Recent trends in Free Boundary Regularity

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Type of lectures and what is discussed

In these lectures I shall discuss at heuristic but also at simple technical level the theory that has been developed to study free boundary problems (FBP) for obstacle problem.

If time allows, I shall also discuss some resent problems and mathematical approaches to the regularity of such problems.

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Type of lectures and what is discussed

FB may arise in many applied but also academic problems. A few of these problems are:

- Hydrodynamic lubrication.
- Filtration through porous medium (Dam problem).
- American type options.
- General Flow problems.
- Composite materials.
- Potential Theory and Quadrature domains
- and many others ...

Contents

- A Catalog of semi-classical problems:
  Modeling, Applications, Academic questions

- Mathematical Theory for obstacle type problems:
  Existence, optimal regularity, non-degeneracy, Hausdorff dim. of FB., Monotonicity formulas, Global solutions, Lipschitz and C^1-regularity of FB

- Other type of FBPs and recent trends: A heuristic tour on:
  Bernoulli type, Semi- Quasi-linear PDEs with FB, non-variational FB, System case, Switching problems, Constraint energy minimizing maps, and nonlocal and thin obstacles

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A Catalog of Obstacle Type Problems

What is a Free Boundary Problem?

Overdetermined problems

Consider the following partial differential equation

$$\Delta u = 1 \qquad \text{in } B_r(0) \qquad u = 0 \quad \text{on } \partial B_r(0)$$

where $B_r(0) = \{|x| < r\}$, and $\Delta = \sum_j \partial^2_{jj}$ is the Laplace operator.

We can immediately solve this problem and obtain

$$u = \frac{(|x|^2 - r^2)}{2n}.$$

Here $n \geq 1$ is the space dimension.

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Now suppose we impose a further condition, say any of these:

a) $\int u = \lambda_1$, with λ_1 given constant,

b) $|\nabla u| = \lambda_2$ on ∂B_1, where $\lambda_2 > 0$.

Then we see that, unless λ_1 in case a) or λ_2 in case b) are taken appropriately, the problem cannot be solved.

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What is a Free Boundary Problem?

Overdetermined problems

An alternative solution to the problem is to change r to accommodate any of our extra conditions.

Changing r, amounts to changing the domain where the equation applies. In other words, we have to let r, i.e., the domain be free in our first equation, if we impose any further condition, that the Dirichlet data.

Such problems are referred to as Free Boundaries (FB).

This is a simple example of a free boundary problem, where extra data is imposed, that makes the problem overdetermined.

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Phase transition: A melting ice block

Consider a big piece of ice block, in the middle of Ocean.

Figure: Courtesy of WWW.

Free Boundaries in Applications

Phase transition: A melting ice block

The heat transfers from water to ice, and melts it gradually. There are two governing PDEs, one in the water region and one in the ice region. The equations depend on the material of water and ice.

On the boundary between water and ice region there is the so-called mushy region, which is a mixture of both materials. At this region a phase-transition takes place.

In this model, the ice melts and turns into water and has a qualitative change in material; hence the name phase transition.

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In this mushy region (if it is taken as sharp boundary) there is the so-called latent heat, which is described by an equation of lower order, than the bulk equation.

In our case it is simply the jump of the normal derivatives of the solution function u from water and ice region

$$|\nabla u^+|^2 - |\nabla u^-|^2 = \lambda^2 > 0 \qquad \text{on } \partial\{u > 0\}.$$

In the rest of the domain we have $\Delta u^{\pm} = 0$ (or some PDE, depending on the ingredients).

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Free Boundaries in Applications

Other flow problems

Several flow problems give rise to free boundaries: Hele-Shaw, Muskat, Stokes, ...

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Hele-Shaw flow

Here, I shall touch upon the the classical Hele-Shaw problem, and describe the model.

Hele-Shaw flow amounts to moving of an initial interface with respect to pressure, in the system.

Specifically we consider pressure from the Green's function with a source $\mu \geq 0$ (usually Dirac source)

$$\Delta G^t = -\mu \quad \text{in } D^t, \qquad G^t = 0 \quad \text{on } \partial D^t,$$

where $D^0 \subset \mathbb{R}^n$ is given and evolves with time, and $\mathrm{supp}(\mu) \subset D^0$.

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Hele-Shaw flow

The flow is governed by the $-\nabla G^t$, which represent pressure in the outward normal direction, i.e. $V = -\partial_\nu G$ is the speed in the normal direction.
Integrating the Green's function in time

$$u(x) = u(x,t) = \int_0^t G^\tau d\tau$$

one obtains a new function u that solves

$$\Delta u = \chi_{D^t} - \chi_{D^0} - t\mu \ .$$

More importantly u admits a variational formulation, and we can talk about weak solutions.

Free Boundaries in Applications

Hele-Shaw flow

To see how we obtain the PDE above, we work with

$$v(x,t) := \mathcal{F} \star (\chi_{D^t} - t\mu)(x),$$

where $\mathcal{F}$ is the (normalized) fundamental solution, $\Delta\mathcal{F} = \delta_0$.

Differentiate v in t to arrive at

$$\frac{d}{dt}v(x,t) = \mathcal{F}\star(Vd\sigma-\mu) = \mathcal{F}\star(-\partial_\nu G^t d\sigma-\mu) = \mathcal{F}\star(\Delta G^t)(x) = G^t(x).$$

In particular $\frac{d}{dt}v(x,t) \geq 0$ with equality if $x \notin D^t$.

$d\sigma$ is the surface measure, and smoothness is assumed.

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Hele-Shaw flow

Since also $\frac{d}{dt}u(x,t) = G^t(x)$, we have $\frac{d}{dt}(u(x,t) - v(x,t)) = 0$
Integrating back, we have (for some $C(x)$, constant in t)
$u(x,t) = v(x,t) - C(x)$. Since

$$v(x,0) = \mathcal{F} \star \chi_{D^0},$$

and $u(x,0) = 0$ (by definition) we have $C(x) = \mathcal{F} \star \chi_{D^0}$ and we have $u(x,t) = \mathcal{F} \star (\chi_{D^t} - t\mu)(x) - \mathcal{F} \star \chi_{D^0}$, as desired.

A Stochastic interpretation

Let μ be as above, and define

$$U_\theta(x) = U_{\theta,t\mu,D^0}(x) := \mathbb{E}^x\left(-\theta + \int_0^\theta \left(t\mu(X_s) + \chi_{D^0}(X_s)\right)\, ds\right),$$

where X_s is standard Brownian motion starting at $x = X_0$, and θ is any finite valued stopping time.

