2)\, dy$$
  The minimum of entropy is identified as the Barenblatt profile.
- Calculate
$$\frac{dE}{ds} = -\int \rho|\nabla \rho^{m-1} + cy|^2\, dy = -D$$
  Moreover, a calculation known as Bakry-Emery gives
$$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$
- *We conclude exponential decay of D in new time s, i.e., a power rate in real time t. It follows that E decays to a minimum $E_\infty > 0$ and we then prove that this is the level of the Barenblatt solution, which attains the functional minimum.*

Cauchy Problem for the Fast Diffusion Equation

- We consider the model problem in $\mathbb{R}^N$

$$\begin{cases} \partial_\tau u = \Delta\left(\dfrac{u^m}{m}\right) = \nabla\cdot\left(u^{m-1}\nabla u\right)\,, & (\tau, y)\in(0,T)\times\mathbb{R}^N \\[2ex] u(0,\cdot) = u_0\,, & u_0\in \mathrm{L}^1_{\mathrm{loc}}\left(\mathbb{R}^N\right) \end{cases}$$

  for any $m < 1$ (i.e. *Fast Diffusion*, FDE)
- We consider non-negative initial data and solutions.
- Note that $m \le 0$ is included and $m = 0$ corresponds to *logarithmic diffusion.*
- Existence and uniqueness of weak solutions by Herrero and Pierre (1985).
- Solutions have different behaviour if $m_c < m < 1$ and if $m < m_c$, where

$$m_c := \frac{N-2}{N}\,, \qquad \text{and} \qquad m_c > 0 \quad \Longleftrightarrow \quad N \ge 3$$

Rates through entropies for Fast Diffusion

Large effort has been invested in making the entropy machinery work for fast diffusion, $-\infty < m < 1$.

The nice properties of the entropies from the view of transport theory (cf. Villani's book) are lost soon, when $m = (N-1)/N$.

Finite entropy is lost when the second moment is infinite, i.e. for $m = (N-1)/(N+1)$.

Finite-mass, stable states (Barenblatt solutions) are lost for $m = (N-2)/N$.

Functional inequalities play a crucial role in the asymptotic analysis, there are so to say "equivalent".

There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Markowich, Otto, Slepcev, Vazquez, ...

Rates through entropies for Fast Diffusion

Large effort has been invested in making the entropy machinery work for fast diffusion, $-\infty < m < 1$.

The nice properties of the entropies from the view of transport theory (cf. Villani's book) are lost soon, when $m = (N-1)/N$.

Finite entropy is lost when the second moment is infinite, i.e. for $m = (N-1)/(N+1)$.

Finite-mass, stable states (Barenblatt solutions) are lost for $m = (N-2)/N$.

Functional inequalities play a crucial role in the asymptotic analysis, there are so to say "equivalent".

There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Markowich, Otto, Slepcev, Vazquez, ...

Rates through entropies for Fast Diffusion

Large effort has been invested in making the entropy machinery work for fast diffusion, $-\infty < m < 1$.

The nice properties of the entropies from the view of transport theory (cf. Villani's book) are lost soon, when $m = (N-1)/N$.

Finite entropy is lost when the second moment is infinite, i.e. for $m = (N-1)/(N+1)$.

Finite-mass, stable states (Barenblatt solutions) are lost for $m = (N-2)/N$.

Functional inequalities play a crucial role in the asymptotic analysis, there are so to say "equivalent".

There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Markowich, Otto, Slepcev, Vazquez, ...

Rates through entropies for Fast Diffusion

Large effort has been invested in making the entropy machinery work for fast diffusion, $-\infty < m < 1$.

The nice properties of the entropies from the view of transport theory (cf. Villani's book) are lost soon, when $m = (N-1)/N$.

Finite entropy is lost when the second moment is infinite, i.e. for $m = (N-1)/(N+1)$.

Finite-mass, stable states (Barenblatt solutions) are lost for $m = (N-2)/N$.

Functional inequalities play a crucial role in the asymptotic analysis, there are so to say "equivalent".

There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Markowich, Otto, Slepcev, Vazquez, ...

Rates through entropies for Fast Diffusion

Large effort has been invested in making the entropy machinery work for fast diffusion, $-\infty < m < 1$.

The nice properties of the entropies from the view of transport theory (cf. Villani's book) are lost soon, when $m = (N-1)/N$.

Finite entropy is lost when the second moment is infinite, i.e. for $m = (N-1)/(N+1)$.

Finite-mass, stable states (Barenblatt solutions) are lost for $m = (N-2)/N$.

Functional inequalities play a crucial role in the asymptotic analysis, there are so to say "equivalent".

There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Markowich, Otto, Slepcev, Vazquez, ...

Rates through entropies for Fast Diffusion

Large effort has been invested in making the entropy machinery work for fast diffusion, $-\infty < m < 1$.

The nice properties of the entropies from the view of transport theory (cf. Villani's book) are lost soon, when $m = (N-1)/N$.

Finite entropy is lost when the second moment is infinite, i.e. for $m = (N-1)/(N+1)$.

Finite-mass, stable states (Barenblatt solutions) are lost for $m = (N-2)/N$.

Functional inequalities play a crucial role in the asymptotic analysis, there are so to say "equivalent".

There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Markowich, Otto, Slepcev, Vazquez, ...

References to our work

- Juan Luis Vázquez, "Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type". Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006.
- M.B., J. L. Vázquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation*, J. Funct. Anal., 240 (2006), pp. 399–428.
- A. Blanchet, M.B., J. Dolbeault, G. Grillo, J. L. Vázquez, *Hardy-Poincaré inequalities and applications to nonlinear diffusions*, C. R. Math. Acad. Sci. Paris, **344** (2007), 431–436.
- A. Blanchet, M.B., J. Dolbeault, G. Grillo, J. L. Vázquez, *Asymptotics of the fast diffusion equation via entropy estimates,* Arch. Rat. Mech. Anal. **191** (2009), 347–385.
- M.B., J. Dolbeault, G. Grillo, J. L. Vázquez, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, PNAS (Proc. Nat. Acad. Sci.), **107** n. 38, (2010) 16459–16464.
- M.B., G. Grillo, J. L. Vázquez, *Special fast diffusion with slow asymptotics. Entropy method and flow on a Riemannian manifold*, Arch. Rat. Mech. Anal. **196,** (2010), 631–680

Nonlocal diffusion

Fractional diffusion

Nonlinear Models and Entropies

Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms it replaces next-neighbour interaction and Brownian motion by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

- Basic evolution equation

$$\boxed{\partial_t u + (-\Delta)^s u = 0}$$

- Intense work in Stochastic Processes for some decades, but not in Analysis of PDEs until 10 years ago, initiated around Prof. Caffarelli in Texas.

- A basic theory and survey for PDE people: M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1 [math.AP], to appear.

Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms it replaces next-neighbour interaction and Brownian motion by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.
- Basic evolution equation

$$\boxed{\partial_t u + (-\Delta)^s u = 0}$$

- Intense work in Stochastic Processes for some decades, but not in Analysis of PDEs until 10 years ago, initiated around Prof. Caffarelli in Texas.
- A basic theory and survey for PDE people: M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1 [math.AP], to appear.

Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms it replaces next-neighbour interaction and Brownian motion by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

- Basic evolution equation

$$\boxed{\partial_t u + (-\Delta)^s u = 0}$$

- Intense work in Stochastic Processes for some decades, but not in Analysis of PDEs until 10 years ago, initiated around Prof. Caffarelli in Texas.

- A basic theory and survey for PDE people: M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1 [math.AP], to appear.

Asymptotic behavior

for the nonlocal PME [1]

[1] *Asymptotic behavior of a porous medium equation with fractional diffusion,* Luis Caffarelli, Juan Luis Vázquez, Discrete Cont. Dynam. Systems, special issue 2011.

Rescaling for the NL-PME

We now begin the study of the large time behavior.

- Inspired by the asymptotics of the standard porous medium equation, we define the renormalized flow through the transformation

(1) $$u(x,t) = t^{-\alpha} v(x/t^{\beta}, \tau)$$

with new time $\tau = \log(1+t)$. We also put $y = x/t^{\beta}$ as rescaled space variable. In order to cancel the factors including t explicitly, we get the condition on the exponents

(2) $$\alpha + (2-2s)\beta = 1$$

- We use the homogeneity of K in the form

(3) $$(Ku)(x,t) = t^{-\alpha+2s\beta}(Kv)(y,\tau).$$

- If we also want conservation of (finite) mass, then we must put $\alpha = n\beta$, and we arrive at the the precise value of the exponents:

$$\beta = 1/(n+2-2s), \quad \alpha = n/(n+2-2s).$$

Rescaling for the NL-PME

We now begin the study of the large time behavior.

- Inspired by the asymptotics of the standard porous medium equation, we define the renormalized flow through the transformation

(1) $$u(x,t) = t^{-\alpha} v(x/t^{\beta}, \tau)$$

with new time $\tau = \log(1+t)$. We also put $y = x/t^{\beta}$ as rescaled space variable. In order to cancel the factors including t explicitly, we get the condition on the exponents

(2) $$\alpha + (2-2s)\beta = 1$$

- We use the homogeneity of K in the form

(3) $$(Ku)(x,t) = t^{-\alpha+2s\beta}(Kv)(y,\tau).$$

- If we also want conservation of (finite) mass, then we must put $\alpha = n\beta$, and we arrive at the the precise value of the exponents:

$$\beta = 1/(n+2-2s), \quad \alpha = n/(n+2-2s).$$

Rescaling for the NL-PME

We now begin the study of the large time behavior.

