A short course on Mean field games

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Chapter 1

Introduction

Mean field game theory is devoted to the analysis of differential games with infinitely many players. For such large population dynamic games, it is unrealistic for a player to collect detailed state information about all other players. Fortunately this impossible task is useless: mean field game theory explains that one just needs to implement strategies based on the distribution of the other players. Such a strong simplification is well documented in the (static) game community since the seminal works of Aumann [2]. However, for differential games, this idea has been considered only very recently: the starting point is a series of papers by Lasry and Lions [23, 24, 25], who introduced the terminology in around 2005. The term mean field comes for an analogy with the mean field models in mathematical physics, which analyse the behavior of many identical particles (see for instance Sznitman's notes [34]). Here the particules are replaced by agents or players, whence the name of mean field games. Related ideas have been developed independently, and at about the same time, by Caines, Huang and Malhamé [17, 18, 19, 20], under the name of Nash certainty equivalence principle.

This text aims at a short (and very incomplete) presentation of mean field games, as self-contained as possible.

Chapter 2

An appetizer: Nonatomic games

Before starting the analysis of differential games with a large number of players, it is not uninteresting to have a look at this question for classical (one-shot) games.

The general framework is the following: let N be a (large) number of players. We assume that the players are symmetric. In particular, the set of strategies Q is the same for all players. We denote by $F_i^N = F_i^N(x_1, \ldots, x_N)$ the payoff (= the cost) of player $i \in \{1, \ldots, N\}$. Our symmetry assumption means that

$$F^N_{\sigma(i)}(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = F_i(x_1, \ldots, x_N)$$

for all permutation σ on $\{1, \ldots, N\}$. We consider Nash equilibria for this game and want to analyze their behavior as $N \to +\infty$.

For this we first recall the notion of Nash equilibria for one shot games. In order to proceed with the analysis of large population games, we describe next the limit of maps of many variable. Then we explain the limit, as the number of players tends to infinity, of Nash equilibria in pure, and then in mixed, strategies. We finally discuss the uniqueness of the solution of the limit equation and present some examples.

2.1 Nash equilibria in classical differential games

In this section, we introduce the notion of Nash equilibria in one-shot games.

Let $S_1, \ldots, S_N$ be compact metric spaces, $J_1, \ldots, J_N$ be continuous real valued functions on $\prod_{i=1}^N S_i$.

Definition 2.1.1. *A Nash equilibrium in pure strategies is a $N-$tuple $(\bar{s}_1, \ldots, \bar{s}_N) \in \prod_{i=1}^N S_i$ such that, for any $i = 1, \ldots, N$,*

$$J_i(\bar{s}_1, \ldots, \bar{s}_N) \leq J_i\left(s_i, (\bar{s}_j)_{j\neq i}\right) \qquad \forall s_i \in S_i\ .$$

Unfortunately Nash equilibria in pure strategies seldom exist and we have to introduce the notion of mixed strategies. For this we denote by $\mathcal{P}(S_i)$ the compact metric space of all Borel probability measures defined on S_i.

Definition 2.1.2. *A Nash equilibrium in mixed strategies is a $N-$tuple $(\bar{\pi}_1, \ldots, \bar{\pi}_N) \in \prod_{i=1}^N \mathcal{P}(S_i)$ such that, for any $i = 1, \ldots, N$,*

$$J_i(\bar{\pi}_1, \ldots, \bar{\pi}_N) \leq J_i\left((\bar{\pi}_j)_{j\neq i}, \pi_i\right) \qquad \forall \pi_i \in \mathcal{P}(S_i)\ . \tag{2.1}$$

where by abuse of notation

$$J_i(\pi_1, \ldots, \pi_N) = \int_{S_1\times\cdots\times S_N} J_i(s_1, \ldots, s_N) d\pi_1(s_1) \ldots d\pi_N(s_N)\ .$$

Remark 2.1.3. Note that condition (2.1) is equivalent to

$$J_i(\bar{\pi}_1, \ldots, \bar{\pi}_N) \leq J_i\left((\bar{\pi}_j)_{j\neq i}, s_i\right) \qquad \forall s_i \in S_i\ .$$

This later characterization is very convenient and used throughout the notes.

Theorem 2.1.4 (Nash (1950), Glicksberg (1952)). *Under the above assumptions, there exists at least one equilibrium point in mixed strategies.*

Proof. It is a straightforward application of Fan's fixed point Theorem [11]: let X be a non-empty, compact and convex subset of a locally convex topological vector space. Let $\phi : X \to 2^X$ be an upper semicontinuous set-valued map such that $\phi(x)$ is non-empty, compact and convex for all $x \in X$. Then ϕ has a fixed point: $\exists \bar{x} \in X$ with $\bar{x} \in \phi(\bar{x})$.

Let us recall that the upper semicontinuity of set-valued function $\phi : X \to 2^X$ means that, for every open set $W \subset X$, the set $\{x \in X\ ,\ \phi(x) \cap W\}$ is open in X.

Let us set $X = \prod_{j=1}^N \mathcal{P}(S_i)$ and let us consider the best response map $\mathcal{R}_i : X \to \mathcal{P}(S_i)$ of player i defined by

$$\mathcal{R}_i((\pi_j)_{j=1,\dots,N}) = \left\{\pi \in \mathcal{P}(S_i)\ ,\ J_i((\pi_j)_{j\neq i}, \pi) = \min_{\pi' \in \mathcal{P}(S_i)} J_i((\pi_j)_{j\neq i}, \pi')\right\} .$$

Then the map $\phi((\pi_j)_{j=1,\dots,N}) = \prod_{i=1}^N \mathcal{R}_i((\pi_j)_{j=1,\dots,N})$ is upper semicontinuous with non-empty, compact and convex values. Therefore it has a fixed point, which is a Nash equilibrium. □

We now consider the case where the game is symmetric. Namely, we assume that, for all $i \in \{1, \dots, N\}$, $S_i = S$ and $J_i(s_1, \dots, s_N) = J_{\theta(s_i)}(s_{\theta(1)}, \dots, s_{\theta(N)})$ for all and all permutation θ on $\{1, \dots, N\}$.

Theorem 2.1.5 (Symmetric games). *If the game is symmetric, then there is an equilibrium of the form $(\bar{\pi}, \dots, \bar{\pi})$, where $\bar{\pi} \in \mathcal{P}(S)$ is a mixed strategy.*

Proof. Let $X = \mathcal{P}(S)$ and $\mathcal{R} : X \to 2^X$ be the set-valued map defined by

$$\mathcal{R}(\pi) = \left\{\sigma \in X\ ,\ J_i(\sigma, \pi, \dots, \pi) = \min_{\sigma' \in X} J_i(\sigma', \pi, \dots, \pi)\right\} .$$

Then $\mathcal{R}$ is upper semicontinuous with nonempty convex compact values. By Fan's fixed point Theorem, it has a fixed point $\bar{\pi}$ and, from the symmetry of the game, the $N-$tuple $(\bar{\pi}, \dots, \bar{\pi})$ is a Nash equilibrium. □

2.2 Symmetric functions of many variables

Let Q be a compact metric space and $u_N : Q^N \to \mathbb{R}$ be a symmetric function:

$$u_N(x_1, \dots, x_N) = u_N(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \qquad \text{for any permutation } \sigma \text{ on } \{1, \dots, n\}.$$

Our aim is to define a limit for the u_N.

For this let us introduce the set $\mathcal{P}(Q)$ of Borel probability measures on Q. This set is endowed with the topology of weak-* convergence: a sequence (m_N) of $\mathcal{P}(Q)$ converges to $m \in \mathcal{P}(Q)$ if

$$\lim_N \int_Q \varphi(x) dm_N(x) = \int_Q \varphi(x) dm(x) \qquad \forall \varphi \in \mathcal{C}^0(Q) .$$

Let us recall that $\mathcal{P}(Q)$ is a compact metric space for this topology, which can be metrized by the distance (often called the Kantorowich-Rubinstein distance)

$$\mathbf{d}_1(\mu, \nu) = \sup\{\int_Q f d(\mu - \nu) \text{ where } f : Q \to \mathbb{R} \text{ is } 1\text{−Lipschitz continuous}\} .$$

Other formulations for this distance will be given later (section 3.2).

In order to show that the (u_N) have a limit (at least up to a subsequence), we assume the following:

1. (Uniform bound) there is some $C > 0$ with

$$\|u_N\|_{L^\infty(Q)} \leq C \tag{2.2}$$

2. (Uniform continuity) there is a modulus of continuity ω independent of n such that

$$|u_N(X) - u_N(Y)| \leq \omega(\mathbf{d}_1(m_X^N, m_Y^N)) \qquad \forall X, Y \in Q^N,\ \forall N \in \mathbb{N}, \tag{2.3}$$

where $m_X^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and $m_Y^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$ if $X = (x_1, \ldots, x_N)$ and $Y = (y_1, \ldots, y_N)$.

Theorem 2.2.1. *If the u_N are symmetric and satisfy (2.2) and (2.3), then there is a subsequence (u_{N_k}) of (u_N) and a continuous map $U : \mathcal{P}(Q) \to \mathbb{R}$ such that*

$$\lim_{k\to+\infty} \sup_{X\in Q^{N_k}} |u_{N_k}(X) - U(m_X^{N_k})| = 0 \ .$$

Proof of Theorem 2.2.1. Without loss of generality we can assume that the modulus ω is concave. Let us define the sequence of maps $U^N : \mathcal{P}(Q) \to \mathbb{R}$ by

$$U^N(m) = \inf_{X\in Q^N} \left\{u_N(X) + \omega(\mathbf{d}_1(m_X^N, m))\right\} \qquad \forall m \in \mathcal{P}(Q) \ .$$

Then, by condition (2.3), $U^N(m_X^N) = u_N(X)$ for any $X \in Q^N$. Let us show that the U^N have ω for modulus of continuity on $\mathcal{P}(Q)$: indeed, if $m_1, m_2 \in \mathcal{P}(Q)$ and if $X \in Q^N$ is $\epsilon-$optimal in the definition of $U^N(m_2)$, then

$$\begin{array}{rcl}
U^N(m_1) & \leq & u_N(X) + \omega(\mathbf{d}_1(m_X^N, m_1)) \\
& \leq & U^N(m_2) + \epsilon + \omega(\mathbf{d}_1(m_X^N, m_2) + \mathbf{d}_1(m_1, m_2)) - \omega(\mathbf{d}_1(m_X^N, m_2)) \\
& \leq & U^N(m_2) + \omega(\mathbf{d}_1(m_1, m_2)) + \epsilon
\end{array}$$

because ω is concave. Hence the U^N are equicontinuous on the compact set $\mathcal{P}(Q)$ and uniformly bounded. We complete the proof thanks to Ascoli Theorem. □

Remark 2.2.2. Some uniform continuity condition is needed: for instance if Q is a compact subset of $\mathbb{R}^d$ and $u_N(X) = \max_i |x_i|$, then u_N "converges" to $U(m) = \sup_{x\in spt(m)} |x|$ which is not continuous. Of course the convergence is not uniform.

Remark 2.2.3. If Q is a compact subset of some finite dimensional space $\mathbb{R}^d$, a typical condition which ensures (2.3) is the existence of a constant $C > 0$, independent of N, such that

$$\sup_{i=1,\ldots,N} \|D_{x_i} u_N\|_\infty \leq \frac{C}{N} \qquad \forall N \ .$$

2.3 Limits of Nash equilibria in pure strategies

Let Q be a compact metric space and $\mathcal{P}(Q)$ be the set of Borel probability measures on Q.

We consider a one-shot game with a large number N of players. Our main assumption is that the payoffs $F_1^N, \ldots, F_N^N$ of the players are symmetric. In particular, under suitable bounds and uniform continuity, we know from Theorem 2.2.1 that the F_i^N have a limit, which has the form $F(x, m)$ (the dependence on x is here to keep track of the fact of the dependence in i of the function F_i^N). So the payoffs of the players are very close to payoffs of the form $F(x_1, \frac{1}{N-1}\sum_{j\geq 2}\delta_{x_j}), \ldots, F(x_N, \frac{1}{N-1}\sum_{j\leq N-1}\delta_{x_j})$.