Then function

$$u(x) := \sup_\theta U_\theta(x)$$

solves the geometric flow problem described above.

Observe that if $\theta \equiv 0$, then $U_\theta(x) = 0$, and hence $u(x) \geq 0$.

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Variational Inequalities

Obstacle problem

The complementary (variational) problem

$$\min(-\Delta u, u - \psi) = 0, \quad \text{in } D \qquad u = g \text{ on } \partial D$$

where ψ is a given obstacle, with $g \geq \psi$ on ∂D, represents the most basic obstacle problem.

It also relates to an optimal stopping problem

$$u(x) := \sup_{\theta} \mathbb{E}^x \left(\psi(X_{\theta \wedge \tau}) \right),$$

where θ is any bounded stoping time, and $\tau = \tau(x)$ is exit time for D.

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A surface, constraint to an obstacle

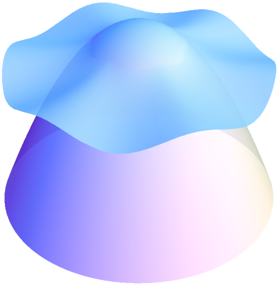

Figure: Equilibrium state of the membrane over the obstacle!
Extracted from book by Petrosyan-Sh-Uraltseva.

Free Boundaries in Applications

Valuation of American Option

Arbitrage theory tells us that:

$$\boxed{\text{Return from Bank} \geq \text{Return from the Hedged portfolio}}$$

Let V be the value of an option for an underlying asset with value S. Suppose Π is a portfolio with one option LONG and m assets SHORT. Then we have

$$\Pi = V - mS, \qquad d\Pi = dV - mdS.$$

This, along with $dS = \sigma SdW + rSdt$ and Ito's formula, leads to

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV \leq 0.$$

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American Option

Value of the Option: Conclusion

Two important ingredients, to the benefit of option-writer are

$$V \geq \psi, \qquad V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV \leq 0$$

A last ingredient (to buyer's benefit) is that V should be the smallest possible value! Otherwise another writer will offer a smaller value and still make profit.

In other words, the smallest "super-solution" to the PDE above.

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An unexpected application: Random Walk

Diaconis-Fulton Smash sum

The smash sum C of two sets $A, B \subset \mathbb{Z}^2$ is a certain random set defined as follows. Let $A \cap B = \{x_1, \cdots, x_k\}$. Each point in $A \cap B$, is sent out through random walks.
Once the point hits *first time* any point y_j outside the region $A \cup B$ it stops and adds y_j to the union:

$$C_j = A \cup B \cup \{y_j\}.$$

The process continuous until all points x_j are sent out. The resulting region $C := C_k$ is the Smash sum, and is independent of the order of x_j (it is Abelian).
Now considering this on the lattice $\sqrt{1/k}\mathbb{Z}^2$ and letting $k \to \infty$, we have a scaling limit which is a solution to an obstacle problem (also called Quadrature domain).

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The obstacle problem

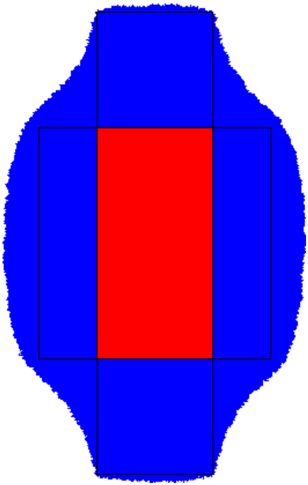

Figure: Smash sum of two rectangles (Courtesy of H. Aleksanyan).

Mathematical Theory for obstacle type problems

Mathematical preliminaries for FBP

What shall be discussed

- Existence, uniqueness.
- Optimal regularity of solutions, Non-degeneracy.
- Lebesgue and Hausdorff measure of FB.

Mathematical preliminaries for FBP

Existence

Existence theory for (scalar) variational problems are well developed. One can use many different approaches, depending on the problem. For the obstacle problem that we are involved with in these lectures one has the following approaches:

- Penalization.
- Perron's method.
- minimization.
- Variational inequality (projection).
- Iterative scheme (good for numerics, system).

Mathematical preliminaries for FBP

Existence: Penalization

Let $v_{\epsilon,k}$ solve

$$\Delta v_{\epsilon,k} = \beta_{\epsilon,k}(v_{\epsilon,k} - \psi) \qquad \text{in } B_1$$

where, e.g., $\beta_{\epsilon,k}(t)$ can be taken as

$$\beta_{\epsilon,k}(t) = \epsilon \min(1, t^+) - \min(\frac{t^-}{\epsilon}, k),$$

and boundary values $v_{\epsilon,k} = g(x) \geq \psi$ on ∂B_1.

Use fixed point theory to solve this problem, for each ϵ, k.

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Mathematical preliminaries for FBP

Existence: Penalization

A clever use of minimum principle can help to show

$$-C \le \beta_{\epsilon,k}(v_{\epsilon,k} - \psi) \le \epsilon,$$

i.e.

$$|\Delta v_{\epsilon,k}| \le C, \qquad \text{independent of } \epsilon, k,$$

which, by elliptic estimates, gives uniform $W^{2,p}_{loc} \cap C^{1,\alpha}_{loc}$-regularity for $v_{\epsilon,k}$.

Let $k \to \infty$ and $\epsilon \to 0$, and use uniform estimates to conclude $v_{\epsilon,k} \to v$, which solves

$$\Delta v \le 0, \qquad v \ge \psi, \qquad (v-\psi)\Delta v = 0 \quad \text{a.e.}$$

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Mathematical preliminaries for FBP

Existence: Perron's super-solution

From the last properties of the limit function v for the penalization method, we see that

$$\Delta v \leq 0, \qquad v \geq \psi.$$

This suggests that Perron's method of smallest super-solution may well be applicable in this case.

This is indeed the case, and the smallest super-solution satisfying the above inequalities give rise to a unique function which we call a solution to the obstacle problem.

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Existence: Minimization for double obstacle

We discuss shortly the minimization problem

$$\min_{v \in \mathcal{K}} \int_{B_1(0)} |\nabla v|^2 dx,$$

where $\mathcal{K} = W^{1,2}_g \cap \{\psi_1 \le v \le \psi_2\},$ (double obstacle)

and g is boundary data, ψ_1 is the lower obstacle, and ψ_2 the upper obstacle.

When $\psi_2 = \infty$, then this is just a one-sided obstacle problem.