- Inspired by the asymptotics of the standard porous medium equation, we define the renormalized flow through the transformation

(1) $$u(x,t) = t^{-\alpha} v(x/t^{\beta}, \tau)$$

with new time $\tau = \log(1+t)$. We also put $y = x/t^{\beta}$ as rescaled space variable. In order to cancel the factors including t explicitly, we get the condition on the exponents

(2) $$\alpha + (2-2s)\beta = 1$$

- We use the homogeneity of K in the form

(3) $$(Ku)(x,t) = t^{-\alpha+2s\beta}(Kv)(y,\tau).$$

- If we also want conservation of (finite) mass, then we must put $\alpha = n\beta$, and we arrive at the the precise value of the exponents:

$$\beta = 1/(n+2-2s), \quad \alpha = n/(n+2-2s).$$

Renormalized flow

- We thus arrive at the *nonlinear, nonlocal Fokker-Plank equation*

(4) $$v_\tau = \nabla_y \cdot (v\,(\nabla_y K(v) + \beta y))$$

- **Stationary renormalized solutions**. They are the solutions $U(y)$ of

(5) $$\nabla_y \cdot (U\,\nabla_y(P + a|y|^2)) = 0, \quad P = K(U).$$

where $a = \beta/2$, and β defined just above. Since we are looking for asymptotic profiles of the standard solutions of the NL-PME we also want $U \geq 0$ and integrable. The simplest possibility is integrating once and getting the radial version

(6) $$U\,\nabla_y(P + a|y|^2)) = 0, \quad P = K(U), \quad U \geq 0.$$

The first equation gives an alternative choice that reminds of the complementary formulation of the obstacle problems.

Renormalized flow

- We thus arrive at the *nonlinear, nonlocal Fokker-Plank equation*

(4)
$$v_\tau = \nabla_y \cdot (v\,(\nabla_y K(v) + \beta y))$$

- **Stationary renormalized solutions**. They are the solutions $U(y)$ of

(5)
$$\nabla_y \cdot (U\,\nabla_y(P + a|y|^2)) = 0, \quad P = K(U).$$

where $a = \beta/2$, and β defined just above. Since we are looking for asymptotic profiles of the standard solutions of the NL-PME we also want $U \geq 0$ and integrable. The simplest possibility is integrating once and getting the radial version

(6)
$$U\,\nabla_y(P + a|y|^2)) = 0, \quad P = K(U), \quad U \geq 0.$$

The first equation gives an alternative choice that reminds of the complementary formulation of the obstacle problems.

Obstacle problem

- Indeed, if we solve the obstacle problem with fractional Laplacian we will obtain a unique solution $P(y)$ of the problem:

$$\begin{array}{l} P \geq \Phi, \quad U = (-\Delta)^s P \geq 0; \\ \text{either } \ P = \Phi \text{ or } \ U = 0. \end{array} \tag{7}$$

  with $0 < s < 1$. Here we have to choose as obstacle

$$\Phi = C - a|y|^2,$$

  where C is any positive constant and $a = \beta/2$. For uniqueness we also need the condition $P \to 0$ as $|y| \to \infty$.

- The theory is developed in A-C-S, C-S-S, the solution is unique and belongs to the space H^{-s} with pressure in H^s. The solutions have $P \in C^{1,s}$ and $U \in C^{1-s}$.
- Note that for $C \leq 0$ the solution is trivial, $P = 0$, $U = 0$, hence we choose $C > 0$. We also note the pressure is defined but for a constant, so that we may take without loss of generality $C = 0$ and take as pressure $\widehat{P} = P - C$ instead of P. But then $P \to 0$ implies that $\widehat{P} \to -C$ as $|y| \to \infty$, so we get a one parameter family of stationary profiles that we denote $U_C(y)$.

Obstacle problem

- Indeed, if we solve the obstacle problem with fractional Laplacian we will obtain a unique solution $P(y)$ of the problem:

$$
\begin{array}{l} P \geq \Phi, \quad U = (-\Delta)^s P \geq 0; \\ \text{either } \; P = \Phi \text{ or } \; U = 0. \end{array} \tag{7}
$$

  with $0 < s < 1$. Here we have to choose as obstacle

$$
\Phi = C - a|y|^2,
$$

  where C is any positive constant and $a = \beta/2$. For uniqueness we also need the condition $P \to 0$ as $|y| \to \infty$.

- The theory is developed in A-C-S, C-S-S, the solution is unique and belongs to the space H^{-s} with pressure in H^s. The solutions have $P \in C^{1,s}$ and $U \in C^{1-s}$.
- Note that for $C \leq 0$ the solution is trivial, $P = 0$, $U = 0$, hence we choose $C > 0$. We also note the pressure is defined but for a constant, so that we may take without loss of generality $C = 0$ and take as pressure $\widehat{P} = P - C$ instead of P. But then $P \to 0$ implies that $\widehat{P} \to -C$ as $|y| \to \infty$, so we get a one parameter family of stationary profiles that we denote $U_C(y)$.

Obstacle problem

- Indeed, if we solve the obstacle problem with fractional Laplacian we will obtain a unique solution $P(y)$ of the problem:

$$
(7) \qquad \begin{array}{l} P \geq \Phi, \quad U = (-\Delta)^s P \geq 0; \\ \text{either } \; P = \Phi \text{ or } \; U = 0. \end{array}
$$

  with $0 < s < 1$. Here we have to choose as obstacle

$$
\Phi = C - a\,|y|^2,
$$

  where C is any positive constant and $a = \beta/2$. For uniqueness we also need the condition $P \to 0$ as $|y| \to \infty$.

- The theory is developed in A-C-S, C-S-S, the solution is unique and belongs to the space H^{-s} with pressure in H^s. The solutions have $P \in C^{1,s}$ and $U \in C^{1-s}$.
- Note that for $C \leq 0$ the solution is trivial, $P = 0$, $U = 0$, hence we choose $C > 0$. We also note the pressure is defined but for a constant, so that we may take without loss of generality $C = 0$ and take as pressure $\widehat{P} = P - C$ instead of P. But then $P \to 0$ implies that $\widehat{P} \to -C$ as $|y| \to \infty$, so we get a one parameter family of stationary profiles that we denote $U_C(y)$.

Barenblatt solutions of new type.

- The above solution of this obstacle problem produces a correct weak solution of the NL-PM equation with initial data a Dirac delta for the density u, i.e., it is the source-type or Barenblatt solution for this problem, which is a profile $U \geq 0$.
  It is positive in the contact set of the obstacle problem, $\mathcal{C} = \{|y| \leq R(C)\}$, and is zero outside, hence it has compact support.

- It is clear that R is smaller than the intersection of the parabola Φ with the axis $R_1 = (C/a)^{1/2}$.

- On the other hand, the pressure $P(|y|)$ is always positive and decays to zero as $|y| \to \infty$ according to fractional potential theory, cf Landau, Stein.
  The rate is $P = O(|y|^{2s-n})$.

Barenblatt solutions of new type.

- The above solution of this obstacle problem produces a correct weak solution of the NL-PM equation with initial data a Dirac delta for the density u, i.e., it is the source-type or Barenblatt solution for this problem, which is a profile $U \geq 0$.
  It is positive in the contact set of the obstacle problem, $\mathcal{C} = \{|y| \leq R(C)\}$,
  and is zero outside, hence it has compact support.

- It is clear that R is smaller than the intersection of the parabola Φ with the axis $R_1 = (C/a)^{1/2}$.

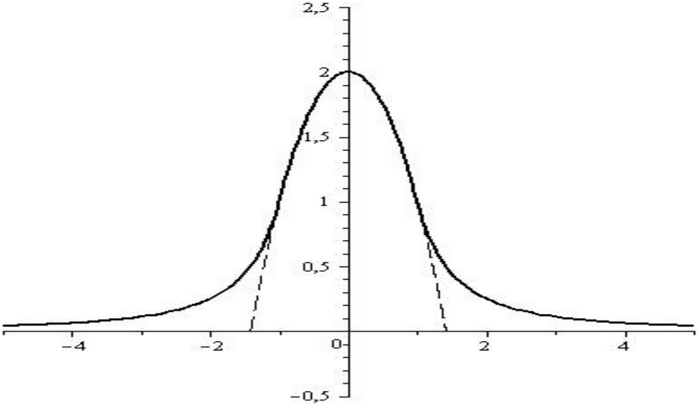
- On the other hand, the pressure $P(|y|)$ is always positive and decays to zero as $|y| \to \infty$ according to fractional potential theory, cf Landau, Stein.
  The rate is $P = O(|y|^{2s-n})$.

Barenblatt solutions of new type.

- The above solution of this obstacle problem produces a correct weak solution of the NL-PM equation with initial data a Dirac delta for the density u, i.e., it is the source-type or Barenblatt solution for this problem, which is a profile $U \geq 0$.
  It is positive in the contact set of the obstacle problem, $\mathcal{C} = \{|y| \leq R(C)\}$,
  and is zero outside, hence it has compact support.

- It is clear that R is smaller than the intersection of the parabola Φ with the axis $R_1 = (C/a)^{1/2}$.

- On the other hand, the pressure $P(|y|)$ is always positive and decays to zero as $|y| \to \infty$ according to fractional potential theory, cf Landau, Stein.
  The rate is $P = O(|y|^{2s-n})$.



The solution of the obstacle problem with parabolic obstacle

Estimates for the renormalized problem. Entropy dissipation.