In order to keep the presentation as simple as possible, we suppose that the payoffs have already this form. That is, we suppose that there is a continuous map $F : Q \times \mathcal{P}(Q) \to \mathbb{R}$ such that, for any $i \in \{1, \ldots, N\}$

$$F_i^N(x_1, \ldots, x_N) = F\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right) \qquad \forall (x_1, \ldots, x_N) \in Q^N \ .$$

Let us recall that a Nash equilibrium for the game $(F_1^N, \ldots, F_N^N)$ is an element $(\bar{x}_1^N, \ldots, \bar{x}_N^N) \in Q^N$ such that

$$F_i^N(\bar{x}_1^N, \ldots, \bar{x}_{i-1}^N, y_i, \bar{x}_{i+1}^N, \ldots, \bar{x}_N^N) \geq F_i^N(\bar{x}_1^N, \ldots, \bar{x}_N^N) \qquad \forall y_i \in Q \ .$$

We set

$$\bar{X}^N = (\bar{x}_1^N, \dots, \bar{x}_N^N) \qquad \text{and} \qquad m_{\bar{X}^N}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{x}_i^N} \ .$$

Theorem 2.3.1. *Assume that, for any N, $\bar{X}^N = (\bar{x}_1^N, \dots, \bar{x}_N^N)$ is a Nash equilibrium in pure strategies for the game $F_1^N, \dots, F_N^N$. Then up to a subsequence, the sequence of measures $(m_{\bar{X}^N}^N)$ converges to a measure $\bar{m} \in \mathcal{P}(Q)$ such that*

$$\int_Q F(y,\bar{m})d\bar{m}(y) = \inf_{m\in\mathcal{P}(Q)} \int_Q F(y,\bar{m})dm(y) \ . \tag{2.4}$$

Remark 2.3.2. The "mean field equation" (2.4) is equivalent to saying that the support of $\bar{m}$ is contained in the set of minima of $F(y,\bar{m})$.

Indeed, if Spt$\bar{m} \subset \arg-\min_{y\in Q} F(y,\bar{m})$, then clearly $\bar{m}$ satisfies (2.4). Conversely, if (2.4) holds, then choosing $m = \delta_x$ shows that $\int_Q F(y,\bar{m})d\bar{m}(y) \leq F(x,\bar{m})$ for any $x \in Q$. Therefore $\int_Q F(y,\bar{m})d\bar{m}(y) \leq \min_{x\in Q} F(x,\bar{m})$, which implies that $\bar{m}$ is supported in $\arg-\min_{y\in Q} F(y,\bar{m})$.

Remark 2.3.3. The result is not completely satisfying because it requires the existence of Nash equilibria in the $N-$player game, which does not always hold. However there always exists Nash equilibria in mixed strategies, i.e., when the player are allowed to randomize their behavior by playing strategies in $\mathcal{P}(Q)$ instead of Q. We discuss this point below.

Proof. Without loss of generality we can assume that the sequence $(m_{\bar{X}^N}^N)$ converges to some $\bar{m}$. Let us check that $\bar{m}$ satisfies (2.4).

For this we note that, by definition, the measure $\delta_{\bar{x}_i^N}$ is a minimum of the problem

$$\inf_{m\in\mathcal{P}(Q)} \int_Q F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{\bar{x}_j^N})dm(y) \ .$$

Since

$$\mathbf{d}_1\left(\frac{1}{N-1}\sum_{j\neq i}\delta_{\bar{x}_j^N}, m_{\bar{X}^N}^N\right) \leq \frac{2}{N} \ ,$$

and since F is uniformly continuous, the measure $\delta_{\bar{x}_i^N}$ is also $\epsilon-$optimal for the problem

$$\inf_{m\in\mathcal{P}(Q)} \int_Q F(y, m_{\bar{X}^N}^N)dm(y)$$

as soon as N is sufficiently large. By linearity, so is $m_{\bar{X}^N}^N$:

$$\int_Q F(y, m_{\bar{X}^N}^N)dm_{\bar{X}^N}^N(y) \leq \inf_{m\in\mathcal{P}(Q)} \int_Q F(y, m_{\bar{X}^N}^N)dm(y) + \epsilon \ .$$

Letting $N \to +\infty$ gives the result. $\square$

2.4 Limit of Nash equilibria in mixed strategies

We now assume that the players play the same game $F_1^N, \dots, F_N^N$ as before, but there are allowed to play in *mixed strategies*, i.e., they minimize over elements of $\mathcal{P}(Q)$ instead of minimizing over elements of Q (which are now viewed as *pure strategies*). If the players play the mixed strategies $\pi_1, \dots, \pi_N \in \mathcal{P}(Q)$, then the outcome of Player i (still denoted, by abuse of notation, F_N^i) is

$$F_i^N(\pi_1, \dots, \pi_N) = \int_{Q^N} F\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right) d\pi_1(x_1)\dots d\pi_N(x_N) \ . \tag{2.5}$$

Let us recall that the notion of Nash equilibria in mixed strategies is defined in Definition 2.1.2 and that we explained in Theorem 2.1.5 that symmetric Nash equilibria do exists.

Theorem 2.4.1. *We assume that F is Lipschitz continuous. Let, for any N, $(\bar{\pi}^N, \dots, \bar{\pi}^N)$ be a symmetric Nash equilibrium in mixed strategies for the game $F_1^N, \dots, F_N^N$. Then, up to a subsequence, $(\bar{\pi}^N)$ converges to a measure $\bar{m}$ satisfying (2.4).*

Remark 2.4.2. In particular the above Theorem proves the existence of a solution to the "mean field equation" (2.4).

Proof. Let $\bar{m}$ be a limit, up to subsequences, of the $(\bar{\pi}^N)$. Since the map $x_j \to F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j})$ is $\mathrm{Lip}(F)/(N-1)-$Lipschitz continuous, we have, by definition of the distance $\mathbf{d}_1$,

$$\begin{array}{l} \left| \int_{Q^{N-1}} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) \prod_{j\neq i} d\bar{\pi}^N(x_j) - \int_{Q^{N-1}} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) \prod_{j\neq i} d\bar{m}(x_j) \right| \\ \qquad \leq \mathrm{Lip}(F)\mathbf{d}_1(\bar{\pi}^N, \bar{m}) \qquad \forall y \in Q\ . \end{array} \tag{2.6}$$

A direct application of the law of large numbers (see Theorem 3.2.5 below) gives

$$\lim_{N\to+\infty} \int_{Q^{N-1}} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) \prod_{j\neq i} d\bar{m}(x_j) = F(y, \bar{m})\ , \tag{2.7}$$

where the convergence is uniform with respect to $y \in Q$ thanks to the (Lipschitz) continuity of F. Since $(\bar{\pi}_1, \dots, \bar{\pi}_N)$ is a Nash equilibrium, inequality (2.6) implies that, for any $\epsilon > 0$ and if we choose N large enough,

$$\int_{Q^N} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) \prod_{j\neq i} d\bar{m}(x_j) d\bar{m}(x_i) \leq \int_{Q^N} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) \prod_{j\neq i} d\bar{m}(x_j) dm(x_i) + \epsilon\ ,$$

for any $m \in \mathcal{P}(Q)$. Letting $N \to +\infty$ on both sides of the inequality gives, in view of (2.7),

$$\int_Q F(x_i, \bar{m}) d\bar{m}(x_i) \leq \int_Q F(x_i, \bar{m}) dm(x_i) + \epsilon \qquad \forall m \in \mathcal{P}(Q)\ ,$$

which gives the result, since ϵ is arbitrary. □

We can also investigate the converse statement: suppose that a measure $\bar{m}$ satisfying the equilibrium condition (2.4) is given. In what extend can it be used in $N-$player games?

Theorem 2.4.3. *Let F be as in Theorem 2.4.1. For any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}^*$ such that, if $N \geq N_0$, the symmetric mixed strategy $\bar{m}, \cdot, \bar{m})$ is $\epsilon-$optimal in the $N-$player game with costs (F_i^N) defined by* (2.5)*. Namely,*

$$F_i^N(\bar{m}, \dots, \bar{m}) \leq F_i^N(x_i, (\bar{m})_{j\neq i}) \qquad \forall x_i \in Q.$$

Proof. Indeed, as explained in the proof of Theorem 2.4.1, we have

$$\lim_{N\to+\infty} F_i^N(x_i, (\bar{m})_{j\neq i}) = F(x_i, \bar{m})$$

and this limit holds uniformly with respect to $x_i \in Q$. So we can find N_0 such that

$$\sup_{x_i\in Q} \left| F_i^N(x_i, (\bar{m})_{j\neq i}) - F(x_i, \bar{m}) \right| \leq \epsilon/2 \qquad \forall N \geq N_0. \tag{2.8}$$

Then, for any $x_i \in Q$, we have

$$F_i^N(x_i, (\bar{m})_{j\neq i}) \geq F(x_i, \bar{m}) - \epsilon/2 \geq \int_Q F(y_i, \bar{m}) d\bar{m}(y_i) - \epsilon/2$$

where the last inequality comes from the equilibrium condition (2.4) on $\bar{m}$. Using again (2.8) we finally get

$$F_i^N(x_i, (\bar{m})_{j\neq i}) \geq \int_Q F(y_i, \bar{m}) d\bar{m}(y_i) - \epsilon/2 \geq F_i^N(\bar{m}, \dots, \bar{m}) - \epsilon.$$

□

2.5 A uniqueness result

One obtains the full convergence of the measure $m^N_{\bar{X}^N}$ (or $\bar{\pi}^N$) if there is a unique measure $\bar{m}$ satisfying the condition (2.4). This is the case under the following (very strong) assumption:

Proposition 2.5.1. *Assume that F satisfies*

$$\int_Q (F(y,m_1) - F(y,m_2))d(m_1 - m_2)(y) > 0 \qquad \forall m_1 \neq m_2 \ . \tag{2.9}$$

Then there is at most one measure satisfying (2.4).

Remark 2.5.2. Requiring at the same time the continuity of F and the above monotonicity condition seems rather restrictive for applications.

Condition (2.9) is more easily fulfilled for mapping defined on strict subsets of $\mathcal{P}(Q)$. For instance, if Q is a compact subset of $\mathbb{R}^d$ of positive measure and $\mathcal{P}_{ac}(Q)$ is the set of absolutely continuous measures on Q (absolutely continuous with respect to the Lebesgue measure), then

$$F(y,m) = \begin{cases} G(m(y)) & \text{if } m \in \mathcal{P}_{ac}(Q) \\ +\infty & \text{otherwise} \end{cases}$$

satisfies (2.9) as soon as $G : \mathbb{R} \to \mathbb{R}$ is continuous and increasing.

If we assume that Q is the closure of some smooth open bounded subset Ω of $\mathbb{R}^d$, another example is given by

$$F(y,m) = \begin{cases} u_m(y) & \text{if } m \in \mathcal{P}_{ac}(Q) \cap L^2(Q) \\ +\infty & \text{otherwise} \end{cases}$$

where u_m is the solution in $H^1(Q)$ of

$$\begin{cases} -\Delta u_m = m & \text{in } \Omega \\ u_m = 0 & \text{on } \partial\Omega \end{cases}$$

Note that in this case the map $y \to F(y,m)$ is continuous.