More general: allowing terms of type $f(x, v)$, with f convex in v, and $\mathcal{K}$ a suitable convex space.

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More general: allowing terms of type $f(x, v)$, with f convex in v, and $\mathcal{K}$ a suitable convex space.

Mathematical preliminaries for FBP

Uniqueness

Due to lower semi-continuity of the functional we always have minimizers. It is not hard to see that any minimizer v will be unique. Suppose v_1, v_2 are minimizers, with $\nabla v_1 \neq \nabla v_2$, then

$$\int_{B_1} \left|\nabla\left(\frac{v_1+v_2}{2}\right)\right|^2 < \int_{B_1} \frac{1}{2}|\nabla v_1|^2 + \int_{B_1} \frac{1}{2}|\nabla v_2|^2 \leq$$

$$\{\text{by minimality}\} \quad \leq \int_{B_1} \left|\nabla\left(\frac{v_1+v_2}{2}\right)\right|^2$$

which is a contradiction, unless $\nabla v_1 = \nabla v_2$, a.e., which gives $v_1 = v_2 + c$ in L^2. Since they have the same boundary values, we must have $c = 0$ and hence $v_1 = v_2$.

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Mathematical preliminaries for FBP

Properties of Minimizers: Reformulation

It is not hard to see that any minimizer v will be harmonic in the set $\psi_1 < v < \psi_2$ (since then one can make variations both upwards and downwards). Therefore our minimizer will satisfy $\psi_1 \leq v \leq \psi_2$, and

$$\Delta v = (\Delta\psi_1)\chi_{\{int(v=\psi_1)\}} + (\Delta\psi_2)\chi_{\{int(v=\psi_2)\}} + \mu$$

where (as we'll show below) μ is a measure supported on the free boundary, i.e., where the solution detaches from the obstacle

$$support(\mu) \subset \partial\{\psi_1 < v < \psi_2\}.$$

Here I assume ψ_i are smooth enough.
Would be nice to show $\mu \equiv 0$! Because then we are in business.

Properties of Minimizers: Reformulation

For simplicity we assume $\psi_2 = +\infty$, so we only have one obstacle, i.e the one from below. For $u := v - \psi_1$ we have (in a weak sense)

$$\Delta u = f\chi_{\{u>0\}} + \mu, \qquad u \geq 0,$$

where $f(x) = -\Delta\psi_1(x)$ and $support(\mu) \subset \partial\{u > 0\}$.

Observe also that for the complementarity problem

$$\Delta v \leq 0, \qquad v \geq \psi, \qquad (v - \psi)\Delta v = 0 \quad \text{a.e.}$$

we also have the same situation

$$\Delta v = \Delta\psi\chi_{\{int(v=\psi)\}} + \mu.$$

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Properties of Minimizers: $\mu \geq 0$

Let $\eta \in C_0^\infty(B_1)$, $\eta \geq 0$. For $\epsilon > 0$ define

$$\eta_\epsilon = \eta \chi_\epsilon,$$

where

$$\chi_\epsilon = \begin{cases} 1 & \text{if } u(x) \geq 2\epsilon \\ \frac{u(x)}{\epsilon} - 1 & \text{if } \epsilon < u(x) < 2\epsilon \\ 0 & \text{if } u(x) \leq \epsilon. \end{cases}$$

This is meaningful, since u is continuous when $u > 0$.

Mathematical preliminaries for FBP

Properties of Minimizers: $\mu \geq 0$

Set $\Omega_+ = \{u > 0\}$ (assume u continuos) then we have

$$-\langle \eta_\epsilon, f\chi_{\Omega_+} \rangle = \int_{\Omega_+} \nabla u \cdot \nabla \eta_\epsilon \, dx$$

$$= \int_{B_1} (\nabla u \cdot \nabla \eta)\chi_\epsilon \, dx + \frac{1}{\epsilon} \int_{\epsilon < u < 2\epsilon} |\nabla u|^2 \eta \, dx$$

$$\geq \int_{B_1} (\nabla u \cdot \nabla \eta)\chi_\epsilon \, dx.$$

Properties of Minimizers: $\mu \geq 0$

Since $0 \leq \eta_\epsilon \leq \eta$ and

$$\int_{B_1} |\nabla u||\nabla \eta| \, dx < \infty,$$

we can let $\epsilon \to 0$, to obtain

$$-\langle \eta, f\chi_{\Omega_+} \rangle \geq \int_{\Omega_+} \nabla u \cdot \nabla \eta \, dx = \int_{B_1} \nabla u \cdot \nabla \eta \, dx.$$

Where we have used that $\nabla u = 0$ a.e. on $B_1 \setminus \Omega_+$. The last inequality is equivalent to $\mu \geq 0$, in the equation

$$\Delta u = f\chi_{\{u>0\}} + \mu.$$

Properties of Minimizers: $\mu \le f\chi_{\partial\{u>0\}}$

Make a variation upwards, with $0 \le \varphi \in W_0^{1,2}(B_1)$, to obtain

$$\int \nabla u \cdot \nabla \varphi + f\varphi \ge 0 \quad \Longrightarrow \quad \Delta u \le f\chi_{\overline{\Omega_+}} \qquad \text{in } B_1$$

where we used $u = 0$ in $(\overline{\Omega_+})^c$. From here we have

$$f\chi_{\Omega_+} + \mu = \Delta u \le f\chi_{\overline{\Omega_+}}$$

$$\text{i.e.} \qquad 0 \le \mu \le f\chi_{\partial\Omega_+} \in L^\infty.$$

$$\text{Hence} \qquad \Delta u = h \qquad \text{where} \qquad f\chi_{\{u>0\}} \le h \le f\chi_{\overline{\{u>0\}}}.$$

Mathematical preliminaries for FBP

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Mathematical preliminaries for FBP

Initial regularity of u

Next question we want to treat is the regularity of solutions, minimizers. Having written the solution as

$$\Delta u = h, \qquad u \geq 0, \qquad \text{in } B_1,$$

$$f\chi_{\{u>0\}} \leq h \leq f\chi_{\overline{\{u>0\}}}, \qquad f \in C^\alpha(B_1).$$

Since $|h| \leq C$ we have $u \in W^{2,p}_{loc} \cap C^{1,\alpha}_{loc}(B_1)$ (for all $p < \infty$, $\alpha < 1$). In particular $D^2 u = 0$ a.e. in $B_1 \setminus \Omega_+$, and hence $\mu = 0$ a.e. But μ being bounded and zero a.e. it must be zero.