- Our main problem is now to prove that these profiles are attractors for the renormalized flow.
- We review the estimates of Main Estimates Section above in order to adapt them to the renormalized problem.
- There is no problem is reproving mass conservation or positivity.
- First energy estimate becomes (recall that $H = K^{1/2}$)

$$\begin{aligned} &\frac{d}{d\tau}\int v(y,\tau)\log v(y,\tau)\,dy \\ &= -\textstyle\int |\nabla Hv|^2\,dy - \beta \textstyle\int \nabla v\cdot y \\ &= -\int |\nabla Hv|^2\,dy + \alpha \int v. \end{aligned} \tag{8}$$

Estimates for the renormalized problem. Entropy dissipation.

- Our main problem is now to prove that these profiles are attractors for the renormalized flow.
- We review the estimates of Main Estimates Section above in order to adapt them to the renormalized problem.
- There is no problem is reproving mass conservation or positivity.
- First energy estimate becomes (recall that $H = K^{1/2}$)

$$\begin{aligned} &\frac{d}{d\tau}\int v(y,\tau)\log v(y,\tau)\,dy \\ &= -\textstyle\int |\nabla Hv|^2\,dy - \beta \int \nabla v\cdot y \\ &= -\int |\nabla Hv|^2\,dy + \alpha\int v. \end{aligned} \tag{8}$$

Estimates for the renormalized problem. Entropy dissipation.

- However, the second energy estimate has an essential change. We need to define the entropy of the renormalized flow as

$$(9) \qquad \mathcal{E}(v(\tau)) := \frac{1}{2}\int_{\mathbb{R}^n} (v\,K(v) + \beta y^2 v)\,dy$$

The entropy contains two terms. The first is

$$E_1(v(\tau)) := \int_{\mathbb{R}^n} v\,K(v)\,dy = \int_{\mathbb{R}^n} |Hv|^2\,dy, \quad H = K^{1/2}$$

hence positive. The second is the moment $E_2(v(\tau)) = M_2(v(\tau)) := \int y^2 v\,dy$, also positive. By differentiation we get

$$(10) \qquad \frac{d}{d\tau}\mathcal{E}(v) = -\mathcal{I}(v), \qquad \mathcal{I}(v) := \int \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 vdy.$$

This means that whenever the initial entropy is finite, then $\mathcal{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathcal{I}(v)$ is integrable in $(0,\infty)$ and

$$\mathcal{E}(v(\tau)) + \iint \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 vdy\,dt \le \mathcal{E}(v_0).$$

Estimates for the renormalized problem. Entropy dissipation.

- However, the second energy estimate has an essential change. We need to define the entropy of the renormalized flow as

$$\mathcal{E}(v(\tau)) := \frac{1}{2}\int_{\mathbb{R}^n} (v\,K(v) + \beta y^2 v)\,dy \tag{9}$$

The entropy contains two terms. The first is

$$E_1(v(\tau)) := \int_{\mathbb{R}^n} v\,K(v)\,dy = \int_{\mathbb{R}^n} |Hv|^2\,dy, \quad H = K^{1/2}$$

hence positive. The second is the moment $E_2(v(\tau)) = M_2(v(\tau)) := \int y^2 v\,dy$, also positive. By differentiation we get

$$\frac{d}{d\tau}\mathcal{E}(v) = -\mathcal{I}(v), \qquad \mathcal{I}(v) := \int \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 vdy. \tag{10}$$

This means that whenever the initial entropy is finite, then $\mathcal{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathcal{I}(v)$ is integrable in $(0,\infty)$ and

$$\mathcal{E}(v(\tau)) + \iint \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 vdy\,dt \le \mathcal{E}(v_0).$$

Estimates for the renormalized problem. Entropy dissipation.

- However, the second energy estimate has an essential change. We need to define the entropy of the renormalized flow as

$$\mathcal{E}(v(\tau)) := \frac{1}{2}\int_{\mathbb{R}^n} (v\,K(v) + \beta y^2 v)\,dy \tag{9}$$

The entropy contains two terms. The first is

$$E_1(v(\tau)) := \int_{\mathbb{R}^n} v\,K(v)\,dy = \int_{\mathbb{R}^n} |Hv|^2\,dy, \quad H = K^{1/2}$$

hence positive. The second is the moment $E_2(v(\tau)) = M_2(v(\tau)) := \int y^2 v\,dy$, also positive. By differentiation we get

$$\frac{d}{d\tau}\mathcal{E}(v) = -\mathcal{I}(v), \qquad \mathcal{I}(v) := \int \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 v dy. \tag{10}$$

This means that whenever the initial entropy is finite, then $\mathcal{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathcal{I}(v)$ is integrable in $(0, \infty)$ and

$$\mathcal{E}(v(\tau)) + \iint \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 v dy\,dt \le \mathcal{E}(v_0).$$

Convergence.

- The standard idea is to let $t \to \infty$ in the renormalized flow. Since the entropy goes down there is a limit
$$E_* = \lim_{t\to\infty} \mathcal{E}(t) \geq 0.$$
Since u is bounded in L^1_x unif. in t, and also ux^2 is bounded in L^1_x unif. in t, and moreover $|\nabla H(u)| \in L^2_x$ unif in t, we have that $u(t)$ is a compact family that there is a subsequence $t_j \to \infty$ that converges in L^1_x and almost everywhere to a limit $u_* \geq 0$. The mass of u_* is the same mass of u. One consequence is that the lim inf of the component $E_2(u(t_j))$ is equal or larger that $M_2(u_*)$.
- We also have $H(u) \in L^2_x$ uniformly in t. The boundedness of $\nabla H(u)$ in L^2_x implies the compactness of $H(u)$ in space, so that it converges along a subsequence to v_* . This allows to pass to the limit in $E_1(u(t_j))$ and obtain a correct limit. We have $v_* = H(u_*)$.
- Since the mass of u for large x is small On the other hand, also

Convergence.

- The standard idea is to let $t \to \infty$ in the renormalized flow. Since the entropy goes down there is a limit
$$E_* = \lim_{t\to\infty} \mathcal{E}(t) \geq 0.$$
Since u is bounded in L^1_x unif. in t, and also ux^2 is bounded in L^1_x unif. in t, and moreover $|\nabla H(u)| \in L^2_x$ unif in t, we have that $u(t)$ is a compact family that there is a subsequence $t_j \to \infty$ that converges in L^1_x and almost everywhere to a limit $u_* \geq 0$. The mass of u_* is the same mass of u. One consequence is that the lim inf of the component $E_2(u(t_j))$ is equal or larger that $M_2(u_*)$.
- We also have $H(u) \in L^2_x$ uniformly in t. The boundedness of $\nabla H(u)$ in L^2_x implies the compactness of $H(u)$ in space, so that it converges along a subsequence to v_* . This allows to pass to the limit in $E_1(u(t_j))$ and obtain a correct limit. We have $v_* = H(u_*)$.
- Since the mass of u for large x is small On the other hand, also

Convergence.

- The standard idea is to let $t \to \infty$ in the renormalized flow. Since the entropy goes down there is a limit
$$E_* = \lim_{t\to\infty} \mathcal{E}(t) \ge 0.$$
Since u is bounded in L^1_x unif. in t, and also ux^2 is bounded in L^1_x unif. in t, and moreover $|\nabla H(u)| \in L^2_x$ unif in t, we have that $u(t)$ is a compact family that there is a subsequence $t_j \to \infty$ that converges in L^1_x and almost everywhere to a limit $u_* \ge 0$. The mass of u_* is the same mass of u. One consequence is that the lim inf of the component $E_2(u(t_j))$ is equal or larger that $M_2(u_*)$.
- We also have $H(u) \in L^2_x$ uniformly in t. The boundedness of $\nabla H(u)$ in L^2_x implies the compactness of $H(u)$ in space, so that it converges along a subsequence to v_* . This allows to pass to the limit in $E_1(u(t_j))$ and obtain a correct limit. We have $v_* = H(u_*)$.
- Since the mass of u for large x is small On the other hand, also

Convergence

- Now we get the consequence that for every $h > 0$ fixed

$$\int_{t_j}^{t_j+h} \int \left| \nabla (Ku + \frac{\beta}{2} x^2) \right|^2 u dx\, dt \to 0.$$

  This implies that if $w(x,t) = Ku + \frac{\beta}{2}x^2$ and $w_h(x,t) = w(x, t+h)$, then $u_h |\nabla w_h|^2$ converges to zero as $h \to \infty$ in $L^1(\mathbb{R}^n \times (0,T))$. Then w_h converges to a constant in space wherever u is not zero, and that constant must be $Ku_* + \frac{\beta}{2}x^2$ along the said subsequence, hence constant also in time

- This means that the limit is a solution of the Barenblatt obstacle problem.

Convergence

- Now we get the consequence that for every $h > 0$ fixed

$$\int_{t_j}^{t_j+h} \int \left| \nabla (Ku + \frac{\beta}{2} x^2) \right|^2 u dx\, dt \to 0.$$

  This implies that if $w(x,t) = Ku + \frac{\beta}{2}x^2$ and $w_h(x,t) = w(x,t+h)$, then $u_h|\nabla w_h|^2$ converges to zero as $h \to \infty$ in $L^1(\mathbb{R}^n \times (0,T))$. Then w_h converges to a constant in space wherever u is not zero, and that constant must be $Ku_* + \frac{\beta}{2}x^2$ along the said subsequence, hence constant also in time

- This means that the limit is a solution of the Barenblatt obstacle problem.