Proof of Proposition 2.5.1. Let $\bar{m}_1, \bar{m}_2$ satisfying (2.4). Then

$$\int_Q F(y,\bar{m}_1)d\bar{m}_1(y) \leq \int_Q F(y,\bar{m}_1)d\bar{m}_2(y)$$

and

$$\int_Q F(y,\bar{m}_2)d\bar{m}_2(y) \leq \int_Q F(y,\bar{m}_2)d\bar{m}_1(y) \ .$$

Therefore

$$\int_Q (F(y,\bar{m}_1) - F(y,\bar{m}_2))d(\bar{m}_1 - \bar{m}_2)(y) \leq 0 \ ,$$

which implies that $\bar{m}_1 = \bar{m}_2$ thanks to assumption (2.9). □

2.6 Example: potential games

We now consider a class of nonatomic games for which the equilibria can be reached by minimizing a functional. To fix the idea, we now assume that $Q \subset \mathbb{R}^d$. The heuristic idea is that, if $F(x,m)$ can somehow be represented as the derivative of some mapping $\Phi(x,m)$ with respect to the $m-$variable, and if the problem

$$\inf_{m \in \mathcal{P}(Q)} \int_Q \Phi(x,m)dx$$

has a minimum $\bar{m}$, then

$$\int_Q \Phi'(x,\bar{m})(m-\bar{m}) \geq 0 \qquad \forall m \in \mathcal{P}(Q) \ .$$

So

$$\int_Q F(x,\bar{m})dm \geq \int_Q F(x,\bar{m})d\bar{m} \qquad \forall m \in \mathcal{P}(Q) \ ,$$

which shows that $\bar{m}$ is an equilibrium.

For instance let us assume that

$$F(x,m) = \begin{cases} V(x)m(x) + G(m(x)) & \text{if } m \in \mathcal{P}_{ac}(Q) \\ +\infty & \text{otherwise} \end{cases}$$

where $V : Q \to \mathbb{R}$ is continuous and $G : (0,+\infty) \to \mathbb{R}$ is continuous, strictly increasing, with $G(0) = 0$ and $G(s) \geq cs$ for some $c > 0$. Then let

$$\Phi(x,m) = V(x)m(x) + H(m(x)) \qquad \text{if } m \text{ is a.c.}$$

where H is a primitive of G with $H(0) = 0$. Note that G is strictly convex with $G(s) \geq \frac{c}{2}s^2 - ds$. Hence the problem

$$\inf_{m \in \mathcal{P}_{ac}(Q)} \int_Q V(x)m(x) + H(m(x)) \ dx$$

has a unique solution $\bar{m} \in L^2(Q)$. Then we have, for any $m \in \mathcal{P}_{ac}(Q)$,

$$\int_Q (V(x) + G(\bar{m}(x)))m(x)dx \geq \int_Q (V(x) + G(\bar{m}(x)))\bar{m}(x)dx \ ,$$

so that $\bar{m}$ satisfies (a slightly modified version of) the mean field equation (2.4). In particular, we have $V(x) + G(m(x)) = \min_y V(y) + G(\bar{m}(y))$ for any $x \in Spt(\bar{m})$. Let us set $\lambda = \min_y V(y) + G(\bar{m}(y))$. Then

$$\bar{m}(x) = G^{-1}((\lambda - V(x))_+)$$

For instance, if we plug formally $Q = \mathbb{R}^d$, $V(x) = \frac{|x|^2}{2}$ and $G(s) = \log(s)$ into the above equality, we get $m(x) = e^{-|x|^2/2}/(2\pi)^{d/2}$.

2.7 Comments

There is a huge literature on games with a continuum of players, starting from the seminal work by Aumann [2]. Schmeidler [32], and then Mas-Colell [27], introduced a notion of non-cooperative equilibrium in games with a continuum of agents and established several existence results in a much more general framework where the agents have *types*, i.e., personal characteristics; in that set-up, the equilibria are known under the name of Cournot-Nash equilibria. Blanchet and Carlier [4] investigated classes of problems in which such equilibrium is unique and can be fully characterized.

Theorem 2.3.1 is borrowed from [26]. The variational approach described in section 2.6 presents strong similarities with the potential games of Monderer and Shapley [28].

Chapter 3

Preliminaries

In this section we give a crash course on optimal control, distance on the space of probability measures and mean field limit. As mean field games consist in a combination of these three topics, it is important to have a hint on all these domains separately.

3.1 Optimal control

3.1.1 The controlled problem

Let us consider a stochastic controlled problem where the state (X_s) of the system is governed by the stochastic differential equation (SDE) with values in $\mathbb{R}^d$:

$$X_s = x + \int_t^s b(r, X_r, \alpha_r)dr + \int_t^s \sigma(r, X_r, \alpha_r)dB_r. \tag{3.1}$$

In the above equation, $B = (B_s)_{s\geq 0}$ is a $N-$dimensional Brownian motion (starting at 0) adapted to a fixed filtration $(\mathcal{F}_t)_{t\geq 0}$, $b : [0, T] \times \mathbb{R}^d \times A \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times N}$ satisfy some regularity conditions given below and the process $\alpha = (\alpha_s)$ is progressively measurable with values in some set A. We denote by $\mathcal{A}$ the set of such processes. The elements of $\mathcal{A}$ are called the control processes.

The controller controls the process X through the control α in order to reach some goal: here we consider optimal control problems, in which the controller aims at minimizing some cost J. We will mostly focus on the finite horizon problem, where J takes the form:

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^T L(s, X_s, \alpha_s)ds + g(X_T)\right].$$

Here $T > 0$ is the finite horizon of the problem, $L : [0, T] \times \mathbb{R}^d \times A \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are given continuous maps (again we are more precise in the next section on the assumptions on L and g). The controller minimizes J by using controls in $\mathcal{A}$.

We introduce the value function the map $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$u(t, x) = \inf_{\alpha\in\mathcal{A}} J(t, x, \alpha) \ .$$

3.1.2 Dynamic programming and the verification Theorem

The main interest of the value function is that it indicates how the controller should choose her control in order to play in an optimal way. We explain the key ideas in a very informal way. A rigorous treatment of the question is described in the references indicated below.

Let us start with the **dynamic programming principle**, which states the following identity: for any $t_1 \leq t_2$,

$$u(t_1, x) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_{t_1}^{t_2} L(s, X_s, \alpha_s)ds + u(t_2, X_{t_2})\right]. \tag{3.2}$$

The interpretation is that, to play optimally at time t_2, the controller can forget the past trajectory and only remember the position X_{t_2}, thus reducing the problem to the computation of $u(t_2, X_{t_2})$.

Fix now $t \in [0, T)$. Choosing $t_1 = t$, $t_2 = t + h$ (for $h > 0$ small) and assuming that u is smooth enough, we obtain by Itô's formula and (3.2) that

$$u(t,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\int_t^{t+h} L(s, X_s, \alpha_s)ds + u(t,x) + \int_t^{t+h} (\partial_t u(s, X_s) + Du(s, X_s) \cdot b(s, X_s, \alpha_s) \\ + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(s, X_s, \alpha_s) D^2 u(s, X_s)))ds\Big].$$

Simplifying by $u(t,x)$, dividing by h and letting $h \to 0^+$ gives (informally) the Hamilton-Jacobi equation

$$0 = \inf_{a \in A} \left[L(t,x,a) + \partial_t u(t,x) + Du(t,x) \cdot b(t,x,a) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,a) D^2 u(t,x))\right].$$

Let us introduce the Hamiltonian H of our problem: for $p \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d \times d}$,

$$H(t,x,p,M) := \sup_{a \in A} \left[-L(t,x,a) - p \cdot b(t,x,a) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,a) M)\right].$$

Then the **Hamilton-Jacobi equation** can be rewritten as a terminal value problem:

$$\begin{cases} -\partial_t u(t,x) + H(t,x,Du(t,x), D^2 u(t,x)) = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d. \end{cases}$$

The first equation is backward in time (the map H being nonincreasing with respect to $D^2 u$). The terminal condition comes just from the definition of u for $t = T$.

Let us now introduce $\alpha^*(t,x) \in A$ as a maximum point in the definition of H when $p = Du(t,x)$ and $M = D^2(t,x)$. Namely

$$H(t,x,Du(t,x), D^2 u(t,x)) = -L(t,x,\alpha^*(t,x)) - Du(t,x) \cdot b(t,x,\alpha^*(t,x)) \\ - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,\alpha^*(t,x)) D^2 u(t,x)). \tag{3.3}$$

We assume that α^* is sufficiently smooth to justify the computation below. We are going to show that α^* is the **optimal feedback**, namely the optimal control to play at time t in the state x. Indeed, one has the following "**Verification Theorem**":

$$u(t,x) = J(t,x,\alpha^*).$$

Proof. Let us denote by X^* the solution to

$$X_s^* = x + \int_t^s b(r, X_r^*, \alpha^*(r, X_r^*))dr + \int_t^s \sigma(r, X_r^*, \alpha^*(r, X_r^*))dB_r$$

and set, to simplify the expression, $\alpha_s^* := \alpha^*(s, X_s^*)$ (note now that (α_s^*) is a control, namely it belongs to $\mathcal{A}$. Strictly speaking, (α_t^*) is the optimal control, $\alpha^* = \alpha^*(t,x)$ being the optimal feedback). By Itô's formula, we have

$$g(X_T^*) = u(T, X_T^*) = u(t,x) + \int_t^T (\partial_t u(s, X_s^*) + Du(s, X_s^*) \cdot b(s, X_s^*, \alpha_s^*) \\ + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(s, X_s^*, \alpha_s^*) D^2 u(s, X_s^*)))ds + \int_t^T \sigma^*(s, X_s^*, \alpha_s^*) Du(s, X_s^*) \cdot dB_s.$$

Taking expectation, using first the optimality of α^* in (3.3) and then the Hamilton-Jacobi equation satisfied by u, we obtain

$$\mathbb{E}\left[g(X_T^*)\right] = u(t,x) + \mathbb{E}\left[\int_t^T (\partial_t u(s,X_s^*) - H(s,X_s^*,Du(s,X_s^*),D^2u(s,X_s^*)) - L(s,X_s^*,\alpha_s^*))ds\right]$$
$$= u(t,x) - \mathbb{E}\left[\int_t^T L(s,X_s^*,\alpha_s^*)ds\right].$$

Rearranging, we find

$$u(t,x) = \mathbb{E}\left[\int_t^T L(s,X_s^*,\alpha_s^*)ds + g(X_T^*)\right],$$

which shows the optimality of α^*. □

The above arguments, although largely heuristic, can be partially justified. Surprisingly, the dynamic programming principle is the hardest step to prove, and only holds under strong restrictions on the probability space (but these restrictions are merely theoretical, and one can most often assume that they are met in practice). In general, the value function is smooth only under very strong assumptions on the system. However, under middler conditions, it is at least continuous and then it satisfies the Hamilton-Jacobi equation in the viscosity sense. Besides, the Hamilton-Jacobi has a unique (viscosity) solution so that it characterizes the value function. If the diffusion is strongly non degenerate (i.e., $M = d$ and σ is invertible with a smooth and bounded inverse) and if the Hamiltonian is smooth, then the value function is smooth as well. In this setting the verification Theorem makes perfectly sense. We will illustrate this case in the Subsection 4.2.

3.1.3 Estimates on the SDE

In the previous parts, we were intentionally very fuzzy about the assumptions and the results. A complete rigorous treatment of the problem is far beyond the aim of these notes. However, we need to clarify a bit the definition of our problem. For this, we assume the maps b and σ to be continuous and Lipschitz continuous in x independently of t and a: There is a constant $K > 0$ such that

$$|b(t,x,a) - b(t,y,a)| + |\sigma(t,x,a) - \sigma(t,y,a)| \le K|x-y| \qquad \forall x,y \in \mathbb{R}^d,\ t \in [0,+\infty),\ a \in A.$$

Under these assumptions, for any bounded control $\alpha \in \mathcal{A}$, there exists a unique solution to (3.1). By a solution we mean a progressively measurable process X such that, for any $T > 0$,

$$\mathbb{E}\left[\int_t^T |X_s|^2 ds\right] < +\infty$$

and (3.1) holds $\mathbb{P}-$a.s. More precisely, we have:

Lemma 3.1.1. *Let α be a bounded control in $\mathcal{A}$. Then there exists a unique solution X to* (3.1) *and this solution satisfies, for any $T > 0$ and $p \in [2,+\infty)$,*

$$\mathbb{E}\left[\sup_{t\in[t,T]} |X_t|^p\right] \le C(1 + |x|^p + \|b(\cdot,0,\alpha_\cdot)\|_\infty^p + \|\sigma(\cdot,0,\alpha_\cdot)\|_\infty^p),$$

where $C = C(T,p,d)$ depends only on T, p and d.