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Mathematical preliminaries for FBP

Regularity of classical PDEs.

- RHS $\in C^\alpha$ $\Longrightarrow$ sol. $C^{2,\alpha}$,
- RHS $\in L^\infty$ $\Longrightarrow$ sol. $C^{1,\alpha}$, any $\alpha < 1$,
- RHS $\in$?? $\Longrightarrow$ sol. $C^{1,1}$, and not more.

For FBP, it is common that we also obtain the last missing regularity. In this case we will prove solutions to our problem

$$\Delta u = f\chi_{\Omega_+} \in L^\infty$$

are $C^{1,1}_{loc}$ whenever f is C^α.

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Mathematical preliminaries for FBP

Assumption on the obstacle

Henceforth, we assume $f \geq \lambda_0 > 0$, as this is crucial for regularity of FB, but not for the solutions u. This assumption prevents degeneracy of solutions.

It means we are only interested in FB points where $\Delta\psi_1 \leq -\lambda_0 < 0$.

Observe that due to super-harmonicity of minimizers they do not touch parts of the obstacle where $\Delta\psi_1 > 0$.

Let me make a further reduction and assume $f \equiv 1$, i.e. $\psi_1 = l(x) - |x|^2/2n$, with $l(x)$ linear.

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Quadratic decay of u at FB

Let us see how the proof works. We shall prove the regularity in $B_{1/4}(0)$ only, for simplicity. Choose $z \in \partial\{u > 0\} \cap B_{1/2}(0)$, and let $0 < r < 1/3$. First, we shall prove

$$\sup_{B_r(z)} u(x) \leq C_0 r^2$$

for a universal C_0.

This will give a quadratic growth/decay for solutions from the FB points.

Why is this enough for showing $u \in C^{1,1}$?

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Mathematical preliminaries for FBP

$C^{1,1}$-regularity from quadratic decay

Pick up any point $y \in \{u > 0\} \cap B_{1/4}(0)$, and define, in $B_1(0)$

$$\tilde{u}_r(x) = \frac{u(y + d_y x)}{r^2}$$

where $d_y = dist(y, \{u = 0\}$.

Let further $z \in \partial\{u > 0\}$ be a/the closest point to y. Then

$$0 \leq \sup_{x \in B_{2r}(z)} \tilde{u}_r(x) \leq 4C_0$$

Since also $\Delta \tilde{u}_r = 1 \in C^\alpha(B_1(0))$, we can apply elliptic regularity to obtain

$$|D^2 u(y)| = |D^2 \tilde{u}_r(0)| \leq C_1$$

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Mathematical preliminaries for FBP

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To close the argument, we need to show the quadratic decay estimate. Decompose the function u as $u = u_1 + u_2$, where

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Observe that $u(z) = 0$, and $u \geq 0$ in B_1, $u_2 \leq 0$, and $u_1 \geq 0$.

In particular Harnack's Inequality applies to u_1 and we conclude

$$u_1(x) \leq c u_1(z) \qquad x \in B_{r/2}(z), \qquad c > 0.$$

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Mathematical preliminaries for FBP

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Putting together we have

$$u(x) = u_1(x) + u_2(x) \leq u_1(x) \leq cu_1(z) = c(u(z) - u_2(z)) = -cu_2(z)$$

where we have used that $u_2(x) \leq 0$, and $u(z) = 0$.

So it remains to show $-u_2(z) \leq Cr^2$, which follows by a barrier argument $h(x) = (|x - z|^2 - r^2)/2n$, then by comparison principle $u_2(x) \geq h(x)$ in $B_r(z)$ and hence $u_2(z) \geq -r^2/2n$.

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Mathematical preliminaries for FBP

$C^{1,1}$-regularity

Conclusion

$$u \in C^{1,1}$$

Also something (weaker) that we shall use later is that

$$u(x) \leq c_1 (dist(x, \partial\Omega_+))^2$$

Mathematical preliminaries for FBP

Non-degeneracy

What is next step? Can a solution converge faster to zero than quadratic? And if so, how does it affect our analysis?

Indeed, if $\Delta u = f$, and $f(z) = 0$, then we have good chance to prove for u, a faster than quadratic decay to zero at z. This is just elliptic regularity.

Analysis of such points fall outside the regularity theory developed by L. Caffarelli for obstacle problem.

So we shall prove that for the case $f > \lambda_0 > 0$, or as we have assumed $f \equiv 1$, then the solution has optimal decay r^2, and not faster.

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Non-degeneracy

We shall prove that for any FB point $z \in \partial\{u > 0\}$ we have

$$\sup_{B_r(z)} u \geq \lambda_0 r^2/2n$$

where $f \geq \lambda_0 > 0$. Since we assumed $f \equiv 1$, we set $\lambda_0 = 1$.

Mathematical preliminaries for FBP

Proof of Non-degeneracy

Let $x^0 \in \Omega_+ := \{u > 0\} \cap B_1(0)$, and consider then the auxiliary function

$$w(x) = u(x) - u(x^0) - \frac{|x - x^0|^2}{2n}.$$

We have $\Delta w = 0$ in $B_r(x^0) \cap \Omega_+$. Since $w(x^0) = 0$, by the maximum principle we have that

$$\sup_{\partial(B_r(x^0) \cap \Omega_+)} w \geq 0.$$

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Proof of Non-degeneracy

Besides, $w(x) = -u(x^0) - |x - x^0|^2/(2n) < 0$ on $\partial\Omega$. Therefore, we must have

$$\sup_{\partial B_r(x^0)\cap\Omega_+} w \geq 0.$$

The latter is equivalent to

$$\sup_{\partial B_r(x^0)\cap\Omega_+} u \geq u(x^0) + \frac{r^2}{2n}.$$

Once we let $x^0 \to \partial\Omega_+$ our claim will follow.

Proof of Non-degeneracy

Besides, $w(x) = -u(x^0) - |x - x^0|^2/(2n) < 0$ on $\partial\Omega$. Therefore, we must have

$$\sup_{\partial B_r(x^0)\cap\Omega_+} w \geq 0.$$

The latter is equivalent to

$$\sup_{\partial B_r(x^0)\cap\Omega_+} u \geq u(x^0) + \frac{r^2}{2n}.$$

Once we let $x^0 \to \partial\Omega_+$ our claim will follow.