Recent work

- Biler, Imbert and Karch. In a note just submitted to CRAS (Barenblatt profiles for a nonlocal porous medium equation) the authors study the more general equation

$$u_t = \nabla \cdot (u \Lambda^{\alpha-1} u^m), \quad 0 < \alpha < 2$$

and obtain our type of Barenblatt solutions for every $m > 1$ with a very nice added information, they happen to be explicit of the form

$$u(x,t) = Ct^{-\mu}(R^2 - x^2 t^{-2\nu})_+^{\alpha/2(m-1)}$$

it uses an important identity by Getoor, $(-\Delta)^{\alpha/2}(1-y^2)_+^{\alpha/2} = K$, valid inside the support. Observe the boundary behavior.

- The work with Serfaty (Barcelona 2012, published 2014) shows that the limit $s \to 1$ of the solutions of the fractional diffusion problem produces the solutions of the hydrodynamic limit

$$u_t = \nabla \cdot (u \nabla p), \quad -\Delta p = u\,.$$

This happens in all dimensions $n \geq 2$. The corresponding self-similar solutions take the form

$$U(x,t) = t^{-1} H(x/t^{1/n})$$

and H is the characteristic function of a ball $B_R(0)$. This means in Physics an expanding vortex patch. This type of solutions is NOT continuous.

Recent work

- Biler, Imbert and Karch. In a note just submitted to CRAS (Barenblatt profiles for a nonlocal porous medium equation) the authors study the more general equation

$$u_t = \nabla \cdot (u\Lambda^{\alpha-1} u^m), \quad 0 < \alpha < 2$$

and obtain our type of Barenblatt solutions for every $m > 1$ with a very nice added information, they happen to be explicit of the form

$$u(x,t) = Ct^{-\mu}(R^2 - x^2 t^{-2\nu})_+^{\alpha/2(m-1)}$$

it uses an important identity by Getoor, $(-\Delta)^{\alpha/2}(1-y^2)_+^{\alpha/2} = K$, valid inside the support. Observe the boundary behavior.

- The work with Serfaty (Barcelona 2012, published 2014) shows that the limit $s \to 1$ of the solutions of the fractional diffusion problem produces the solutions of the hydrodynamic limit

$$u_t = \nabla \cdot (u \nabla p), \quad -\Delta p = u\,.$$

This happens in all dimensions $n \geq 2$. The corresponding self-similar solutions take the form

$$U(x,t) = t^{-1} H(x/t^{1/n})$$

and H is the characteristic function of a ball $B_R(0)$. This means in Physics an expanding vortex patch. This type of solutions is NOT continuous.

Recent work

- Biler, Imbert and Karch. In a note just submitted to CRAS (Barenblatt profiles for a nonlocal porous medium equation) the authors study the more general equation

$$u_t = \nabla \cdot (u\Lambda^{\alpha-1}u^m), \quad 0 < \alpha < 2$$

  and obtain our type of Barenblatt solutions for every $m > 1$ with a very nice added information, they happen to be explicit of the form

$$u(x,t) = Ct^{-\mu}(R^2 - x^2 t^{-2\nu})_+^{\alpha/2(m-1)}$$

  it uses an important identity by Getoor, $(-\Delta)^{\alpha/2}(1-y^2)_+^{\alpha/2} = K$, valid inside the support. Observe the boundary behavior.

- The work with Serfaty (Barcelona 2012, published 2014) shows that the limit $s \to 1$ of the solutions of the fractional diffusion problem produces the solutions of the hydrodynamic limit

$$u_t = \nabla \cdot (u\nabla p), \quad -\Delta p = u\,.$$

  This happens in all dimensions $n \geq 2$. The corresponding self-similar solutions take the form

$$U(x,t) = t^{-1}H(x/t^{1/n})$$

  and H is the characteristic function of a ball $B_R(0)$. This means in Physics an expanding vortex patch. This type of solutions is NOT continuous.

Recent work

- Biler, Imbert and Karch. In a note just submitted to CRAS (Barenblatt profiles for a nonlocal porous medium equation) the authors study the more general equation

$$u_t = \nabla \cdot (u\Lambda^{\alpha-1}u^m), \quad 0 < \alpha < 2$$

  and obtain our type of Barenblatt solutions for every $m > 1$ with a very nice added information, they happen to be explicit of the form

$$u(x,t) = Ct^{-\mu}(R^2 - x^2 t^{-2\nu})_+^{\alpha/2(m-1)}$$

  it uses an important identity by Getoor, $(-\Delta)^{\alpha/2}(1-y^2)_+^{\alpha/2} = K$, valid inside the support. Observe the boundary behavior.

- The work with Serfaty (Barcelona 2012, published 2014) shows that the limit $s \to 1$ of the solutions of the fractional diffusion problem produces the solutions of the hydrodynamic limit

$$u_t = \nabla \cdot (u\nabla p), \quad -\Delta p = u\,.$$

  This happens in all dimensions $n \geq 2$. The corresponding self-similar solutions take the form

$$U(x,t) = t^{-1}H(x/t^{1/n})$$

  and H is the characteristic function of a ball $B_R(0)$. This means in Physics an expanding vortex patch. This type of solutions is NOT continuous.

Our first project. Last results

- Limit $s \to 1$ S. Serfaty, and J. L. Vazquez, *Hydrodynamic Limit of Nonlinear. Diffusion with Fractional Laplacian Operators*, Calc. Var. PDEs **526**, online; arXiv:1205.6322v1 [math.AP], may 2012.

  A presentation of this topic and results for the Proceedings from the Abel Symposium 2010.

- J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators.* in "Nonlinear partial differential equations: the Abel Symposium 2010", Holden, Helge & Karlsen, Kenneth H. eds., Springer, 2012. Pp. 271–298.

  Proving that the self-similar solutions of Barenblatt type (Caffareli-Vazquez, Biler-Karch-Monneau) are attractors with calculated rate in 1D.

- *Exponential Convergence Towards Stationary States for the 1D Porous Medium Equation with Fractional Pressure*, by J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vázquez. JDE, 2015.

  Uses entropy analysis and Bakry Emery method. Problem is open (and quite interesting in higher dimensions). Previous work by Biler, Karch and Monneau in 1D with no rate.

See JLV's webpage, preprints, arxiv.

Our first project. Last results

- Limit $s \to 1$ S. Serfaty, and J. L. Vazquez, *Hydrodynamic Limit of Nonlinear. Diffusion with Fractional Laplacian Operators*, Calc. Var. PDEs **526**, online; arXiv:1205.6322v1 [math.AP], may 2012.

  A presentation of this topic and results for the Proceedings from the Abel Symposium 2010.

- J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators.* in "Nonlinear partial differential equations: the Abel Symposium 2010", Holden, Helge & Karlsen, Kenneth H. eds., Springer, 2012. Pp. 271–298.

  Proving that the self-similar solutions of Barenblatt type (Caffareli-Vazquez, Biler-Karch-Monneau) are attractors with calculated rate in 1D.

- *Exponential Convergence Towards Stationary States for the 1D Porous Medium Equation with Fractional Pressure*, by J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vázquez. JDE, 2015.

  Uses entropy analysis and Bakry Emery method. Problem is open (and quite interesting in higher dimensions). Previous work by Biler, Karch and Monneau in 1D with no rate.

See JLV's webpage, preprints, arxiv.

Our first project. Last results

- Limit $s \to 1$ S. Serfaty, and J. L. Vazquez, *Hydrodynamic Limit of Nonlinear. Diffusion with Fractional Laplacian Operators*, Calc. Var. PDEs **526**, online; arXiv:1205.6322v1 [math.AP], may 2012.

  A presentation of this topic and results for the Proceedings from the Abel Symposium 2010.

- J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators.* in “Nonlinear partial differential equations: the Abel Symposium 2010”, Holden, Helge & Karlsen, Kenneth H. eds., Springer, 2012. Pp. 271–298.

  Proving that the self-similar solutions of Barenblatt type (Caffareli-Vazquez, Biler-Karch-Monneau) are attractors with calculated rate in 1D.

- *Exponential Convergence Towards Stationary States for the 1D Porous Medium Equation with Fractional Pressure*, by J. A. Carrillo, Y. Huang, M. C. Santos, and J. L. Vázquez. JDE, 2015.

  Uses entropy analysis and Bakry Emery method. Problem is open (and quite interesting in higher dimensions). Previous work by Biler, Karch and Monneau in 1D with no rate.

  See JLV’s webpage, preprints, arxiv.

Work to Do

- Study the regularity of the solutions
- Study the equation and regularity of the free boundary
- Study fine asymptotic behavior, and extend to other classes of data
- Study these nonlocal problems in bounded domains
- Decide conditions of uniqueness
- Decide conditions of comparison
- Write a performing numerical code
- Consider different nonlocal nonlinear diffusion problems
- Discuss the Stochastic Particle Models in the literature that involve long-range effects and anomalous diffusion parameters.

Work to Do

- Study the regularity of the solutions
- Study the equation and regularity of the free boundary
- Study fine asymptotic behavior, and extend to other classes of data
- Study these nonlocal problems in bounded domains
- Decide conditions of uniqueness
- Decide conditions of comparison
- Write a performing numerical code
- Consider different nonlocal nonlinear diffusion problems
- Discuss the Stochastic Particle Models in the literature that involve long-range effects and anomalous diffusion parameters.