Remark 3.1.2. In view of the above result, the cost J is well-defined provided, for instance, that the maps $L : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are continuous with at most a polynomial growth.

Proof. The existence can be proved by a fixed point argument, exactly as for the McKean-Vlasov equation (see below). Let us show the bound. We set $M := \|b(\cdot,0,\alpha.)\|_\infty + \|\sigma(\cdot,0,\alpha.)\|_\infty$. We have, by Hölder's inequality

$$|X_s|^p \leq 3^{p-1}\left(|x|^p + \int_t^s |b(r,X_r,\alpha_r)|^p dr + \left|\int_t^s \sigma(r,X_r,\alpha_r)dB_r\right|^p\right).$$

Thus

$$\mathbb{E}\left[\sup_{t\leq r\leq s}|X_r|^p\right] \leq 3^{p-1}\left(|x|^p + \int_t^s \mathbb{E}\left[|b(r,X_r,\alpha_r)|^p\right]dr + \mathbb{E}\left[\sup_{t\leq r\leq s}\left|\int_t^r \sigma(u,X_u,\alpha_u)dB_u\right|^p\right]\right).$$

Note that

$$|b(s,X_s,\alpha_s)| \leq |b(s,0,\alpha_s)| + L|X_s| \leq M + L|X_s|$$

and, in the same way,

$$|\sigma(s,X_s,\alpha_s)| \leq M + L|X_s|. \tag{3.4}$$

So we have

$$\int_t^s \mathbb{E}\left[|b(r,X_r,\alpha_r)|^p\right]dr \leq 2^{p-1}(M^p(s-t) + L^p\int_t^s \mathbb{E}\left[|X_r|^p\right]dr).$$

By the Burkholder-Davis-Gundy inequality (see Theorem IV.4.1 in [30]), we have

$$\mathbb{E}\left[\sup_{t\leq r\leq s}\left|\int_t^r \sigma(u,X_u,\alpha_u)dB_u\right|^p\right] \leq C_p\mathbb{E}\left[\left(\int_t^s \mathrm{Tr}(\sigma\sigma^*(r,X_r,\alpha_r))dr\right)^{p/2}\right],$$

where the constant C_p depends on p only. Combining Hölder's inequality (since $p/2 \geq 1$) with (3.4), we then obtain

$$\mathbb{E}\left[\sup_{t\leq r\leq s}\left|\int_t^r \sigma(u,X_u,\alpha_u)dB_u\right|^p\right] \leq C_p(s-t)^{p/2-1}2^{p-1}\left(M^p(s-t) + L^p\int_t^s \mathbb{E}\left[|X_r|^p\right]dr\right).$$

Putting together the different estimates we get therefore, for $s \in [t,T]$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq r\leq s}|X_r|^p\right] &\leq C(p,T,d)\left(1 + |x|^p + M^p + \int_t^s \mathbb{E}\left[|X_r|^p\right]dr\right)\\ &\leq C(p,T,d)\left(1 + |x|^p + M^p + \int_t^s \mathbb{E}\left[\sup_{t\leq u\leq r}|X_u|^p\right]dr\right),\end{aligned}$$

where the constant $C(p,T,d)$ depends only on p, T and d. We can then conclude by Gronwall's Lemma.
□

3.1.4 Further reading

Several introductory courses on optimal control can be found online: for instance

- B. Bouchard, "Introduction to stochastic control of mixed diffusion processes, viscosity solutions and applications in finance and insurance",
  https://www.ceremade.dauphine.fr/~bouchard/bouchard.htm
- N. Touzi, "Deterministic and Stochastic Control, Application to Finance",
  http://www.cmap.polytechnique.fr/~touzi/#Lecture

Classical references on stochastic optimal control problems are the monographs by Fleming and Rischel [12], Fleming and Soner [13], Yong and Zhou [38].

3.2 A distance on the space of measures

3.2.1 The Monge-Kantorovitch distance

Let (X,d) be a Polish space (= complete metric space). We have mostly in mind $X=\mathbb{R}^d$ endowed with the usual distance. We denote by $\mathcal{P}(X)$ the set of Borel probability measures m on X. Let us recall that a sequence (m_n) of $\mathcal{P}(X)$ narrowly converges to a measure $m\in\mathcal{P}(X)$ if, for any test function $\phi\in C_b^0(X)$ (= the set of continuous and bounded maps on X), we have

$$\lim_n \int_X \phi(x)m_n(dx)=\int_X \phi(x)m(dx).$$

Let us recall *Prokhorov compactness criterium:* a subset $\mathcal{K}$ of $\mathcal{P}(X)$ is relatively compact for the sequential narrow convergence if and only if it is tight: for any $\epsilon>0$ there exists a compact subset K of X such that

$$\sup_{\mu\in\mathcal{K}} m(X\backslash K)\leq \epsilon.$$

We fix from now on a point $x_0\in X$ and $\mathcal{P}_1(X)$ is the set measures $m\in\mathcal{P}(X)$ such that

$$\int_X d(x,x_0)m(dx)<+\infty.$$

By the triangle inequality, it is easy to check that the set $\mathcal{P}_1(X)$ does not depend on the choice of x_0. We endow $\mathcal{P}_1(X)$ with the Monge-Kantorovitch distance:

$$\mathbf{d}_1(m_1,m_2)=\sup_\phi \int_X \phi(x)(m_1-m_2)(dx)\qquad \forall m_1,m_2\in\mathcal{P}_1(X),$$

where the supremum is taken over the set of maps $\phi:X\to\mathbb{R}$ such that ϕ is $1-$Lipschitz continuous. Note that such a map ϕ is integrable against any measure over $\mathcal{P}_1(X)$ because it has a linear growth.

We note for later use that, if $\phi:X\to\mathbb{R}$ is $Lip(\phi)-$Lipschitz continuous, then

$$\int_X \phi(x)(m_1-m_2)(dx)\leq Lip(\phi)\mathbf{d}_1(m_1,m_2).$$

Moreover, if X_1 and X_2 are random variables on some probability space $(\Omega,\mathcal{F},\mathcal{P})$ such that the law of X_i is m_i, then

$$\mathbf{d}_1(m_1,m_2)\leq \mathbb{E}\left[|X_1-X_2|\right],$$

because, for any $1-$Lipschitz map $\phi:X\to\mathbb{R}$,

$$\int_X \phi d(m_1-m_2)=\mathbb{E}\left[\phi(X_1)-\phi(X_2)\right]\leq \mathbb{E}\left[|X_1-X_2|\right].$$

Taking the supremum in ϕ gives the result. Actually one can show that, if the probability space $(\Omega,\mathcal{F},\mathcal{P})$ is "rich enough" (namely is a "standard probability space"), then

$$\mathbf{d}_1(m_1,m_2)=\inf_{X_1,X_2}\mathbb{E}\left[|X_1-X_2|\right],$$

where the infimum is taken over random variables X_1 and X_2 such that the law of X_i is m_i.

Lemma 3.2.1. $\mathbf{d}_1$ *is a distance over* $\mathcal{P}_1(X)$.

Proof. We first note that $\mathbf{d}_1(m_1,m_2)=\mathbf{d}_1(m_2,m_1)\geq 0$ since one can always replace ϕ by $-\phi$ in the definition. Let us show that $\mathbf{d}_1(m_1,m_2)=0$ implies that $m_1=m_2$. Indeed, if $\mathbf{d}_1(m_1,m_2)=0$, then, for any $1-$Lipschitz continuous map ϕ, one has $\int_X\phi(x)(m_1-m_2)(dx)\leq 0$. Replacing ϕ by $-\phi$, one has therefore $\int_X\phi(x)(m_1-m_2)(dx)=0$. It remains to show that this equality holds for any continuous, bounded map $\phi:X\to\mathbb{R}$. Let $\phi\in C_b^0(X)$. We show in Lemma 3.2.2 below that there exists a sequence of maps (ϕ_k) such that ϕ_k is $k-$Lipschitz continuous, with $\|\phi_k\|_\infty\leq\|\phi\|_\infty$, and the sequence (ϕ_k) converges locally uniformly to ϕ. By Lipschitz continuity of ϕ_k, we have $\int_X \phi_k d(m_1 - m_2) = 0$. Since we can apply Lebesgue convergence theorem (because the ϕ_k are uniformly bounded and m_1 and m_2 are probability measures), we obtain that $\int_X \phi d(m_1 - m_2) = 0$. This proves that $m_1 = m_2$.

It remains to show the triangle inequality, which is immediate since

$$\begin{aligned}\mathbf{d}_1(m_1, m_3) &= \sup_\phi \int_X \phi(x)(m_1 - m_2 + m_2 - m_3)(dx) \\ &\leq \sup_\phi \int_X \phi(x)(m_1 - m_2)(dx) + \sup_\phi \int_X \phi(x)(m_2 - m_3)(dx) \\ &= \mathbf{d}_1(m_1, m_2) + \mathbf{d}_1(m_2, m_3).\end{aligned}$$

□

Lemma 3.2.2. *Let $\phi \in C_b^0(X)$ and let us define the sequence of maps (ϕ_k) by*

$$\phi_k(x) = \inf_{y \in X} \phi(y) + kd(y, x) \qquad \forall x \in X$$

Then $\phi_k \leq \phi$, ϕ_k is $k-$Lipschitz continuous with $\|\phi_k\|_\infty \leq \|\phi\|_\infty$, and the sequence (ϕ_k) converges locally uniformly to ϕ.

Proof. We have

$$\phi_k(x) = \inf_{y \in X} \phi(y) + kd(y, x) \leq \phi(x) + kd(x, x) = \phi(x),$$

so that $\phi_k \leq \phi$. Let us now check that ϕ_k is $k-$Lipschitz continuous. Indeed, let $x_1, x_2 \in X$, $\epsilon > 0$ and y_1 be $\epsilon-$optimal in the definition of $\phi_k(x_1)$. Then

$$\phi_k(x_2) \leq \phi(y_1) + kd(y_1, x_2) \leq \phi(y_1) + kd(y_1, x_1) + kd(x_1, x_2) \leq \phi_k(x_1) + \epsilon + kd(x_1, x_2).$$

As ϵ is arbitrary, this shows that ϕ_k is $k-$Lipschitz continuous. Note that

$$\phi_k(x) \geq \inf_{y \in X} -\|\phi\|_\infty + kd(y, x) = -\|\phi\|_\infty.$$

As $\phi_k \leq \phi$, this shows that $\|\phi_k\|_\infty \leq \|\phi\|_\infty$.

Finally, let $x_k \to x$ and y_k be $(1/k)-$optimal in the definition of $\phi_k(x_k)$. Our aim is to show that $(\phi_k(x_k))$ converges to $\phi(x)$, which will show the local uniform convergence of (ϕ_k) to ϕ. Let us first remark that, by definition of y_k, we have

$$kd(y_k, x_k) \leq \phi_k(x_k) - \phi(y_k) + 1/k \leq 2\|\phi\|_\infty + 1.$$

Therefore

$$d(y_k, x) \leq d(y_k, x_k) + d(x_k, x) \to 0 \qquad \text{as } k \to +\infty.$$

This shows that

$$\liminf_k \phi_k(x_k) \geq \liminf_k \phi(y_k) + kd(y_k, x_k) + 1/k \geq \liminf_k \phi(y_k) + 1/k = \phi(x).$$

As, on the other hand, $\phi_k \leq \phi$, we immediately have

$$\limsup_k \phi_k(x_k) \leq \phi(x),$$

from which we conclude the convergence of $(\phi_k(x_k))$ to $\phi(x)$. □

Proposition 3.2.3. *Let (m_n) a sequence in $\mathcal{P}_1(X)$ and $m \in \mathcal{P}_1(X)$. There is an equivalence between:*
i) $\mathbf{d}_1(m_n, m) \to 0$,
ii) (m_n) *weakly converges to* m *and* $\displaystyle\int_X d(x, x_0) m_n(dx) \to \int_X d(x, x_0) m(dx)$.
iii) (m_n) *weakly converges to* m *and* $\displaystyle\limsup_{R \to +\infty} \int_{B_R(x_0)^c} d(x, x_0) m_n(dx) = 0$.