Mathematical preliminaries for FBP

Conclusion so far

So far we have shown that for $x^0 \in \partial\Omega_+$

$$c_0 r^2 \leq \sup_{\partial B_r(x^0)\cap\Omega_+} u \leq c_1 r^2 \qquad (c_0, c_1 \text{ universal})$$

and that also $u \in C^{1,1}_{loc}(B_1)$.

Why did we insist on finding these extremes estimates on both sides, and how will we be using them later on?

The reason is to find the invariant scaling of our equation.

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Mathematical preliminaries for FBP

Finding invariant scaling of equation

Indeed if we look at the scaling

$$v_r := \frac{u(rx)}{r^a} \qquad \text{(some } a\text{)}$$

then we see that

$$\Delta v_r = r^{2-a}\chi_{\{u(rx)>0\}} = r^{2-a}\chi_{\{v_r(x)>0\}}.$$

Hence to keep the equation invariant under scaling, we need $a = 2$, so that we can retain the original equation, during any scaling.

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Finding invariant scaling of equation

Now after ensuring that the equation is invariant under this scaling, we also need to make sure, that the scaled function $u_r = u(rx + z)/r^2$ (at FB point z) is

1) Uniformly bounded on compact sets of $\mathbb{R}^n$.
2) Does not flat out to zero, in the limit.

These are guaranteed by optimal growth and non-degeneracy as we showed above.
Just think of examples such as $u(x) \approx o(|x - z|^2)$ near a FB point z, then scaling u_r will give us

$$u_r(x) \approx o(|rx|^2)/r^2 = o(1) \to 0.$$

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Mathematical preliminaries for FBP

Lebesgue and Hausdorff measure of FB

- Let us now discuss very low regularity of FB.
- Can it have positive measure?
- Is $(n-1)$-Hausdorff measure of FB finite?

Mathematical preliminaries for FBP

Lebesgue measure of FB

Let me recall the fact that the Lebesgue upper density of a set E must be 1 for a.a. points of E:

$$\limsup_{r\to 0} \frac{|E \cap B_r(x^0)|}{|B_r|} = 1, \qquad a.a.\ x^0 \in E.$$

We shall use this along with the optimal quadratic growth and non-degeneracy to conclude the FB has zero Lebesgue measure.

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Mathematical preliminaries for FBP

Lebesgue measure of FB

Let $z \in \partial\Omega_+$, and take $r > 0$ (not too big). Then we know that on $\partial B_{r/2}(z)$ there is a point y_r such that

$$c_0(r/2)^2 \leq u(y_r) \leq c_1(dist(y, \partial\Omega_+))^2,$$

where the second inequality uses optimal quadratic decay.

This means that for any r, and $B_r(z)$ (as above) we have a point $y \in B_{r/2}(z)$ such that

$$dist(y_r, \partial\Omega_+) \geq c_2 r$$

In other words

$$B_{c_2 r}(y_r) \ \subset \ \Omega_+ \cap B_r(z).$$

Mathematical preliminaries for FBP

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Mathematical preliminaries for FBP

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This means that for any $z \in \partial\Omega_+$

$$\frac{|\partial\Omega_+ \cap B_r(z)|}{|B_r|} = \frac{|\partial\Omega_+ \cap (B_r(z) \setminus B_{c_2 r}(y_r))|}{|B_r|} \leq 1 - c_3 < 1.$$

But according to Lebesgue density theorem this set is negligible.

Actually we have proved that the FB is porous, i.e. it has dimension less than n.

Mathematical preliminaries for FBP

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$(n-1)$-Hausdorff measure of $\partial\Omega_+$

We want to show that $\partial\Omega_+$ is a set of finite $(n-1)$-dimensional Hausdorff measure locally in $B_{1/2}$. I.e., we want to cover the FB with $(1/\epsilon)^{n-1}$-balls of radius ϵ.

Let

$$v_i = \partial_{x_i} u, \quad i = 1, \ldots, n, \qquad E_\varepsilon = \{|\nabla u| < \varepsilon\} \cap \Omega_+.$$

Observe that

$$1 \leq |\Delta u|^2 \leq c_n \sum_{i=1}^{n} |\nabla v_i|^2 \quad \text{in } \Omega_+.$$

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Mathematical preliminaries for FBP

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Thus, for an arbitrary compact set $K \Subset B_1$ we have

$$|K \cap E_\epsilon| \leq c_n \int_{K\cap E_\epsilon} \sum_i |\nabla v_i|^2 dx \leq c_n \sum_i \int_{K\cap\{|v_i|<\epsilon\}\cap\Omega_+} |\nabla v_i|^2 dx.$$

To estimate the right hand side, we notice that

$$\Delta v_i^\pm = \Delta(\partial_{x_i} u)^\pm \geq 0,$$

$$\text{i.e.} \qquad \int_{B_1} \nabla v_i^\pm \nabla \eta \, dx \leq 0, \quad i = 1, ..., n$$

for any non-negative $\eta \in C_0^\infty(B_1)$, and by continuity, for any nonnegative $\eta \in W_0^{1,2}(B_1)$.

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Mathematical preliminaries for FBP

$(n-1)$-Hausdorff measure of $\partial\Omega_+$

Let $\eta = \psi_\epsilon(v_i^\pm)\phi$, with $0 \le \phi \in C_0^\infty(B_1)$, $\phi = 1$ on K,

$$\psi_\varepsilon(t) = \begin{cases} 0, & t \le 0 \\ \epsilon^{-1} t, & 0 \le t \le \epsilon \\ 1, & t \ge \epsilon. \end{cases}$$

Then

$$\int_{B_1} \nabla v_i^\pm \nabla(\psi_\epsilon(v_i^\pm)\phi) dx =$$

$$\int_{\{0<v_i^\pm<\epsilon\}} \epsilon^{-1} |\nabla v_i^\pm|^2 \phi \, dx + \int_{B_1} \nabla v_i^\pm \psi_\epsilon(v_i^\pm) \nabla\phi \, dx \le 0.$$

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Mathematical preliminaries for FBP

$(n-1)$-Hausdorff measure of $\partial\Omega_+$

For $M := \|D^2 u\|_{L^\infty(B_1)}$, we use the above estimate to arrive at

$$\epsilon^{-1}\int_{K\cap\{|v_i|<\epsilon\}\cap\Omega_+(u)} |\nabla v_i|^2 dx \le \epsilon^{-1}\int_{\{0<|v_i|<\epsilon\}} |\nabla v_i|^2 \phi\, dx$$
$$\le \int_{B_1} |\nabla v_i||\nabla\phi|\, dx \le c_n M \int_{B_1} |\nabla\phi|\, dx.$$

Thus, summing over $i = 1, \dots, n$, we will arrive at an estimate

$$c_0 |K \cap E_\epsilon| \le C\epsilon M,$$

where $C = C(n, K, B_1)$.