Work to Do

- Study the regularity of the solutions
- Study the equation and regularity of the free boundary
- Study fine asymptotic behavior, and extend to other classes of data
- Study these nonlocal problems in bounded domains
- Decide conditions of uniqueness
- Decide conditions of comparison
- Write a performing numerical code
- Consider different nonlocal nonlinear diffusion problems
- Discuss the Stochastic Particle Models in the literature that involve long-range effects and anomalous diffusion parameters.

Work to Do

- Study the regularity of the solutions
- Study the equation and regularity of the free boundary
- Study fine asymptotic behavior, and extend to other classes of data
- Study these nonlocal problems in bounded domains
- Decide conditions of uniqueness
- Decide conditions of comparison
- Write a performing numerical code
- Consider different nonlocal nonlinear diffusion problems
- Discuss the Stochastic Particle Models in the literature that involve long-range effects and anomalous diffusion parameters.

Work to Do

- Study the regularity of the solutions
- Study the equation and regularity of the free boundary
- Study fine asymptotic behavior, and extend to other classes of data
- Study these nonlocal problems in bounded domains
- Decide conditions of uniqueness
- Decide conditions of comparison
- Write a performing numerical code
- Consider different nonlocal nonlinear diffusion problems
- Discuss the Stochastic Particle Models in the literature that involve long-range effects and anomalous diffusion parameters.

There is much more going on,
but let us stop here.

End

Thank you

♠ ♠ ♠

Gracias

There is much more going on,
but let us stop here.

End

Thank you

♠ ♠ ♠

Gracias

There is much more going on,
but let us stop here.

End

Thank you

♠ ♠ ♠

Gracias

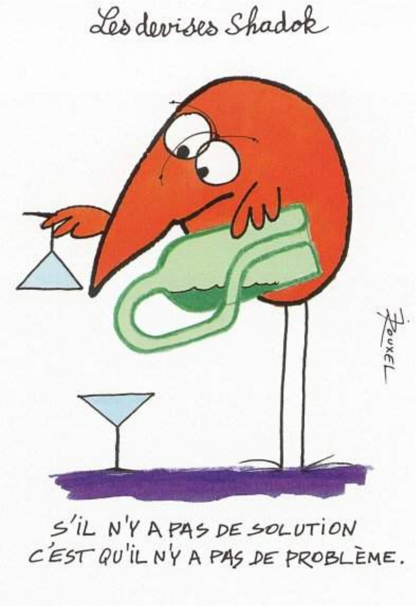
There is much more going on,
but let us stop here.

End

Thank you

♠ ♠ ♠

Gracias


Les devises Shadok
S'IL N'Y A PAS DE SOLUTION
C'EST QU'IL N'Y A PAS DE PROBLÈME.
ROUXEL

The theory of nonlinear diffusion with fractional operators II

JUAN LUIS VÁZQUEZ

Departamento de Matemáticas
Universidad Autónoma de Madrid
and Royal Academy of Sciences

CIME Summer Course

“Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions”

Cetraro, Italia July 7, 2016

Outline

Outline

Outline

FPME: Second model for fractional Porous Medium Flows

- An alternative natural equation is the equation that we will call FPME:

$$\partial_t u + (-\Delta)^s u^m = 0. \tag{1}$$

- This model arises from stochastic differential equations when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling.
  Understanding the physical situation looks difficult to me , but the modelling on linear an non linear fractional heat equations is done by Stefano Olla, Milton Jara and collaborators, see for instance *M. D. Jara, T. Komorowski, S. Olla*, Ann. Appl. Probab. **19** (2009), no. 6, 2270–2300. *M. Jara, C. Landim, S. Sethuraman*, Probab. Theory Relat. Fields **145** (2009), 565–590.

- Another derivation comes from boundary control problems and it appears in
  Athanasopoulos, I.; Caffarelli, L. A. *Continuity of the temperature in boundary heat control problems*, *Adv. Math.* **224** (2010), no. 1, 293–315, where they prove C^α regularity of the solutions.

FPME: Second model for fractional Porous Medium Flows

- An alternative natural equation is the equation that we will call FPME:

$$\partial_t u + (-\Delta)^s u^m = 0. \tag{1}$$

- This model arises from stochastic differential equations when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling.
  Understanding the physical situation looks difficult to me , but the modelling on linear an non linear fractional heat equations is done by Stefano Olla, Milton Jara and collaborators, see for instance
  M. D. Jara, T. Komorowski, S. Olla, Ann. Appl. Probab. **19** (2009), no. 6, 2270–2300. *M. Jara, C. Landim, S. Sethuraman*, Probab. Theory Relat. Fields **145** (2009), 565–590.

- Another derivation comes from boundary control problems and it appears in
  Athanasopoulos, I.; Caffarelli, L. A. *Continuity of the temperature in boundary heat control problems*, *Adv. Math.* **224** (2010), no. 1, 293–315, where they prove C^α regularity of the solutions.

FPME: Second model for fractional Porous Medium Flows

- An alternative natural equation is the equation that we will call FPME:

$$\partial_t u + (-\Delta)^s u^m = 0. \tag{1}$$

- This model arises from stochastic differential equations when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling.
  Understanding the physical situation looks difficult to me , but the modelling on linear an non linear fractional heat equations is done by Stefano Olla, Milton Jara and collaborators, see for instance
  M. D. Jara, T. Komorowski, S. Olla, Ann. Appl. Probab. **19** (2009), no. 6, 2270–2300. *M. Jara, C. Landim, S. Sethuraman,* Probab. Theory Relat. Fields **145** (2009), 565–590.

- Another derivation comes from boundary control problems and it appears in
  Athanasopoulos, I.; Caffarelli, L. A. *Continuity of the temperature in boundary heat control problems*, *Adv. Math.* **224** (2010), no. 1, 293–315, where they prove C^α regularity of the solutions.

Mathematical theory of the Fractional Heat Equation

- The Linear Problem is

$$\boxed{u_t + (-\Delta)^s(u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This model represents the linear flow generated by the so-called Lévy processes in Stochastic PDEs, where the transition from one site x_j of the mesh to another site x_k has a probability that depends on the distance $|x_k - x_j|$ in the form of an inverse power for $j \neq k$. The power we take is $c\,|x_k - x_j|^{-n-2s}$. The range is $0 < s < 1$. The limit from random walk to the continuous equation is done by E. Valdinoci, in *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. 49 (2009), 33-44.

- The solution of the linear equation can be obtained in $\mathbb{R}^n$ by means of convolution with the fractional heat kernel

$$u(x,t) = \int u_0(y) P_t(x-y)\, dy,$$

and people in probability (like Blumental and Getoor) proved in the 1960s that

$$P_t(x) \asymp \frac{t}{\left(t^{1/s} + |x|^2\right)^{(n+2s)/2}} \qquad \Rightarrow \text{look at the fat tail.}$$

Mathematical theory of the Fractional Heat Equation

- The Linear Problem is

$$\boxed{u_t + (-\Delta)^s(u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This model represents the linear flow generated by the so-called Lévy processes in Stochastic PDEs, where the transition from one site x_j of the mesh to another site x_k has a probability that depends on the distance $|x_k - x_j|$ in the form of an inverse power for $j \neq k$. The power we take is $c\,|x_k - x_j|^{-n-2s}$. The range is $0 < s < 1$. The limit from random walk to the continuous equation is done by E. Valdinoci, in *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. 49 (2009), 33-44.

- The solution of the linear equation can be obtained in $\mathbb{R}^n$ by means of convolution with the fractional heat kernel

$$u(x,t) = \int u_0(y) P_t(x-y)\, dy,$$

and people in probability (like Blumental and Getoor) proved in the 1960s that

$$P_t(x) \asymp \frac{t}{\left(t^{1/s} + |x|^2\right)^{(n+2s)/2}} \qquad \Rightarrow \text{look at the fat tail.}$$

Mathematical theory of the Fractional Heat Equation

- The Linear Problem is

$$\boxed{u_t + (-\Delta)^s(u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.
This model represents the linear flow generated by the so-called Lévy processes in Stochastic PDEs, where the transition from one site x_j of the mesh to another site x_k has a probability that depends on the distance $|x_k - x_j|$ in the form of an inverse power for $j \neq k$. The power we take is $c\,|x_k - x_j|^{-n-2s}$. The range is $0 < s < 1$. The limit from random walk to the continuous equation is done by E. Valdinoci, in *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. 49 (2009), 33-44.

- The solution of the linear equation can be obtained in $\mathbb{R}^n$ by means of convolution with the fractional heat kernel

$$u(x,t) = \int u_0(y) P_t(x-y)\,dy,$$

and people in probability (like Blumental and Getoor) proved in the 1960s that

$$P_t(x) \asymp \frac{t}{\left(t^{1/s} + |x|^2\right)^{(n+2s)/2}} \qquad \Rightarrow \text{look at the fat tail.}$$

- The paper

  B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629-650, studies the theory in classes of (maybe) large functions and studies the question: is every solution representable by the convolution formula. The answer is yes if the solutions are 'nice' strong solutions and the growth in x is no more that $u(x,t) \le (1+|x|)^a$ with $a < 2s$.

- Our recent paper

  M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1

  solves the problem of existence and uniqueness of solutions when the initial data is a locally finite Radon measure with the condition

$$\int_{\mathbb{R}^n} (1+|x|)^{-(n+2s)}\, d\mu(x) < \infty\,. \qquad (2)$$

  Moreover we prove that any constructed solution by convolution, or any very weak solution $u \ge 0$, has an initial trace μ which is a measure in the above class $\mathcal{M}_s$. So the result closes the problem of the Widder theory for the fractional heat equation posed in $\mathbb{R}^n$.