Sketch of proof. $(i) \Rightarrow (ii)$. Let us assume that $\mathbf{d}_1(m_n, m) \to 0$. Then, for any Lipschitz continuous map ϕ, we have $\int \phi dm_n \to \int \phi dm$ by definition of $\mathbf{d}_1$. In particular, if we chose $\phi(x) = d(x, x_0)$, we have $\displaystyle\int_X d(x, x_0) m_n(dx) \to \int_X d(x, x_0) m(dx)$. We now prove the weak convergence of (m_n). Let $\phi : X \to \mathbb{R}$ be continuous and bounded and let (ϕ_k) be the sequence defined in Lemma 3.2.2. Then

$$\int \phi d(m_n - m) = \int \phi_k d(m_n - m) + \int (\phi - \phi_k) d(m_n - m).$$

Fix $\epsilon > 0$. As $(\int_X d(x, x_0) m_n(dx))$ converges and $m \in \mathcal{P}_1(X)$, we can find $R > 0$ large such that

$$\sup_n m_n(X \backslash B_R(x_0)) + m(X \backslash B_R(x_0)) \leq \epsilon.$$

On the other hand, we can find k large enough such that $\|\phi_k - \phi\|_{L^\infty(B_R(x_0))} \leq \epsilon$, by local uniform convergence of (ϕ_k). Finally, if n is large enough, we have $|\int \phi_k d(m_n - m)| \leq \epsilon$, by the convergence of (m_n) to m in $\mathbf{d}_1$. So

$$\begin{aligned}
\left|\int \phi d(m_n - m)\right| &\leq \left|\int \phi_k d(m_n - m)\right| + \left|\int_{X \backslash B_R(x_0)} (\phi - \phi_k) d(m_n - m)\right| + \left|\int_{B_R(x_0)} (\phi - \phi_k) d(m_n - m)\right| \\
&\leq \left|\int \phi_k d(m_n - m)\right| + (\|\phi_k\|_\infty + \|\phi\|_\infty)(m_n(X \backslash B_R(x_0)) + m(X \backslash B_R(x_0))) \\
&\qquad\qquad + \int_{B_R(x_0)} |\phi - \phi_k| d(m_n + m) \\
&\leq \epsilon + 2\|\phi\|_\infty \epsilon + 2\epsilon.
\end{aligned}$$

This shows the convergence of (m_n) to m.

$(ii) \Rightarrow (iii)$. This is straightforward since the convergence $\displaystyle\int_X d(x, x_0) m_n(dx) \to \int_X d(x, x_0) m(dx)$ implies the limit $\displaystyle\limsup_{R \to +\infty} \int_{B_R(x_0)^c} d(x, x_0) m_n(dx) = 0$.

$(iii) \Rightarrow (i)$. Let us assume that (m_n) weakly converges to m and that $\displaystyle\limsup_{R \to +\infty} \int_{B_R(x_0)^c} d(x, x_0) m_n(dx) = 0$. We first assume that X is compact and remove this assumption at the end of the proof. Let $\mathcal{K}_0$ be the set of $1-$Lipschitz continuous maps which vanish at x_0. Then, using the fact that m_n and m are two probability measures, we have

$$\mathbf{d}_1(m_n, m) = \sup_\phi \int_X (\phi(x) - \phi(x_0))(m_n - m)(dx) = \sup_{\phi \in \mathcal{K}_0} \int_X \phi(x)(m_n - m)(dx).$$

As, by Ascoli-Arzelà, $\mathcal{K}_0$ is compact for the uniform convergence, there exists $\phi_n \in \mathcal{K}_0$ optimal in the right-hand side. Again by Ascoli-Arzelà, (ϕ_n) converges, up to a subsequence, to some $\phi \in \mathcal{K}_0$. Then

$$\begin{aligned}
\mathbf{d}_1(m_n, m) &\leq \left|\int_X \phi(x)(m_n - m)(dx)\right| + \left|\int_X (\phi_n - \phi)(x)(m_n - m)(dx)\right| \\
&\leq \left|\int_X \phi(x)(m_n - m)(dx)\right| + 2\|\phi_n - \phi\|_\infty,
\end{aligned}$$

which tends to 0 as $n \to +\infty$ (note that we are cheating a little since the convergence actually holds only up to a subsequence. We leave the details of the complete proof to the reader).

It remains to remove the assumption on the compactness of X. By Prokhorov Theorem, for any $\epsilon > 0$, there is compact subset K of X such that

$$\sup_n m_n(X \backslash K) \leq \epsilon.$$

On the other hand, by assumption (iii), we can find $R > 0$ large enough such that

$$\sup_n \int_{B_R(x_0)^c} d(x, x_0) m_n(dx) \le \epsilon \qquad \text{and} \qquad \int_{B_R(x_0)^c} d(x, x_0) m(dx) \le \epsilon.$$

Without loss of generality, we can assume that $K \subset B_R(x_0)$.

Then, for any $\phi : X \to \mathbb{R}$ 1−Lipschitz continuous (with $\phi(x_0) = 0$, which is without loss of generality), we have

$$\begin{aligned}
\int_X \phi d(m_n - m) &\le \int_{B_R(x_0)^c} d(x, x_0)(m_n + m) + \int_{B_R(x_0)\setminus K} d(x, x_0)(m_n + m) + \int_K \phi d(m_n - m) \\
&\le 2\epsilon + 2R\epsilon + \int_K \phi d(m_n - m).
\end{aligned}$$

We can prove (almost) as in the compact case that

$$\sup_\phi \int_K \phi d(m_n - m) = 0.$$

This shows that $\mathbf{d}_1(m_n, m) \to 0$. □

In the case where $X = \mathbb{R}^d$, we repeatedly use the following compactness criterium:

Lemma 3.2.4. *Let $r > 1$ and $\mathcal{K} \subset \mathcal{P}_1(\mathbb{R}^d)$ be such that*

$$\sup_{\mu \in \mathcal{K}} \int_{\mathbb{R}^d} |x|^r \mu(dx) < +\infty \ .$$

Then $\mathcal{K}$ is relatively compact for the $\mathbf{d}_1$ distance.

Note that bounded subsets of $\mathcal{P}_1(\mathbb{R}^d)$ are not relatively compact for the $\mathbf{d}_1$ distance.

Proof of Lemma 3.2.4. Let $\epsilon > 0$ and $R > 0$ sufficiently large. We have for any $\mu \in \mathcal{K}$:

$$\mu(\mathbb{R}^d \setminus B_R(0)) \le \int_{\mathbb{R}^d \setminus B_R(0)} \frac{|x|^r}{R^r} \mu(dx) \le \frac{C}{R^r} < \epsilon \ , \tag{3.5}$$

where $C = \sup_{\mu \in \mathcal{K}} \int_{\mathbb{R}^d} |x|^r \mu(dx) < +\infty$. So $\mathcal{K}$ is tight.

Let now (μ_n) be a sequence in $\mathcal{K}$. From the previous step we know that (μ_n) is tight and therefore there is a subsequence, again denoted (μ_n), which narrowly converges to some μ. By (3.5) and (iii) in Proposition 3.2.3 the convergence also holds for the distance $\mathbf{d}_1$. □

3.2.2 The Glivenko-Cantelli law of large numbers

Next we consider (X_n) a sequence of i.i.d. random variables on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}[|X_1|] < +\infty$. We denote by m the law of X_1. The law of large numbers states that, a.s. and in L^1,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^N X_n = \mathbb{E}[X_1].$$

Our aim is to show that a slightly stronger convergence holds: let

$$m_X^N := \frac{1}{N} \sum_{n=1}^N \delta_{X_n}$$

Note that m_X^N is a random measure, in the sense that m_X^N is a.s. a measure and that, for any Borel set $A \subset X$, $m_X^N(A)$ is a random variable. The following result is (sometimes) known as the Glivenko-Cantelli Theorem:

Theorem 3.2.5. *If $\mathbb{E}[|X_1|] < +\infty$, then, a.s. and in L^1,*

$$\lim_{N\to+\infty} \mathbf{d}_1(m_X^N, m) = 0.$$

Sketch of proof. Let $\phi \in C_b^0(X)$. Then, by the law of large numbers,

$$\int_{\mathbb{R}^d} \phi(x) m_X^N(dx) = \frac{1}{N}\sum_{n=1}^N \phi(X_n) = \mathbb{E}[\phi(X_1)] \qquad a.s..$$

By a separability argument, it is not difficult to check that the set of zero probability in the above convergence can be chosen independent of ϕ. So (m_X^N) converge weakly to m a.s. Note also that

$$\mathbb{E}\left[\int_{\mathbb{R}^d} d(x,x_0) m_X^N(dx)\right] = \frac{1}{N}\sum_{n=1}^N \mathbb{E}\left[d(X_n,x_0)\right] = \mathbb{E}\left[d(X_1,x_0)\right], \tag{3.6}$$

while, for any $M > 0$ and a.s.

$$\int_{\mathbb{R}^d} (d(x,x_0)\wedge M) m(dx) = \lim_N \int_{\mathbb{R}^d} (d(x,x_0)\wedge M) m_X^N(dx) \le \liminf_N \int_{\mathbb{R}^d} d(x,x_0) m_X^N(dx).$$

Letting $M \to +\infty$, we get

$$\int_{\mathbb{R}^d} d(x,x_0) m(dx) \le \liminf_N \int_{\mathbb{R}^d} d(x,x_0) m_X^N(dx) \qquad \text{a.s..} \tag{3.7}$$

Combining (3.6) and (3.7) proves that

$$\int_{\mathbb{R}^d} d(x,x_0) m_X^N(dx) \to \int_{\mathbb{R}^d} d(x,x_0) m(dx) \qquad \text{a.s..}$$

By Proposition 3.2.3, (m_X^N) converges a.s. in $\mathbf{d}_1$ to m. It remains to show that this convergence also holds in expectation. For this we note that

$$\mathbf{d}_1(m_X^N, m) = \sup_\phi \int_{\mathbb{R}^d} \phi d(m_X^N - m) \le \sup_\phi \frac{1}{N}\sum_{i=1}^N \phi(X_i) - \int_{\mathbb{R}^d} \phi(x) m(dx),$$

where the supremum is taken over the 1−Lipschitz continuous maps ϕ with $\phi(0) = 0$. So

$$\mathbf{d}_1(m_X^N, m) \le \frac{1}{N}\sum_{i=1}^N |X_i| + \int_{\mathbb{R}^d} |x| m(dx).$$

As the right-hand side converges in L^1 to $2\mathbb{E}[|X_1|]$, $\mathbf{d}_1(m_X^N, m)$ is uniformly integrable which implies its convergence in expectation to 0. □

3.2.3 The $\mathbf{d}_2$ distance

In many application, the Wasserstein distance is much more natural than the Monge-Kantorovitch one. We define this distance only on $\mathbb{R}^d$, the generalization to metric spaces being straightforward. The Wasserstein distance is defined on the space $\mathcal{P}_2(\mathbb{R}^d)$ of Borel probability measures m with a finite second order moment: $\int_{\mathbb{R}^d} |x|^2 m(dx) < +\infty$. It is given by

$$\mathbf{d}_2(m_1, m_2) := \inf_\pi \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(x,y)\right)^{1/2},$$

where the infimum is taken over the Borel probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal given by m_1 and second marginal by m_2:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\pi(dx,dy) = \int_{\mathbb{R}^d} \phi(x)m_1(dx), \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)\pi(dx,dy) = \int_{\mathbb{R}^d} \phi(y)m_2(dy) \qquad \forall \phi \in C_b^0(\mathbb{R}^d).$$

Given a "sufficiently rich" probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the distance can be defined equivalently by

$$\mathbf{d}_2(m_1, m_2) = \inf_{X,Y} \left(\mathbb{E}\left[|X-Y|^2\right]\right)^{1/2},$$

where the infimum is taken over random variables X, Y over Ω with law m_1 and m_2 respectively.