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Consider now a covering of $\partial\Omega_+ \cap K$ by a finite family $\{B^i\}_{i\in I}$ of balls of radius ϵ centered at $\partial\Omega_+ \cap K$, such that no more than $N = N_n$ balls from this family overlap (Besicovitch covering).

For $\epsilon > 0$ sufficiently small, we may assume that $B^i \subset K'$ for a slightly larger compact K' so that $K \Subset \text{Interior}(K') \Subset B_1$. Now notice that $|\nabla u| < M\epsilon$ in each B^i, implying that

$$B^i \cap \Omega_+ \subset E_{M\epsilon} = \{x : |\nabla u| < M\epsilon\}.$$

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Mathematical preliminaries for FBP

$(n-1)$-Hausdorff measure of $\partial\Omega_+$

Then, using density argument and estimate above, we obtain

$$\sum_{i\in I}|B^i| \leq C_1 \sum_{i\in I}|B^i \cap \Omega_+| \leq C_1 \sum_{i\in I}|B^i \cap E_{M\epsilon}| \leq NC_1|K' \cap E_{M\epsilon}|$$

But we have already shown that

$$c_0\,|K \cap E_\epsilon| \leq C\epsilon M, \qquad \text{and} \qquad K \approx K'$$

which combined with the above estimate (with ϵ replaced by $M\epsilon$) implies

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Mathematical preliminaries for FBP

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Mathematical preliminaries for FBP

$(n-1)$-Hausdorff measure of $\partial\Omega_+$

This gives the estimate

$$\sum_{i\in I} \mathrm{diam}(B^i)^{n-1} = \sum_{i\in I} \epsilon^{n-1} \leq C(n, M, K, B_1),$$

which by letting $\epsilon \to 0$, and taking infimum, implies

$$H^{n-1}(FB) \cap K) \leq C(n, M, K, B_1).$$

Regularity theory for obstacle problem

Regularity Theory

What is to be discussed

- Scaling and blow-ups
- Global solutions, and their role for regularity theory.
- Good points of FB $\equiv$ No cusps.
- At good points of FB: Lipschitz regularity (Use directional monotonicity).
- Lipschitz points are C^1: Improved Lip. norm.

What does it mean FB being Regular?

Since FB in our problems are boundary of a set, and as such they are regular if locally they can be as graph of regular functions.

Now how to prove something is locally a (smooth) graph?

Observe that the FB is $\partial\{u > 0\}$. So we need somehow to involve the function u in our analysis

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Regularity Theory

Illustration within an example

As an example consider (for m fixed positive number)

$$u(x) := \left[(x_2 - (\cos mx_1) + 1)_+\right]^2.$$

Then $\Delta u = m^2 f(x)\chi_{\{u>0\}}$, for some smooth f, with $f(0) > 0$.

Moreover for $e = (a_1, a_2) \approx e_2$,

$$D_e u \approx c_1 a_2 (x_2)_+ \approx c_2 a_2 \sqrt{u} \geq c_3 a_2 u \geq 0$$

if we are close to FB. Hence we have $\partial\{u > 0\}$ is a graph in e-direction. Since e can vary (at least slightly) we may deduce Lipschitz regularity of FB.

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This suggests that for $e \approx e_2$ we need to prove

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Regularity Theory

Scaling and blow-ups

So far we have learned that solutions of one of our problems, which in its simplest form solves

$$\Delta u = \chi_{\{u>0\}}, \qquad u \geq 0, \qquad \text{in } B_1,$$

have some good properties such as

$$c_0 r^2 \leq \sup_{B_r(z)} u(x) \leq c_1 r^2, \qquad z \in \Gamma \cap B_{1/2}$$

where $\Gamma = \partial\{u > 0\}$.

Scaling and blow-ups

Thes bounds give us that the scaled function

$$u_r(x) = \frac{u(rx+z)}{r^2}$$

satisfies

$$\Delta u_r = \chi_{\{u_r>0\}}, \qquad u_r \geq 0, \qquad \text{in } B_{1/r},$$

and

$$c_0 \leq \sup_{B_1(0)} u_r(x) \leq c_1, \qquad 0 \in \Gamma_r$$

where $\Gamma_r = \frac{1}{r}(\Gamma - z)$ is scaled version $\Gamma - z$.

The last two conditions guarantee that u_r (for a subsequence) will converge to a solution in $\mathbb{R}^n$, for the same problem

$$\Delta u_0 = \chi_{\{u_0>0\}}, \qquad u_0 \geq 0, \qquad \text{in } \mathbb{R}^n,$$

and moreover

$$\sup_{B_R} u_0(x) \leq CR^2 \qquad \forall\ R > 0.$$

To see the latter we set $x = Ry$, with $|y| = 1$. Then for some subsequence r_j

$$u_0(x) \approx \frac{u(r_j x + z)}{r_j^2} = \frac{u(r_j Ry + z)}{(Rr_j)^2} R^2 \leq c_1 R^2.$$

Regularity Theory

What does this tell us?

Suppose the FB, Γ at point z has some irregularities, say a cusp, or a cone shape singularity. If so, then this property must be reflected in the scaled version Γ_r, and hence for the blow-up Γ_0.

On the other hand if Γ at point z is regular, say C^1, then Γ_0 must be a hyperplane, with $u_0 > 0$ on one side and $u_0 = 0$ on the other side.

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Regularity Theory

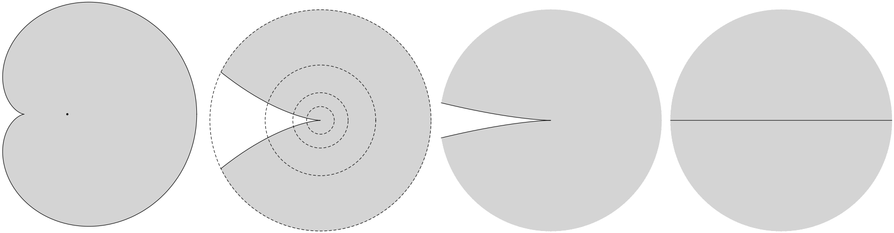

Figure: The Cardioid from left to right scaled at its cusp point

Regularity Theory

The above pictures suggests that the local analysis of FB has to take into consideration the global pictures, after scaling, and blow-up.
Therefore we need to see what kind of global solutions we shall have after blow-up of a solution.