- The paper goes on to tell what you want to know about this semigroup for nonnegative solutions. Arxiv is free.

- The paper

  B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629-650, studies the theory in classes of (maybe) large functions and studies the question: is every solution representable by the convolution formula. The answer is yes if the solutions are 'nice' strong solutions and the growth in x is no more that $u(x,t) \leq (1+|x|)^a$ with $a < 2s$.

- Our recent paper

  M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1

  solves the problem of existence and uniqueness of solutions when the initial data is a locally finite Radon measure with the condition

$$\int_{\mathbb{R}^n} (1+|x|)^{-(n+2s)}\, d\mu(x) < \infty\,. \tag{2}$$

  Moreover we prove that any constructed solution by convolution, or any very weak solution $u \geq 0$, has an initial trace μ which is a measure in the above class $\mathcal{M}_s$. So the result closes the problem of the Widder theory for the fractional heat equation posed in $\mathbb{R}^n$.

- The paper goes on to tell what you want to know about this semigroup for nonnegative solutions. Arxiv is free.

- The paper

  B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629-650, studies the theory in classes of (maybe) large functions and studies the question: is every solution representable by the convolution formula. The answer is yes if the solutions are 'nice' strong solutions and the growth in x is no more that $u(x,t) \le (1+|x|)^a$ with $a < 2s$.

- Our recent paper

  M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1

  solves the problem of existence and uniqueness of solutions when the initial data is a locally finite Radon measure with the condition

$$\int_{\mathbb{R}^n} (1+|x|)^{-(n+2s)}\, d\mu(x) < \infty\,. \tag{2}$$

  Moreover we prove that any constructed solution by convolution, or any very weak solution $u \ge 0$, has an initial trace μ which is a measure in the above class $\mathcal{M}_s$. So the result closes the problem of the Widder theory for the fractional heat equation posed in $\mathbb{R}^n$.

- The paper goes on to tell what you want to know about this semigroup for nonnegative solutions. Arxiv is free.

- The paper

  B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629-650, studies the theory in classes of (maybe) large functions and studies the question: is every solution representable by the convolution formula. The answer is yes if the solutions are 'nice' strong solutions and the growth in x is no more that $u(x,t) \le (1+|x|)^a$ with $a < 2s$.

- Our recent paper

  M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1

  solves the problem of existence and uniqueness of solutions when the initial data is a locally finite Radon measure with the condition

  $$\int_{\mathbb{R}^n} (1+|x|)^{-(n+2s)}\, d\mu(x) < \infty\,. \tag{2}$$

  Moreover we prove that any constructed solution by convolution, or any very weak solution $u \ge 0$, has an initial trace μ which is a measure in the above class $\mathcal{M}_s$. So the result closes the problem of the Widder theory for the fractional heat equation posed in $\mathbb{R}^n$.

- The paper goes on to tell what you want to know about this semigroup for nonnegative solutions. Arxiv is free.

Recent coauthors

Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 2 papers appeared in Advances in Mathematics (2011) and Comm. Pure Appl. Math. (2012).

  In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

  C^α regularity (if m is not near 0),

  Nonnegative solutions have infinite speed of propagation for all m and $s \Rightarrow$ no compact support. But Model 1 with Caffarelli did have the compact support property.

Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 2 papers appeared in Advances in Mathematics (2011) and Comm. Pure Appl. Math. (2012).

In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

C^α regularity (if m is not near 0),

Nonnegative solutions have infinite speed of propagation for all m and $s \Rightarrow$ no compact support. But Model 1 with Caffarelli did have the compact support property.

Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 2 papers appeared in Advances in Mathematics (2011) and Comm. Pure Appl. Math. (2012).

In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

C^α regularity (if m is not near 0),

Nonnegative solutions have infinite speed of propagation for all m and $s \Rightarrow$ no compact support. But Model 1 with Caffarelli did have the compact support property.

Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 2 papers appeared in Advances in Mathematics (2011) and Comm. Pure Appl. Math. (2012).

In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

C^α regularity (if m is not near 0),

Nonnegative solutions have infinite speed of propagation for all m and $s \Rightarrow$ no compact support. But Model 1 with Caffarelli did have the compact support property.

Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 2 papers appeared in Advances in Mathematics (2011) and Comm. Pure Appl. Math. (2012).

In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

C^α regularity (if m is not near 0),

Nonnegative solutions have infinite speed of propagation for all m and $s \Rightarrow$ no compact support. But Model 1 with Caffarelli did have the compact support property.

Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 2 papers appeared in Advances in Mathematics (2011) and Comm. Pure Appl. Math. (2012).

In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

C^α regularity (if m is not near 0),

Nonnegative solutions have infinite speed of propagation for all m and $s \Rightarrow$ no compact support. But Model 1 with Caffarelli did have the compact support property.

Outline

Outline of work done for model M2

Comparison of models M1 and M2 is quite interesting

- Existence of self-similar solutions, paper JLV, JEMS 2014. The fractional Barenblatt solution is constructed:

  $$U(x,t) = t^{-\alpha} F(xt^{-\beta})$$

  The difficulty is to find F as the solution of an elliptic nonlinear equation of fractional type.

  F has behaviour like a Blumental tail $F(r) \sim r^{-(n+2s}$ for $m \geq 1$, but not for some fast diffusion $m < 1$. Asymptotic behaviour follows: the Barenblatt solution is an attractor.

- A priori upper and lower estimates of intrinsic, local type. Paper with Matteo Bonforte in Advances Math., 2014 for problems posed in $\mathbb{R}^n$.

  - *Quantitative positivity and Harnack Inequalities follow.*

  Against some prejudice due to the nonlocal character of the diffusion, we are able to obtain them here for fractional PME/FDE using a technique of weighted integrals to control the tails of the integrals in a uniform way. The novelty is the weighted functional inequalities.

  Work on bounded domains is more recent and very interesting.

Outline of work done for model M2

Comparison of models M1 and M2 is quite interesting

- Existence of self-similar solutions, paper JLV, JEMS 2014. The fractional Barenblatt solution is constructed:

$$U(x,t) = t^{-\alpha} F(xt^{-\beta})$$

The difficulty is to find F as the solution of an elliptic nonlinear equation of fractional type.

F has behaviour like a Blumental tail $F(r) \sim r^{-(n+2s}$ for $m \geq 1$, but not for some fast diffusion $m < 1$. Asymptotic behaviour follows: the Barenblatt solution is an attractor.

- A priori upper and lower estimates of intrinsic, local type. Paper with Matteo Bonforte in Advances Math., 2014 for problems posed in $\mathbb{R}^n$.

- *Quantitative positivity and Harnack Inequalities follow.*

Against some prejudice due to the nonlocal character of the diffusion, we are able to obtain them here for fractional PME/FDE using a technique of weighted integrals to control the tails of the integrals in a uniform way. The novelty is the weighted functional inequalities.

Work on bounded domains is more recent and very interesting.

Outline of work done for model M2

Comparison of models M1 and M2 is quite interesting

- Existence of self-similar solutions, paper JLV, JEMS 2014. The fractional Barenblatt solution is constructed:

  $$U(x,t) = t^{-\alpha} F(xt^{-\beta})$$

  The difficulty is to find F as the solution of an elliptic nonlinear equation of fractional type.

  F has behaviour like a Blumental tail $F(r) \sim r^{-(n+2s}$ for $m \geq 1$, but not for some fast diffusion $m < 1$. Asymptotic behaviour follows: the Barenblatt solution is an attractor.

- A priori upper and lower estimates of intrinsic, local type. Paper with Matteo Bonforte in Advances Math., 2014 for problems posed in $\mathbb{R}^n$.

  - *Quantitative positivity and Harnack Inequalities follow.*

  Against some prejudice due to the nonlocal character of the diffusion, we are able to obtain them here for fractional PME/FDE using a technique of weighted integrals to control the tails of the integrals in a uniform way. The novelty is the weighted functional inequalities.

  Work on bounded domains is more recent and very interesting.

- Existence of classical solutions and higher regularity for the FPME and the more general model

$$\boxed{\partial_t u + (-\Delta)^s \Phi(u) = 0}$$

Two works by group PQRV. The first appeared at J. Math. Pures Appl. *treats the model case* $\Phi(u) = \log(1+u)$, *which is interesting.* Second is general Φ and is accepted 2015 in J. Eur. Math. Soc. It proves higher regularity for nonnegative solutions of this fractional porous medium equation.

Recent extension of C^∞ regularity to solutions in bounded domains by M. Bonforte, A. Figalli, X. Ros-Oton, *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*, arxiv1510.03758.

- Symmetrization (Schwarz and Steiner). Collaboration with Bruno Volzone, two papers at JMPA. *Applying usual symmetrization techniques is not easy and we have many open problems.* Recent collaboration with Sire and Volzone on the Faber Krahn inequality.

- Existence of classical solutions and higher regularity for the FPME and the more general model

$$\boxed{\partial_t u + (-\Delta)^s \Phi(u) = 0}$$

Two works by group PQRV. The first appeared at J. Math. Pures Appl. *treats the model case* $\Phi(u) = \log(1+u)$, *which is interesting.* Second is general Φ and is accepted 2015 in J. Eur. Math. Soc. It proves higher regularity for nonnegative solutions of this fractional porous medium equation.

Recent extension of C^∞ regularity to solutions in bounded domains by M. Bonforte, A. Figalli, X. Ros-Oton, *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*, arxiv1510.03758.