3.2.4 Further reading

Classical references on the distances over the space of probability measures are the monographs by Ambrosio, Gigli and Savaré [1], by Rachev and Rüschendorf [29], Santambrogio [31], and Villani [37], [36].

It is often useful to quantify the convergence speed in the law of large numbers. Such results can be found in the text books [29] or, in a sharper form, in [8].

3.3 Mean field limits

We now consider system of $N-$particles (where $N \in \mathbb{N}^*$ is a large number) and want to understand the behavior of the system as the number N tends to infinity. We work (for simplicity) with the following system: for $i = 1, \dots, N$,

$$\text{(3.8)} \qquad \begin{cases} dX_t^i = \dfrac{1}{N}\displaystyle\sum_{j=1}^N b(X_t^i, X_t^j)dt + dB_t^i \\ X_0^i = Z^i \end{cases}$$

where the (B^i) are independent Brownian motions, the Z^i are i.i.d. random variables in $\mathbb{R}^d$ which are also independent of the (B^i). The map $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be Lipschitz continuous. Note that, under these assumptions, the solution (X^i) to (3.8) exists and is unique. An important remark is that, because the (Z^i) have the same law and the equations satisfied by the X^i are symmetric, the X^i have the same law (they are actually "exchangeable").

We want to understand the limit of the (X^i) as $N \to +\infty$. The heuristic idea is that, as N is large, the (X^i) become more and more independent, so that they become almost i.i.d. The law of large numbers then implies that

$$\frac{1}{N}\sum_{j=1}^N b(X_t^i, X_t^j) \approx \tilde{\mathbb{E}}\left[b(X_t^i, \tilde{X}_t^i)\right] = \int_{\mathbb{R}^d} b(X_t^i, y)\mathbb{P}_{X_t^i}(dy),$$

where $\tilde{X}_t^i$ is an independent copy of X_t^i and $\tilde{\mathbb{E}}$ is the expectation with respect to this independent copy. Therefore we expect the X^i to be close to the solution $\bar{X}^i$ to the solution to the McKean-Vlasov equation

$$\text{(3.9)} \qquad \begin{cases} d\bar{X}_t^i = \displaystyle\int_{\mathbb{R}^d} b(\bar{X}_t^i, y)\mathbb{P}_{\bar{X}_t^i}(dy)dt + dB_t^i \\ \bar{X}_0^i = Z^i \end{cases}$$

This is exactly what we are going to show. Before this, we need to define a distance on the space of probability measures on $\mathbb{R}^d$.

3.3.1 The well-posedness of the McKean-Vlasov equation

Theorem 3.3.1. *Let us assume that $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is globally Lipschitz continuous and let $Z \in L^2(\Omega)$. Then the McKean-Vlasov equation*

$$\begin{cases} dX_t = \int_{\mathbb{R}^d} b(X_t, y)\mathbb{P}_{X_t}(dy)dt + dB_t \\ X_0 = Z \end{cases}$$

has a unique solution, i.e., a progressively measurable process such that $\mathbb{E}\left[\int_0^T |X_s|^2 ds\right] < +\infty$ for any $T > 0$.

Remark 3.3.2. By Itô's formula, the law m_t of a solution X_t solves in the sense of distributions the McKean-Vlasov equation

$$\begin{cases} \partial_t m_t - \Delta m_t + \operatorname{div}(m_t b(x, m_t)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d \\ m_0 = \mathcal{L}(Z) \qquad \text{in } \mathbb{R}^d \end{cases}$$

One can show that this equation has a unique solution, which proves the uniqueness in law of the process X.

Proof. Let $\alpha > 0$ to be chosen later and E be the set of progressively measurable processes (X_t) such that

$$\|X\|_E := \mathbb{E}\left[\int_0^\infty e^{-\alpha t}|X_t|dt\right] < +\infty.$$

Then $(E, \|\cdot\|_E)$ is a Banach space. On E we define the map Φ by

$$\Phi(X)_t = Z + \int_0^t \int_{\mathbb{R}^d} b(X_s, y)\mathbb{P}_{X_s}(dy)ds + B_t, \qquad t \geq 0.$$

Let us check that the map Φ is well defined from E to E. Note first that $\Phi(X)$ is indeed progressively measurable. By the $L-$Lipschitz continuity of b (for some $L > 0$),

$$\begin{aligned} |\Phi(X)_t| &\leq |Z| + \int_0^t \int_{\mathbb{R}^d} |b(X_s, y)|\mathbb{P}_{X_s}(dy)ds + |B_t| \\ &\leq |Z| + t|b(0,0)| + L\int_0^t \int_{\mathbb{R}^d} |X_s|ds + \int_0^t \int_{\mathbb{R}^d} |y|\mathbb{P}_{X_s}(dy)ds + |B_t|, \end{aligned}$$

where $\int_0^t \int_{\mathbb{R}^d} |y|\mathbb{P}_{X_s}(dy)ds = \mathbb{E}\left[\int_0^t |X_s|ds\right]$. So

$$\begin{aligned} \mathbb{E}\left[\int_0^{+\infty} e^{-\alpha t}|\Phi(X)_t|dt\right] &\leq \alpha^{-1}|Z| + 2L\mathbb{E}\left[\int_0^{+\infty} e^{-\alpha t}\int_0^t |X_s|dsdt\right] + \int_0^{+\infty} e^{-\alpha t}\mathbb{E}\left[|B_t|\right]dt \\ &= \alpha^{-1}\mathbb{E}[|Z|] + \frac{2L}{\alpha}\mathbb{E}\left[\int_0^{+\infty} e^{-\alpha s}|X_s|ds\right] + C_d\int_0^{+\infty} t^{1/2}e^{-\alpha t}dt, \end{aligned}$$

where C_d depends only on dimension. This proves that $\Phi(X)$ belongs to E.

Let us finally check that Φ is a contraction. We have, if $X, Y \in E$,

$$\begin{aligned} |\Phi(X)_t - \Phi(Y)_t| &\leq \int_0^t \left|\int_{\mathbb{R}^d} b(X_s, y)\mathbb{P}_{X_s}(dy) - \int_{\mathbb{R}^d} b(Y_s, y)\mathbb{P}_{Y_s}(dy)\right| dt \\ &\leq \int_0^t \left|\int_{\mathbb{R}^d} b(X_s, y)\mathbb{P}_{X_s}(dy) - \int_{\mathbb{R}^d} b(X_s, y)\mathbb{P}_{Y_s}(dy)\right| dt \\ &\quad + \int_0^t \left|\int_{\mathbb{R}^d} b(X_s, y)\mathbb{P}_{Y_s}(dy) - \int_{\mathbb{R}^d} b(Y_s, y)\mathbb{P}_{Y_s}(dy)\right| dt \\ &\leq Lip(b)\left(\int_0^t \mathbf{d}_1(\mathbb{P}_{X_s}, \mathbb{P}_{Y_s})dt + \int_0^t |X_s - Y_s|dt\right). \end{aligned}$$

Recall that $\mathbf{d}_1(\mathbb{P}_{X_s}, \mathbb{P}_{Y_s}) \le \mathbb{E}\left[|X_s - Y_s|\right]$. So multiplying by $e^{-\alpha t}$ and taking expectation, we obtain:

$$\begin{aligned}
\|\Phi(X) - \Phi(Y)\|_E &= \mathbb{E}\left[\int_0^{+\infty} e^{-\alpha t}\,|\Phi(X)_s - \Phi(Y)_s|\,dt\right] \\
&\le 2Lip(b)\int_0^{+\infty} e^{-\alpha t}\int_0^t \mathbb{E}\left[|X_s - Y_s|\right] ds dt \\
&\le \frac{2Lip(b)}{\alpha}\|X - Y\|_E.
\end{aligned}$$

If we choose $\alpha > 2Lip(b)$, then Φ is a contraction in the Banach space E and therefore has a unique fixed point. It is easy to check that this fixed point is the unique solution to our problem. □

3.3.2 The mean field limit

Let (X^i) be the solution to the particle system (3.8) and $(\bar{X}^i)$ be the solution to (3.9). Let us note that, as the (B^i) and the (Z^i) are independent with the same law, the $(\bar{X}^i_t)$ are i.i.d. for any $t \ge 0$.

Theorem 3.3.3. *We have, for any $T > 0$,*

$$\lim_{N\to+\infty} \mathbb{E}\left[\sup_{i=1,\ldots,N}\sup_{t\in[0,T]} |X^i_t - \bar{X}^i_t|\right] = 0.$$

Remark: a similar result holds when there is a non constant volatility term σ in front of the Brownian motion. The proof is then slightly more intricate.

Proof. We consider

$$X^i_t - \bar{X}^i_t = \int_0^t \left(\frac{1}{N}\sum_{j=1}^N b(X^i_t, X^j_t) - \int_{\mathbb{R}^d} b(\bar{X}^i_t, y)\mathbb{P}_{\bar{X}^i_t}(dy)\right) dt.$$

By the uniqueness in law of the solution to the McKean-Vlasov equation we can denote by $m(t, dy) := \mathbb{P}_{\bar{X}^i_t}(dy)$. Then, using the definition of $m^N_{\bar{X}_t} := \frac{1}{N}\sum_{j=1}^N \delta_{\bar{X}^n_t}$ and the triangle inequality, we have

$$\begin{aligned}
|X^i_t - \bar{X}^i_t| \le &\int_0^t \left|\frac{1}{N}\sum_{j=1}^N (b(X^i_s, X^j_s) - b(\bar{X}^i_s, \bar{X}^j_s))\right| ds \\
&+ \int_0^t \left|\int_{\mathbb{R}^d} b(\bar{X}^i_s, y) m^N_{\bar{X}_s}(dy) - \int_{\mathbb{R}^d} b(\bar{X}^i_s, y) m(s, dy)\right| ds \\
\le\ & Lip(b)\int_0^t (|X^i_s - \bar{X}^i_s| + \frac{1}{N}\sum_{j=1}^N |X^j_s - \bar{X}^j_s|) ds + Lip(b)\int_0^t \mathbf{d}_1(m^N_{\bar{X}_s}, m(s,\cdot)) ds.
\end{aligned}$$

Summing over $i = 1, \ldots, N$, we get

$$\frac{1}{N}\sum_{i=1}^N |X^i_t - \bar{X}^i_t| \le 2Lip(b)\int_0^t \frac{1}{N}\sum_{j=1}^N |X^j_s - \bar{X}^j_s| ds + Lip(b)\int_0^t \mathbf{d}_1(m^N_{\bar{X}_s}, m(s,\cdot)) ds.$$

Using Gronwall Lemma, we find, for any $T > 0$,

$$\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N |X^i_t - \bar{X}^i_t| \le C_T \int_0^T \mathbf{d}_1(m^N_{\bar{X}_s}, m(s,\cdot)) ds,$$

where C_T depends on T and $Lip(b)$ (but not on N). As the $X^i - \bar{X}^i$ have the same law, we have, for any $t \in [0,T]$,

$$\mathbb{E}\left[|X_t^i - \bar{X}_t^i|\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N |X_t^i - \bar{X}_t^i|\right] \leq C_T \int_0^T \mathbb{E}\left[\mathbf{d}_1(m_{\bar{X}_s}^N, m(s,\cdot))ds\right].$$

One can finally check exactly as in the proof of Theorem 3.2.5 that the right-hand side tends to 0. □

3.3.3 Further reading

An introductive exposition of the topic can be found in the notes of Villani [35]. Classical references for the mean field limit of particle systems are the monographs or texbooks by Sznitman [34], Spohn [33] and Golse [14].

Chapter 4

The second order MFG systems

This chapter is devoted to the most classical mean field game system. We first explain—in a very general framework—the derivation and the meaning of the system and illustrate the concept through an example. Then we prove the existence of a solution in a very particular setting. Finally we also discuss potential MFG, for which one can obtain the equilibrium as the minimum of a functional.