We shall first make a reasonable observation:

$$\boxed{\text{Blow-ups are homogeneous}}$$

How do we prove this?

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Weiss Monotonicity functional

To prove homogeneity we shall prove the monotonicity of

$$W(r,u,x^0) := \frac{1}{r^{n+2}} \int_{B_r(x^0)} (|\nabla u|^2 + 2u)\, dx - \frac{2}{r^{n+3}} \int_{\partial B_r(x^0)} u^2 dH^{n-1}.$$

$$= \int_{B_1(0)} (|\nabla u_r|^2 + 2u_r)\, dx - 2 \int_{\partial B_1(0)} u_r^2 dH^{n-1},$$

where

$$u_r(x) := \frac{u(rx + x^0)}{r^2}, \qquad x^0 \in \partial\{u > 0\}.$$

Regularity Theory

Weiss Monotonicity functional

$$W'(r) = \int_{B_1(0)} (2\nabla u_r \cdot \nabla u_r' + 2u_r')\, dx - 2\int_{\partial B_1(0)} 2u_r u_r' dH^{n-1}.$$

Integrating by parts and using the PDE $\Delta u_r = \chi_{\{u_r>0\}}$, and that

$$u_r \chi_{\{u_r>0\}} = u_r$$

gives us

$$W'(r) = 2\int_{\partial B_1(0)} (\nabla u_r u_r' - 2u_r u_r')\, dH^{n-1} = \frac{2}{r}\int_{\partial B_1(0)} (u_r')^2\, dH^{n-1},$$

which is strictly positive unless $u_r' = 0$.

Observe that $u_r' = 0$ for $r_1 < r < r_2$ implies (setting $y = rx + x^0$)

$$rx \cdot \nabla u(rx + x^0) - 2u(rx + x^0) = (y - x^0) \cdot \nabla u(y) - 2u(y) = 0$$

in $B_{r_2} \setminus B_{r_1}$. This means u is homogeneous of degree 2.
Now we see that for $s > 0$

$$W(0^+, u, x^0) = \lim_{r\to 0} W(sr, u, x^0) = \lim_{r\to 0} W(s, u_r, 0) = W(s, u_0, 0)$$

and hence $W(s, u_0, 0)$ is constant in s, and hence $W' = 0$, implying that the blow-up u_0 is homogeneous of degree 2.

The conclusion from above is that a blow-up is homogeneous function of degree 2.
So it remains now to classify homogeneous global solutions of degree 2.
Our next task will be to show:

$\boxed{\text{Any 2-homogeneous solution is either a polynomial or 1-dim.}}$

The only one dimensional solutions (after suitable rotations) are either 2-degree polynomials or

$$\mathbf{u}_0 = \frac{1}{2}(x_1^+)^2.$$

To prove the above classification, we can look at two possible situations

- The set $\{u_0 = 0\}$ has empty interior.
- The set $\{u_0 = 0\}$ has non-void interior.

The first case, along with the fact that the FB has zero measure, implies that $\Delta u_0 = 0$ a.e. in $\mathbb{R}^n$, and that it has quadratic growth.
By Liouville's theorem we obtain u_0 is a polynomial of degree two.

Regularity Theory

The second case is more complicated. One can prove it in different ways. Using ACF monotonicity functional, or in this particular case (since $u_0 \geq 0$) one may try to prove convexity of u_0, i.e., to prove $D_{ee}u_0 \geq 0$ for any direction e. We choose the latter method. Hence we claim

$$\boxed{\text{For any global solution } u_0 \text{ we have } D_{ee}u_0 \geq 0}$$

Regularity Theory

If the claim fails, then since u_0 is $C^{1,1}$, we must have

$$D_{ee}u_0(x) \geq -C > -\infty$$

and hence there is a minimizing sequence x^j such that

$$D_{ee}u_0(x^j) \quad \to \quad -m = \inf_{\mathbb{R}^n} D_{ee}u_0(x) > -\infty.$$

Now let $d_j = \mathrm{dist}(x^j, \{u_0 = 0\})$, and set

$$v_j(x) := \frac{u_0(d_j x + x^j)}{d_j^2}.$$

Regularity Theory

Then $\Delta v_j = 1$ in B_1, and v_j are uniformly bounded (due to quadratic growth of u_0). Hence a subsequence of v_j converges to a new global solution v_0 satisfying $\Delta v_0 = 1$ in B_1, and moreover

$$D_{ee} v_0(0) = -m, \qquad \text{and} \qquad D_{ee} v_0(x) \geq -m.$$

By minimum principle $D_{ee} v_0(x) = -m$ in the connected component of $\{v_0 > 0\}$ that contains the origin. Let us rotate and set $e = e_1$, and set $D_{11} v_0(x) = -m$. If the set $\{v_0 = 0\}$ has empty interior then as before v_0 is a polynomial and hence integration gives $v_0 = -m x_1^2/2 + P_2(x')$, which takes negative values, and contradicts $v_0 \geq 0$.

Regularity Theory

On the other hand if the set $\{v_0 = 0\}$ has non-void interior, e.g. it contains $B_r(z)$, then it means that $D_1 v_0$ is decreasing in e_1 direction and hence becomes negative in e_1 direction starting from $B_r(z)$.

Now $D_v 0 \leq 0$ means v_0 is decreasing in e_1-direction and hence becomes negative in e_1 direction starting from $B_r(z)$. This is impossible since $v_0 \geq 0$.

Hence it follows that v_0 must be convex $D_{ee} v_0(x) \geq 0$.

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Regularity Theory

Now we use convexity to show that blow-ups are 1-dimensional.

The convexity of v_0 implies $\{v_0 = 0\}$ is convex, and moreover $D_e v_0 \geq 0$, along any line $L_e := \{z + se, \; s \in \mathbb{R}\}$ that intersects the set $\{v_0 = 0\}$.

From here, the homogeneity and convexity of v_0, and that $\{v_0 = 0\}$ is a cone with largest angle being less than π.

Since $D_1 v_0$ is harmonic, Lipschitz, and non-negative we can use (2-dimensional) barriers to show a contradiction unless $\{v_0 = 0\}$ is a half-space, and v_0 is 1-dimensional, that is half-space solution $(x_1^+)^2/2$.

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Regularity Theory

Let us summarize what we have so far:

A local solution can be scaled and blown up, to obtain a homogeneous global solution, which is either a polynomial or a half-space solution $(x_1^+)^2/2$.