- Symmetrization (Schwarz and Steiner). Collaboration with Bruno Volzone, two papers at JMPA. *Applying usual symmetrization techniques is not easy and we have many open problems.* Recent collaboration with Sire and Volzone on the Faber Krahn inequality.

- The phenomenon of KPP propagation in linear and nonlinear fractional diffusion. Work with Diana Stan based on previous linear work of Cabré and Roquejoffre (2009, 2013).
- Numerics is being done by a number of authors at this moment: Nochetto, Jakobsen, and coll., and with my student Felix del Teso.
- Extension of model M1 to accept a general exponent m so that the comparison of both models happens on equal terms.
Work by P. Biler and collaborators. Work by Stan, Teso and JLV (papers in CRAS, and a Journal Diff. Eqns., 2016) on

$$\boxed{\partial_t u + \nabla(u^{m-1}\nabla(-\Delta)^{-s}u^p) = 0}$$

Interesting question : *separating finite and infinite propagation.*

- The phenomenon of KPP propagation in linear and nonlinear fractional diffusion. Work with Diana Stan based on previous linear work of Cabré and Roquejoffre (2009, 2013).
- Numerics is being done by a number of authors at this moment: Nochetto, Jakobsen, and coll., and with my student Felix del Teso.
- Extension of model M1 to accept a general exponent m so that the comparison of both models happens on equal terms.
  Work by P. Biler and collaborators. Work by Stan, Teso and JLV (papers in CRAS, and a Journal Diff. Eqns., 2016) on

$$\boxed{\partial_t u + \nabla(u^{m-1}\nabla(-\Delta)^{-s}u^p) = 0}$$

Interesting question : *separating finite and infinite propagation.*

- The phenomenon of KPP propagation in linear and nonlinear fractional diffusion. Work with Diana Stan based on previous linear work of Cabré and Roquejoffre (2009, 2013).
- Numerics is being done by a number of authors at this moment: Nochetto, Jakobsen, and coll., and with my student Felix del Teso.
- Extension of model M1 to accept a general exponent m so that the comparison of both models happens on equal terms.
Work by P. Biler and collaborators. Work by Stan, Teso and JLV (papers in CRAS, and a Journal Diff. Eqns., 2016) on

$$\boxed{\partial_t u + \nabla(u^{m-1}\nabla(-\Delta)^{-s}u^p) = 0}$$

Interesting question : *separating finite and infinite propagation.*

- The phenomenon of KPP propagation in linear and nonlinear fractional diffusion. Work with Diana Stan based on previous linear work of Cabré and Roquejoffre (2009, 2013).
- Numerics is being done by a number of authors at this moment: Nochetto, Jakobsen, and coll., and with my student Felix del Teso.
- Extension of model M1 to accept a general exponent m so that the comparison of both models happens on equal terms.
  Work by P. Biler and collaborators. Work by Stan, Teso and JLV (papers in CRAS, and a Journal Diff. Eqns., 2016) on

$$\boxed{\partial_t u + \nabla(u^{m-1}\nabla(-\Delta)^{-s}u^p) = 0}$$

Interesting question : *separating finite and infinite propagation.*

- The phenomenon of KPP propagation in linear and nonlinear fractional diffusion. Work with Diana Stan based on previous linear work of Cabré and Roquejoffre (2009, 2013).
- Numerics is being done by a number of authors at this moment: Nochetto, Jakobsen, and coll., and with my student Felix del Teso.
- Extension of model M1 to accept a general exponent m so that the comparison of both models happens on equal terms.
  Work by P. Biler and collaborators. Work by Stan, Teso and JLV (papers in CRAS, and a Journal Diff. Eqns., 2016) on

$$\boxed{\partial_t u + \nabla(u^{m-1}\nabla(-\Delta)^{-s}u^p) = 0}$$

Interesting question : *separating finite and infinite propagation.*

- The phenomenon of KPP propagation in linear and nonlinear fractional diffusion. Work with Diana Stan based on previous linear work of Cabré and Roquejoffre (2009, 2013).
- Numerics is being done by a number of authors at this moment: Nochetto, Jakobsen, and coll., and with my student Felix del Teso.
- Extension of model M1 to accept a general exponent m so that the comparison of both models happens on equal terms.
  Work by P. Biler and collaborators. Work by Stan, Teso and JLV (papers in CRAS, and a Journal Diff. Eqns., 2016) on

$$\boxed{\partial_t u + \nabla(u^{m-1}\nabla(-\Delta)^{-s}u^p) = 0}$$

Interesting question : *separating finite and infinite propagation.*

Younger collaborators

- A detailed account on such progress is obtained in the papers (cf. arxiv) and in the following reference that is meant as a survey for two-year progress on Model M2

  Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian Operators, by Juan Luis Vázquez. In "Nonlinear elliptic and parabolic differential equations", Disc. Cont. Dyn. Syst. - S **7,** no. 4 (2014), 857–885.

- Fast diffusion and extinction. Very singular fast diffusion. Paper with Bonforte and Segatti in CalcVar. 2016, on non-existence due to instantaneous extinction.

- fractional p-Laplacian flows This is a rather new topic. The definition of the nonlocal p-laplacian operator was given in Mingione's last talk as the Euler-Lagrange operator corresponding to a power-like functional with nonlocal kernel of the s-Laplacian type. There the aim is elliptic theory. Paper by JLV, 2015 in arXiv, appeared JDE 2016, solves parabolic theory on bounded domains.

- Very degenerate nonlinearities, like the Mesa Problem. This is the limit of NLPME with $m \to \infty$. Paper by JLV, Interfaces Free Bound. 2015.

- A detailed account on such progress is obtained in the papers (cf. arxiv) and in the following reference that is meant as a survey for two-year progress on Model M2

  Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian Operators, by Juan Luis Vázquez. In "Nonlinear elliptic and parabolic differential equations", Disc. Cont. Dyn. Syst. - S **7,** no. 4 (2014), 857–885.

- Fast diffusion and extinction. Very singular fast diffusion. Paper with Bonforte and Segatti in CalcVar. 2016, on non-existence due to instantaneous extinction.

- fractional p-Laplacian flows This is a rather new topic. The definition of the nonlocal p-laplacian operator was given in Mingione's last talk as the Euler-Lagrange operator corresponding to a power-like functional with nonlocal kernel of the s-Laplacian type. There the aim is elliptic theory. Paper by JLV, 2015 in arXiv, appeared JDE 2016, solves parabolic theory on bounded domains.

- Very degenerate nonlinearities, like the Mesa Problem. This is the limit of NLPME with $m \to \infty$. Paper by JLV, Interfaces Free Bound. 2015.

- A detailed account on such progress is obtained in the papers (cf. arxiv) and in the following reference that is meant as a survey for two-year progress on Model M2

  Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian Operators, by Juan Luis Vázquez. In "Nonlinear elliptic and parabolic differential equations", Disc. Cont. Dyn. Syst. - S **7,** no. 4 (2014), 857–885.

- Fast diffusion and extinction. Very singular fast diffusion. Paper with Bonforte and Segatti in CalcVar. 2016, on non-existence due to instantaneous extinction.

- fractional p-Laplacian flows This is a rather new topic. The definition of the nonlocal p-laplacian operator was given in Mingione's last talk as the Euler-Lagrange operator corresponding to a power-like functional with nonlocal kernel of the s-Laplacian type. There the aim is elliptic theory. Paper by JLV, 2015 in arXiv, appeared JDE 2016, solves parabolic theory on bounded domains.

- Very degenerate nonlinearities, like the Mesa Problem. This is the limit of NLPME with $m \to \infty$. Paper by JLV, Interfaces Free Bound. 2015.

- A detailed account on such progress is obtained in the papers (cf. arxiv) and in the following reference that is meant as a survey for two-year progress on Model M2

  Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian Operators, by Juan Luis Vázquez. In "Nonlinear elliptic and parabolic differential equations", Disc. Cont. Dyn. Syst. - S **7,** no. 4 (2014), 857–885.

- Fast diffusion and extinction. Very singular fast diffusion. Paper with Bonforte and Segatti in CalcVar. 2016, on non-existence due to instantaneous extinction.

- fractional p-Laplacian flows This is a rather new topic. The definition of the nonlocal p-laplacian operator was given in Mingione's last talk as the Euler-Lagrange operator corresponding to a power-like functional with nonlocal kernel of the s-Laplacian type. There the aim is elliptic theory. Paper by JLV, 2015 in arXiv, appeared JDE 2016, solves parabolic theory on bounded domains.

- Very degenerate nonlinearities, like the Mesa Problem. This is the limit of NLPME with $m \to \infty$. Paper by JLV, Interfaces Free Bound. 2015.

Some future Directions

- Other nonlocal linear operators (hot topic)
- Elliptic theory (main topic, by many authors)
- Geostrophic flows (this is more related to Fluid Mechanics)
- Reaction-diffusion and blowup
- Geometrical flows, fractional Yamabe (MMar Gonzalez, Sire)
- Chemotaxis systems, ...

Some future Directions

- Other nonlocal linear operators (hot topic)
- Elliptic theory (main topic, by many authors)
- Geostrophic flows (this is more related to Fluid Mechanics)
- Reaction-diffusion and blowup
- Geometrical flows, fractional Yamabe (MMar Gonzalez, Sire)
- Chemotaxis systems, ...