4.1 Description of the system

4.1.1 Heuristic derivation of the MFG system

We describe here the simplest, standard class of mean field games. In this control problem with infinitely many agents, each small agent controls her own dynamics:

$$X_s = x + \int_t^s b(r, X_r, \alpha_r, m(r))dr + \int_t^s \sigma(r, X_r, \alpha_r, m(r))dB_r, \tag{4.1}$$

where X lives in $\mathbb{R}^d$, α is the control (taking its values in a fixed set A) and B is a given $M-$dimensional Brownian motion. The difference with Subsection 3.1.1 is the dependence of the coefficients with respect to the distribution $(m(t))$ of the all the players. This (time dependent) distribution $(m(t))$ belongs to the set $\mathcal{P}(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$ and is, at this stage, supposed given: we think at $(m(t))$ as the anticipation made by the agents on their future time dependent distribution. The coefficients $b : [0,T] \times \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times M}$ are assumed to be smooth enough for the solution (X_t) to exist.

The cost of a small player is given by

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T L(s, X_s, \alpha_s)ds + g(X_T, m(T))\right].$$

Here $T > 0$ is the finite horizon of the problem, $L : [0, T \times \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ and $g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ are given continuous maps.

If we define the value function u as

$$u(t,x) = \inf_\alpha J(t,x,\alpha),$$

then, at least in a formal way, u solves the Hamilton-Jacobi equation

$$\left\{ \begin{array}{ll} -\partial_t u(t,x) + H(t,x,Du(t,x),D^2u(t,x),m(t)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ u(T,x) = g(x,m(T)) \qquad \text{in } \mathbb{R}^d. & \end{array}\right.$$

where the Hamiltonian $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is defined by

$$H(t,x,p,M,m) := \sup_{a\in A}\left[-L(t,x,a,m) - p\cdot b(t,x,a,m) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,a,m)M)\right].$$

Let us now introduce $\alpha^*(t,x) \in A$ as a maximum point in the definition of H when $p = Du(t,x)$ and $M = D^2(t,x)$. Namely

$$
\begin{aligned}
H(t,x,Du(t,x),D^2u(t,x),m(t)) &= -L(t,x,\alpha^*(t,x),m(t)) - Du(t,x)\cdot b(t,x,\alpha^*(t,x),m(t)) \\
&\quad - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,\alpha^*(t,x))D^2u(t,x),m(t)).
\end{aligned} \tag{4.2}
$$

Recall from Subsection 3.1.1 that α^* is the optimal feedback for the problem. Let us strongly emphasize that u and α^* depend on the time-dependent family of measures $(m(t))$.

We now discuss the evolution of the population density. For this we make two important assumptions: First we assume that all the agents control the same system (4.1) (although not necessarily starting from the same initial position) and minimize the same cost J. As a consequence, the dynamics at optimum of each player is given by

$$
dX^*_s = b(s,X^*_s,\alpha^*(s,X^*_s),m(s))ds + \sigma(s,X^*_s,\alpha^*(s,X^*_s),m(r))dB_s.
$$

Second we assume that the initial position of the agents and the noise driving their dynamics are independent: in particular, there is no “common noise” impacting all the players. The initial distribution of the agents at time $t=0$ is denoted by $\bar{m}_0 \in \mathcal{P}(\mathbb{R}^d)$. From the analysis of the mean field limit (in the simple case where the coefficients do not depend on the other agents) the actual distribution $(\tilde{m}(s))$ of all agents at time s is simply given by the law of (X^*_s) with $\mathcal{L}(X^*_0) = \bar{m}_0$.

Let us now write the equation satisfied by $(\tilde{m}(s))$. By Itô's formula, we have, for any smooth map $\phi : [0,T)\times\mathbb{R}^d \to \mathbb{R}$ with a compact support:

$$
\begin{aligned}
0 = \mathbb{E}\left[\phi(T,X^*_T)\right] &= \mathbb{E}\left[\phi(0,X^*_0)\right] + \int_0^T \mathbb{E}\Big[\partial_t\phi(s,X^*_s) + b(s,X^*_s,\alpha^*(s,X^*_s),m(s))\cdot D\phi(s,X^*_s) \\
&\qquad + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(s,X^*_s,\alpha^*(s,X^*_s),m(r))D^2\phi(s,X^*_s))\Big]ds \\
&= \int_{\mathbb{R}^d}\phi(0,x)\bar{m}_0(dx) + \int_0^T\int_{\mathbb{R}^d}\Big[\partial_t\phi(s,x) + b(s,x,\alpha^*(s,x),m(s))\cdot D\phi(s,x) \\
&\qquad + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(s,x,\alpha^*(s,x),m(r))D^2\phi(s,x))\Big]\tilde{m}(t,dx)ds
\end{aligned}
$$

After integration by parts, we obtain that $(\tilde{m}(t))$ satisfies, in the sense of distributions,

$$
\left\{
\begin{array}{l}
\partial_t\tilde{m} - \dfrac{1}{2}\displaystyle\sum_{ij} D^2_{ij}(\tilde{m}(t,x)a_{ij}(t,x,\alpha^*(t,x),m(t))) + \mathrm{div}(\tilde{m}(t,x)b(s,x,\alpha^*(s,x),m(s))) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \\
\tilde{m}(0) = \bar{m}_0 \qquad \text{in } \mathbb{R}^d,
\end{array}
\right.
$$

where $a = \sigma\sigma^*$.

At equilibrium, one expect the anticipation $(m(t))$ made by the agents to be correct: $\tilde{m}(t) = m(t)$. Collecting the above equation leads to the MFG system:

$$
\left\{
\begin{array}{l}
-\partial_t u(t,x) + H(t,x,Du(t,x),D^2u(t,x),m(t)) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d, \\
\partial_t m - \dfrac{1}{2}\displaystyle\sum_{ij} D^2_{ij}(m(t,x)a_{ij}(t,x,\alpha^*(t,x),m(t))) + \mathrm{div}(m(t,x)b(s,x,\alpha^*(s,x),m(s))) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \\
m(0) = \bar{m}_0,\ u(T,x) = g(x,m(T)) \qquad \text{in } \mathbb{R}^d,
\end{array}
\right.
$$

where α^* is given by (4.2) and $a = \sigma\sigma^*$.

In order to simplify a little this system, let us assume that $M = d$ and $\sigma = \sqrt{2}I_d$. We set (warning! abuse of notation!)

$$
H(t,x,p,m) := \sup_{a\in A}\left[-L(t,x,a,m) - p\cdot b(t,x,a,m)\right]
$$

and note that, by the Envelope Theorem (see Lemma 4.1.1 below), one can expect to get (under suitable assumptions):

$$
D_pH(t,x,Du(t,x),m(t)) = -b(t,x,\alpha^*(t,x),m(t)).
$$

In this case the MFG system becomes

$$\left\{\begin{array}{l} -\partial_t u(t,x) - \Delta u(t,x) + H(t,x,Du(t,x),m(t)) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d,\\ \partial_t m - \Delta m(t,x) - \mathrm{div}(m(t,x)D_pH(t,x,Du(t,x),m(t))) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d,\\ m(0)=\bar m_0,\ u(T,x)=g(x,m(T)) \qquad \text{in } \mathbb{R}^d, \end{array}\right.$$

Here is the Envelope Theorem:

Lemma 4.1.1. *Let A be a compact metric space, $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ and $f : A\times\mathcal{O}\to\mathbb{R}$ be continuous and with D_xf continuous on $A\times\mathcal{O}$. Then the marginal map*

$$V(x) = \inf_{a\in A} f(a,x)$$

is differentiable at each point $x\in\mathcal{O}$ such that the infimum in $V(x)$ is a unique point $a_x\in A$. Then

$$DV(x) = D_xf(a_x,x).$$

Proof. Let $x\in\mathcal{O}$ be such that the infimum in $V(x)$ is a unique point $a_x\in A$. Then an easy compactness argument shows that, if a_y is a minimum point of $V(y)$ for $y\in\mathcal{O}$ and $y\to x$, then $a_y\to a_x$.

Fix $y\in\mathcal{O}$. Note first that, as $a_x\in A$,

$$V(y)\le f(a_x,y) = f(a_x,x)+D_xf(a_x,z_y)\cdot(y-x) = V(x)+D_xf(a_x,z_y)\cdot(y-x),$$

for some $z_y\in[x,y]$.

Conversely,

$$V(x)\le f(a_y,x) = f(a_y,y)+D_xf(a_y,z_y')\cdot(x-y) = V(y)+D_xf(a_y,z_y')\cdot(x-y),$$

for some $z_y'\in[x,y]$.

By continuity of D_xf and convergence of a_y, we infer that

$$\begin{aligned}&\lim_{y\to x}\frac{|V(y)-V(x)-D_xf(a_x,x)\cdot(y-x)|}{|y-x|}\\ &\qquad\le \liminf_{y\to x}|D_xf(a_x,z_y)-D_xf(a_x,x)| + \left|D_xf(a_y,z_y')-D_xf(a_x,x)\right| = 0.\end{aligned}$$

□

4.1.2 A simple example of optimal trading

We discuss here a toy example of optimal trading. Let us warn the reader that this example does not fit the framework described above. However, it can be solved explicitly and illustrates quite well the mean field interactions.

A continuum of investors decide to buy or sell a given tradable instrument. The decision is a signed quantity Q_0 to buy (in such a case Q_0 is negative: the investor has a negative inventory at the initial time $t=0$) or to sell (when Q_0 is positive). All investors have to buy or sell before a given terminal time T, and we assume that they have the same risk aversion parameters ϕ and A.

Each investor will control its trading speed α_t through time, in order to fulfill its goal. The price S_t of the tradable instrument is submitted to two kinds of moves: an exogenous innovation supported by a standard Wiener process B_t (with its natural probability space and the associated filtration $\mathcal{F}_t$), and the *permanent market impact* generated linearly from the buying or selling pressure $\theta\mu_t$ where μ_t is the net sum of the trading speed of all investors and $\theta>0$ is a fixed parameter.

$$dS_t = \theta\mu_t\,dt+\sigma\,dB_t. \tag{4.3}$$

In principle one should expect the term (μ_t) to be stochastic and adapted to the filtration generate by B, which, in this setting, should be interpreted as a common noise. Because of the specific structure of the problem it is not the case here: so in the rest of the example we consider (μ_t) as deterministic.

The state of each investor is described by two variables: its inventory Q_t and its wealth X_t (starting with $X_0 = 0$ for all investors). The evolution of Q reads

$$dQ_t = \alpha_t \, dt, \tag{4.4}$$

since for a seller, $Q_0 > 0$ (the associated control α will be mostly negative) and the wealth suffers from linear trading costs (or *temporary*, or *immediate market impact*, parametrized by κ):

$$dX_t = -\alpha_t (S_t + \kappa \cdot \alpha_t) \, dt. \tag{4.5}$$

Meaning the wealth of a seller will be positive (and the faster you sell—i.e. α is largely negative—, the smaller the sell price).