Examples of the cardioid type as well as simple polynomials $(x_1^2 + x_2^2)/4$ in dimension 3 suggests that we should pay attention to possible irregularities, that may appear.

Therefore to obtain regularity we shall need to have some restriction on the FB-points. E.g. that the free boundary has no cusp at z, i.e., at least one blow-up cannot be a polynomial, and has to be a half-space solution.

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Regularity Theory

A way of doing so, is to look at the balanced energy,
Let u be a solution of our problem and $x^0 \in \Gamma(u)$. Then

$$x^0 \text{ is regular} \iff W(0^+, u, x^0) = \frac{\alpha_n}{2},$$
$$x^0 \text{ is singular} \iff W(0^+, u, x^0) = \alpha_n,$$

where

$$\alpha_n = W(r, P, 0) = \cdots = \frac{H^{n-1}(\partial B_1)}{2n(n+2)},$$

and $P(x)$ is any polynomial solution to our problem.

Regularity Theory

The conclusion is that if x^0 is a regular point according to the balanced energy, $W(0^+, u, x^0) = \frac{\alpha_n}{2}$, then any blow-up of u at x^0 is a half-space solution. More exactly we obtain (for r small enough)

$$\|u_r - u_0\|_{L^\infty(B_1)} \leq \epsilon. \qquad \text{where } u_0(x) = \frac{1}{2}(x_n^+)^2.$$

This in turn implies (using non-degeneracy)

$$\begin{aligned} u_r &> 0 \quad \text{in } \{x_n > \sqrt{2\epsilon}\} \cap B_1, \\ u_r &= 0 \quad \text{in } \{x_n \leq -2\sqrt{n\epsilon}\} \cap B_{1/2}. \end{aligned}$$

In particular,

$$\Gamma(u_r) \cap B_{1/2} \subset \{|x_n| \leq 2\sqrt{n\epsilon}\}.$$

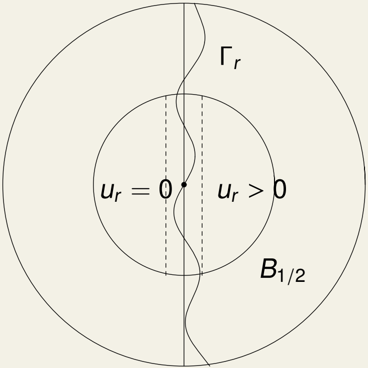

Figure: Flatness of $\Gamma(u)$

Regularity Theory

If the free boundary point x^0 is regular (in the sense of energy $W(0^+, u, x^0) = \alpha_n/2$) then a scaled version of our solution u_r satisfies

$$C\partial_e u_r - r u_r \geq -\epsilon_0 \quad \text{in } B_1, \tag{1}$$

with $\epsilon_0 < 1/8n$, and $e \approx e_1$, or more exactly for some $\delta > 0$

$$e \in \mathcal{C}_\delta := \{x \in \mathbb{R}^n : x_n > \delta|x'|\}, \quad x' = (x_1, \dots, x_{n-1}).$$

This in turn implies

$$C\partial_e u_r - r u_r \geq 0 \quad \text{in } B_{1/2}. \tag{2}$$

Proof of directional monotonicity

We shall assume w.l.o.g. $r = 1$, as the drill is the same. Suppose the conclusion of the lemma fails and let $y \in B_{1/2} \cap \Omega$ be such that $C\partial_e u(y) - u(y) < 0$. Consider then the auxiliary function

$$w(x) = C\partial_e u(x) - u(x) + \frac{1}{2n}|x - y|^2.$$

It is easy to see that w is harmonic in $\Omega \cap B_{1/2}(y)$, $w(y) < 0$, and that $w \geq 0$ on $\partial\Omega$.

Proof of directional monotonicity

Hence by the minimum principle, w has a negative infimum on $\partial B_{1/2}(y) \cap \Omega$, i.e.

$$\inf_{\partial B_{1/2}(y)\cap\Omega} w < 0.$$

This can be rewritten as

$$\inf_{\partial B_{1/2}(y)\cap\Omega} (C\partial_e u - u) < -\frac{1}{8n},$$

which contradicts (1), since $\epsilon_0 < 1/8n$.

Regularity Theory

Then for any $z \in \Gamma(u) \cap B_{1/2}$

$$\begin{aligned} u > 0 \quad &\text{in } (z + \mathcal{C}_\delta) \cap B_{1/2}, \\ u = 0 \quad &\text{in } (z - \mathcal{C}_\delta) \cap B_{1/2}. \end{aligned}$$

In particular, $\Gamma(u) \cap B_{1/2}$ is a Lipschitz graph $x_n = f(x')$ with the Lipschitz constant of f not exceeding δ

Regularity Theory

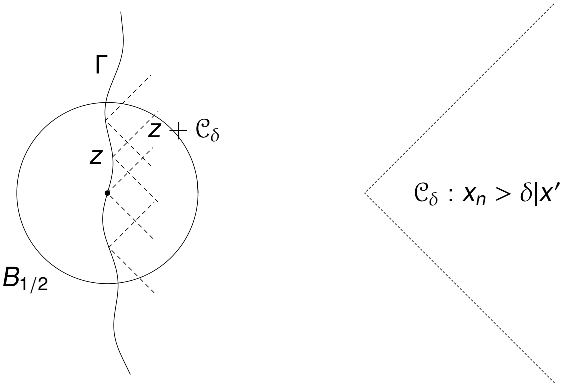

Figure: Cone of monotonicity $\mathcal{C}_\delta$ and Lipschitz regularity of the free boundary

Regularity Theory

In the original scaling, there exists $\rho = \rho(u) > 0$ and a Lipschitz function $f : B'_\rho \to \mathbb{R}$ such that

$$\Omega_+ \cap B_\rho = \{x \in B_\rho : x_n > f(x')\},$$
$$\Gamma_+ \cap B_\rho = \{x \in B_\rho : x_n = f(x')\}.$$

Moreover, if for r_δ, $\delta \in (0, 1]$, is such that we have

$$\|u_r - u_0\|_{L^\infty(B_1)} \le \epsilon. \qquad \text{where } u_0(x) = \frac{1}{2}(x_n^+)^2,$$

then $|\nabla_{x'} f| \le \delta$ a.e. on $B'_{r_\delta/2}$.
Since $\delta > 0$ is arbitrarily, we conclude the existence of a tangent plane to FB at the origin. This can be done similarly for points close to the origin, and hence we obtain C^1-graph, locally.