Outline

Operators and Equations in Bounded Domains

- Work needs a different lecture. It comes from long time collaboration with Matteo Bonforte, and recently with Yannick Sire and Alessio Figalli.
- We develop a new programme for Existence, Uniqueness, Positivity, A priori bounds and Asymptotic behaviour for fractional porous medium equations on bounded domains, after examining very carefully the set of suitable concepts of FLO in a bounded domain.
- But the main issue is how many natural definitions we find of the FLO in a bounded domain.
- Then we use the "dual" formulation of the problem and the concept of weak dual solution. In brief, we use the linearity of the operator L to lift the problem to a problem for the potential function

$$U(x,t)=\int_\Omega u(y,t)G(x,y)dy$$

Where G is the elliptic Green function for L.

Operators and Equations in Bounded Domains

- Work needs a different lecture. It comes from long time collaboration with Matteo Bonforte, and recently with Yannick Sire and Alessio Figalli.
- We develop a new programme for Existence, Uniqueness, Positivity, A priori bounds and Asymptotic behaviour for fractional porous medium equations on bounded domains, after examining very carefully the set of suitable concepts of FLO in a bounded domain.
- But the main issue is how many natural definitions we find of the FLO in a bounded domain.
- Then we use the "dual" formulation of the problem and the concept of weak dual solution. In brief, we use the linearity of the operator L to lift the problem to a problem for the potential function

$$U(x,t) = \int_\Omega u(y,t)G(x,y)dy$$

Where G is the elliptic Green function for L.

Fractional Laplacian operators on bounded domains

- The Restricted Fractional Laplacian operator (RFL) is defined via the hypersingular kernel in $\mathbb{R}^n$, "restricted" to functions that are zero outside Ω.
$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{n+2s}}\,\mathrm{d}z\,, \qquad \text{with } \mathrm{supp}(g)\subset\overline{\Omega}\,.$$
where $s\in(0,1)$ and $c_{n,s}>0$ is a normalization constant.
- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0<\overline{\lambda}_1\le\overline{\lambda}_2\le\ldots\le\overline{\lambda}_j\le\overline{\lambda}_{j+1}\le\ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j\in C^s(\overline{\Omega})$.
- Lateral boundary conditions for the RFL:
$$u(t,x)=0\,, \qquad \text{in } (0,\infty)\times(\mathbb{R}^N\setminus\Omega)\,.$$
- The Green function G of RFL satisfies a strong behaviour condition (K4)
$$G(x,y)\asymp\frac{1}{|x-y|^{N-2s}}\left(\frac{\delta_\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,, \qquad \text{with } \gamma=s$$

References.(K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005).

Fractional Laplacian operators on bounded domains

- The Restricted Fractional Laplacian operator (RFL) is defined via the hypersingular kernel in $\mathbb{R}^n$, “restricted” to functions that are zero outside Ω.
$$(-\Delta_{|\Omega})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{n+2s}}\,\mathrm{d}z\,, \qquad \text{with } \mathrm{supp}(g) \subset \overline{\Omega}\,.$$
where $s \in (0,1)$ and $c_{n,s} > 0$ is a normalization constant.
- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$.
- Lateral boundary conditions for the RFL:
$$u(t,x) = 0\,, \qquad \text{in } (0,\infty)\times\left(\mathbb{R}^N \setminus \Omega\right).$$
- The Green function G of RFL satisfies a strong behaviour condition (K4)
$$G(x,y) \asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta_\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,, \quad \text{with } \gamma = s$$

References.(K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005).

Fractional Laplacian operators on bounded domains

- The The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

where Δ_Ω is the classical Dirichlet Laplacian on the domain Ω,

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{L^2(\Omega)} = 1\,.$$

- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/n}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.
- Lateral boundary conditions for the SFL. They are better defined by using the Caffarelli-Silvestre extension adapted to bounded domain as a cylinder. Then we put $U = 0$ on the lateral boundary $x \in \partial\Omega$, $y > 0$.
- The Green function of SFL satisfies a stronger assumption

$$G(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma}{|x-y|^\gamma} \wedge 1 \right), \qquad \text{with } \gamma = 1$$

Fractional Laplacian operators on bounded domains

- The The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

where Δ_Ω is the classical Dirichlet Laplacian on the domain Ω,

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{L^2(\Omega)} = 1\,.$$

- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/n}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.
- Lateral boundary conditions for the SFL. They are better defined by using the Caffarelli-Silvestre extension adapted to bounded domain as a cylinder. Then we put $U = 0$ on the lateral boundary $x \in \partial\Omega$, $y > 0$.
- The Green function of SFL satisfies a stronger assumption

$$G(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma}{|x-y|^\gamma} \wedge 1 \right), \qquad \text{with } \gamma = 1$$

Fractional Laplacian operators on bounded domains

- The The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

where Δ_Ω is the classical Dirichlet Laplacian on the domain Ω,

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{L^2(\Omega)} = 1\,.$$

- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/n}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.
- Lateral boundary conditions for the SFL. They are better defined by using the Caffarelli-Silvestre extension adapted to bounded domain as a cylinder. Then we put $U = 0$ on the lateral boundary $x \in \partial\Omega$, $y > 0$.
- The Green function of SFL satisfies a stronger assumption

$$G(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma}{|x-y|^\gamma} \wedge 1 \right) , \quad \text{with } \gamma = 1$$

Fractional Laplacian operators on bounded domains

- Censored Fractional Laplacians (CFL)
  This is another option that has been introduced in 2003 by Bogdan, Burdzy and Chen. Definition

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,, \qquad \text{with } \frac{1}{2} < s < 1 \,,$$

  where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.

- The Green function $G(x,y)$ satisfies condition $K4$, proven by Chen, Kim and Song (2010), in the form

$$G(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y))}{|x-y|^\gamma} \wedge 1 \right) \,, \quad \text{with } \gamma = s - \frac{1}{2} \,.$$

- Roughly speaking, $s \in (0, 1/2]$ would correspond to Neumann boundary conditions.

Fractional Laplacian operators on bounded domains

- Censored Fractional Laplacians (CFL)
  This is another option that has been introduced in 2003 by Bogdan, Burdzy and Chen. Definition

$$\mathcal{L}f(x) = \text{P.V.} \int_{\Omega} (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}}\, \mathrm{d}y\,, \qquad \text{with } \frac{1}{2} < s < 1\,,$$

  where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.

- The Green function $G(x,y)$ satisfies condition $K4$, proven by Chen, Kim and Song (2010), in the form

$$G(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^{\gamma}(x)}{|x-y|^{\gamma}} \wedge 1 \right) \left(\frac{\delta^{\gamma}(y))}{|x-y|^{\gamma}} \wedge 1 \right)\,, \quad \text{with } \gamma = s - \frac{1}{2}\,.$$

- Roughly speaking, $s \in (0, 1/2]$ would correspond to Neumann boundary conditions.

Fractional Laplacian operators on bounded domains

- Censored Fractional Laplacians (CFL)
  This is another option that has been introduced in 2003 by Bogdan, Burdzy and Chen. Definition

$$\mathcal{L}f(x) = \mathrm{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, dy\,, \qquad \text{with } \frac{1}{2} < s < 1\,,$$

  where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.

- The Green function $G(x,y)$ satisfies condition $K4$, proven by Chen, Kim and Song (2010), in the form

$$G(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y))}{|x-y|^\gamma} \wedge 1 \right) \,, \quad \text{with } \gamma = s - \frac{1}{2}\,.$$

- Roughly speaking, $s \in (0, 1/2]$ would correspond to Neumann boundary conditions.

- We have presented 3 models of Dirichlet fractional Laplacian.
  Put $\left[\, a(x,y) = const \right]$ in the last case. The estimates (K4) show that they are of course **not equivalent**.

- **References.** K. Bogdan, K. Burdzy, K., Z.-Q. Chen. *Censored stable processes*. Probab. Theory Relat. Fields (2003).
  Z.-Q. Chen, P. Kim, R. Song, *Two-sided heat kernel estimates for censored stable-like processes.* Probab. Theory Relat. Fields (2010).

- There is a crowd of generalizations. See papers or recent talks by M. Bonforte.

- We have presented 3 models of Dirichlet fractional Laplacian. Put $\left[\, a(x,y) = const \right]$ in the last case. The estimates (K4) show that they are of course **not equivalent**.

- **References.** K. Bogdan, K. Burdzy, K., Z.-Q. Chen. *Censored stable processes*. Probab. Theory Relat. Fields (2003).
Z.-Q. Chen, P. Kim, R. Song, *Two-sided heat kernel estimates for censored stable-like processes.* Probab. Theory Relat. Fields (2010).

- There is a crowd of generalizations. See papers or recent talks by M. Bonforte.

- We have presented 3 models of Dirichlet fractional Laplacian.
  Put $\big[\, a(x, y) = const \big]$ in the last case. The estimates (K4) show that they are of course **not equivalent**.

- **References.** K. Bogdan, K. Burdzy, K., Z.-Q. Chen. *Censored stable processes*. Probab. Theory Relat. Fields (2003).
  Z.-Q. Chen, P. Kim, R. Song, *Two-sided heat kernel estimates for censored stable-like processes.* Probab. Theory Relat. Fields (2010).

- There is a crowd of generalizations. See papers or recent talks by M. Bonforte.

But let us stop here

and wish you a happy summer.

Thank you, Gracias, Grazie

But let us stop here

and wish you a happy summer.

Thank you, Gracias, Grazie

But let us stop here

and wish you a happy summer.

Thank you, Gracias, Grazie

But let us stop here

and wish you a happy summer.

Thank you, Gracias, Grazie