The cost function of each investor is made of the wealth at T, plus the value of the inventory penalized by a terminal market impact, and minus a running cost quadratic in the inventory:

$$V_t := \sup_\alpha \mathbb{E}\left[X_T + Q_T(S_T - A \cdot Q_T) - \phi \int_t^T (Q_s)^2 \, ds \right]. \tag{4.6}$$

The Hamilton-Jacobi-Bellman satisfied by the valued function $V = V(t, s, q, x; \mu)$ associated to (4.6) is

$$0 = \partial_t V - \phi \, q^2 + \frac{1}{2}\sigma^2 \partial^2_{ss} V + \theta\mu\partial_s V + \sup_\alpha \{\alpha \partial_q V - \alpha(s + \kappa\, \alpha)\partial_x V\} \tag{4.7}$$

with terminal condition

$$V(T, x, s, q; \mu) = x + q(s - Aq).$$

To find V, we will use the following ansatz:

$$V = x + qs + v(t, q; \mu). \tag{4.8}$$

Thus the HJB on v is

$$-\theta\mu \, q = \partial_t v - \phi \, q^2 + \sup_\alpha \left\{ \alpha \partial_q v - \kappa \, \alpha^2 \right\},$$

with terminal condition

$$v(T, q; \mu) = -Aq^2$$

and the associated optimal feedback is

$$\alpha(t, q) = \frac{\partial_q v(t, q)}{2\kappa}. \tag{4.9}$$

Further simplification In view of the equation, we look for $v(t, q)$ as a quadratic function of q:

$$v(t, q) = h_0(t) + q \, h_1(t) - q^2 \, \frac{h_2(t)}{2}.$$

Then the equation for v can be split in three parts

$$\begin{array}{rcl} 0 &=& -2\kappa h_2'(t) - 4\kappa\phi + (h_2(t))^2, \\ 2\kappa\theta\mu(t) &=& -2\kappa h_1'(t) + h_1(t) h_2(t), \\ -(h_1(t))^2 &=& 4\kappa h_0'(t), \end{array} \tag{4.10}$$

One also has to add the terminal condition: as $V_T = x + q(s - Aq)$, $v(T, q) = -Aq^2$. This implies that

$$h_0(T) = h_1(T) = 0, \;\; h_2(T) = 2A. \tag{4.11}$$

Defining the mean field. In our model, the net trading flow μ is the average of the optimal feedback in the population. If the initial inventories of the investors is distributed as the random variable $\tilde{Q}_0$, by (4.9) the inventories at time t of these investors is the random variable given by

$$\tilde{Q}_t = \tilde{Q}_0 + \int_0^t \frac{\partial_q v(s, \tilde{Q}_s)}{2\kappa} ds = \tilde{Q}_0 + \frac{1}{2\kappa} \int_0^t (-h_2(s)\tilde{Q}_s + h_1(s)) ds.$$

Calling $E(t)$ the expectation of $\tilde{Q}_t$, we find that E solves the ODE

$$E'(t) = -\frac{h_2(t)}{2\kappa} E(t) + \frac{h_1(s)}{2\kappa}, \qquad E(0) = E_0 := \mathbb{E}\left[\tilde{Q}_0\right]$$

while

$$\mu(t) = \mathbb{E}\left[\frac{\partial_q v(s, \tilde{Q}_s)}{2\kappa}\right] = \frac{1}{2\kappa}\mathbb{E}\left[-h_2(t)\tilde{Q}_t + h_1(t)\right] = E'(t).$$

The full system To summarize, we find the system:

$$\begin{array}{rcl}
0 & = & -2\kappa h_2'(t) - 4\kappa\phi + (h_2(t))^2, \\
2\kappa\theta E'(t) & = & -2\kappa h_1'(t) + h_1(t)h_2(t), \\
-(h_1(t))^2 & = & 4\kappa h_0'(t), \\
E'(t) & = & -\frac{h_2(t)}{2\kappa} E(t) + \frac{h_1(t)}{2\kappa}, \\
h_0(T) & = & h_1(T) = 0,\ h_2(T) = 2A,\ E(0) = E_0.
\end{array}$$

Note that this is a forward-backward system and thus the standard theory of ODEs does not directly apply.

An example of resolution The (backward) equation for h_2 can be solved separately. Once h_1 is known, one can also find h_0. Let us now show that one can reduce the system of equations satisfied (h_1, E) to a single equation. We have

$$h_1(t) = 2\kappa E'(t) + h_2(t)E(t),$$

so that

$$\begin{aligned}
h_1'(t) &= 2\kappa E''(t) + h_2(t)E'(t) + h_2'(t)E(t) \\
&= -\theta E'(t) + \frac{h_2(t)}{2\kappa} h_1(t) = -\theta E'(t) + \frac{h_2(t)}{2\kappa}(2\kappa E'(t) + h_2(t)E(t)).
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= -2\kappa E''(t) - \theta E'(t) + (-h_2'(t) + \frac{h_2^2(t)}{2\kappa})E(t) \\
&= -2\kappa E''(t) - \theta E'(t) + 2\phi E(t).
\end{aligned}$$

with boundary conditions

$$E(0) = E_0, \qquad 0 = h_1(T) = 2\kappa E'(T) + h_2(T)E(T) = 2\kappa E'(T) + 2AE(T).$$

To summarize we find the problem

$$\left\{ \begin{array}{ll} -2\kappa E''(t) - \theta E'(t) + 2\phi E(t) = 0 & \text{in } (0,T), \\ E(0) = E_0, \qquad 2\kappa E'(T) + 2AE(T) = 0. & \end{array} \right. \tag{4.12}$$

Let us consider the Hilbert space $H := \{x \in H^1([0,T]),\ x(0) = 0\}$. Then by the equation satisfied by $x(t) = E(t) - E_0$, we have, for any $y \in H$,

$$\begin{aligned}
0 &= \int_0^T y(t)(-2\kappa x''(t) - \theta x'(t) + 2\phi(x(t) + E_0))dt \\
&= \Big[-2y(t)x'(t)\Big]_0^T + \int_0^T (2\kappa x'(t)y'(t)x''(t) - \theta x'(t)y(t) + 2\phi x(t)y(t))dt + E_0)\int_0^T y(t)dt \\
&= \frac{2A}{\kappa}(x(T) + E_0)y(T) + \int_0^T (2\kappa x'(t)y'(t) - \theta x'(t)y(t) + 2\phi x(t)y(t))dt + E_0 \int_0^T y(t)dt
\end{aligned}$$

Let us consider on H the bilinear form

$$\mathcal{A}(x, y) := \frac{2A}{\kappa} x(T)y(T) + \int_0^T (2\kappa x'(t)y'(t)x''(t) - \theta x'(t)y(t) + 2\phi x(t)y(t))dt.$$

One easily checks that it is bilinear continuous on $H \times H$. Moreover,

$$\begin{aligned}
\mathcal{A}(x, x) &= \frac{2A}{\kappa}(x(T))^2 + \int_0^T (2\kappa(x'(t))^2 - \theta x'(t)x(t) + 2\phi(x(t))^2)dt \\
&= (\frac{2A}{\kappa} - \theta)(x(T))^2 + \int_0^T (2\kappa(x'(t))^2 + 2\phi(x(t))^2)dt
\end{aligned}$$

So $\mathcal{A}$ is coercive (at least if) if $\theta \le \frac{2A}{\kappa}$. From Lax-Milgram Theorem we deduce that, if ξ is the linear form

$$\xi(y) := -\frac{2AE_0}{\kappa} y(T) - E_0 \int_0^T y(t)dt \qquad \forall y \in H,$$

there exists a unique $\bar{x} \in H$ such that

$$\mathcal{A}(\bar{x}, y) = \xi(y) \qquad \forall y \in H.$$

One can check by standard arguments that this is equivalent to saying that $\bar{x}$ solves system (4.12).

4.2 Analysis of a second order quadratic MFG

We discuss the existence of a solution of the MFG system in the simple setting of second order equations with a quadratic Hamiltonian:

$$(4.13) \qquad \left\{ \begin{array}{rl}
(i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m) \qquad \text{in } \mathbb{R}^d \times (0,T) \\
(ii) & \partial_t m - \Delta m - \operatorname{div}(m\ Du) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \\
(iii) & m(0) = m_0\ ,\ u(x,T) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d
\end{array}\right.$$

Our aim is to prove the existence of classical solutions for this system and give some interpretation in terms of game with finitely many players.

For this our main assumption is that F and G are regularizing on the set of probability measures on $\mathbb{R}^d$. To make our life simple, we assume that all measures considered in this section have a finite first order moment: let $\mathcal{P}_1$ be the set of such Borel probability measures m on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |x|dm(x) < +\infty$. The set $\mathcal{P}_1$ can be endowed with following (Kantorovitch-Rubinstein) introduced in Subsection 3.2.

Here are our main assumptions on F, G and m_0: we suppose that there is some constant C_0 such that

1. (Bounds on F and G) F and G are uniformly bounded by C_0 over $\mathbb{R}^d \times \mathcal{P}_1$.
2. (Uniform regularity of G) $G(\cdot, m)$ is bounded in $C^{2+\alpha}(\mathbb{R}^d)$ (for some $\alpha \in (0,1)$), uniformly with respect to the measure m.

3. (Lipschitz continuity of F and G)

$$|F(x_1,m_1)-F(x_2,m_2)| \leq C_0\left[|x_1-x_2|+\mathbf{d}_1(m_1,m_2)\right] \qquad \forall (x_1,m_1),(x_2,m_2)\in \mathbb{R}^d\times \mathcal{P}_1\ .$$

and

$$|G(x_1,m_1)-G(x_2,m_2)| \leq C_0\left[|x_1-x_2|+\mathbf{d}_1(m_1,m_2)\right] \qquad \forall (x_1,m_1),(x_2,m_2)\in \mathbb{R}^d\times \mathcal{P}_1\ .$$

4. The probability measure m_0 is absolutely continuous with respect to the Lebesgue measure, has a $C^{2+\alpha}$ density (still denoted m_0) which satisfies $\int_{\mathbb{R}^d}|x|^2m_0(x)dx<+\infty$.

A pair (u,m) is a classical solution to (4.13) if $u,m:\mathbb{R}^d\times[0,T]\to\mathbb{R}$ are continuous, of class $\mathcal{C}^2$ in space and $\mathcal{C}^1$ in time and (u,m) satisfies (4.13) in the classical sense. The main result of this section is the following:

Theorem 4.2.1. *Under the above assumptions, there is at least one classical solution to (4.13).*

The proof is relatively easy and relies on basic estimates for the heat equation as well as some remarks on the Fokker-Planck equation (4.13-(ii)).

4.2.1 On the Fokker-Planck equation

Let $b:\mathbb{R}^d\times[0,T]\to\mathbb{R}$ be a given vector field. Our aim is to analyse the Fokker-Planck equation

$$\left\{\begin{array}{ll} \partial_t m-\Delta m-\operatorname{div}(m\ b)=0 & \text{in } \mathbb{R}^d\times(0,T)\\ m(0)=m_0 \end{array}\right. \tag{4.14}$$

as an evolution equation is the space of probability measures. We assume here that the vector field $b:\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ is continuous, uniformly Lipschitz continuous in space, and bounded.

Definition 4.2.2 (Weak solution to (4.14))**.** *We say that m is a weak solution to (4.14) if $m\in L^1([0,T],\mathcal{P}_1)$ is such that, for any test function $\varphi\in\mathcal{C}^\infty_c(\mathbb{R}^d\times[0,T))$, we have*

$$\int_{\mathbb{R}^d}\phi(x,0)dm_0(x)+\int_0^T\int_{\mathbb{R}^d}\left(\partial_t\varphi(x,t)+\Delta\varphi(x,t)+\langle D\varphi(x,t),b(x,t)\rangle\right)\ dm(t)(x)=0\ .$$

In order to analyse some particular solutions of (4.14), it is convenient to introduce the following stochastic differential equation (SDE)

$$\left\{\begin{array}{ll} dX_t=b(X_t,t)dt+\sqrt{2}dB_t, & t\in[0,T]\\ X_0=Z_0 \end{array}\right. \tag{4.15}$$

where (B_t) is a standard $d-$dimensional Brownian motion over some probability space $(\Omega,\mathcal{A},\mathbb{P})$ and where the initial condition $Z_0\in L^1(\Omega)$ is random and independent of (B_t). Under the above assumptions on b, there is a unique solution to (4.15). This solution is closely related to equation (4.14):

Lemma 4.2.3. *If $\mathcal{L}(Z_0)=m_0$, then $m(t):=\mathcal{L}(X_t)$ a weak solution of (4.14).*

Proof : This is a straightforward consequence of Itô's formula, which says that, if $\varphi:\mathbb{R}^d\times[0,T]\to\mathbb{R}$ is bounded, of class $\mathcal{C}^2$ in space and $\mathcal{C}^1$ in time, then

$$\begin{array}{l}\varphi(X_t,t)=\varphi(Z_0,0)\\ \qquad +\displaystyle\int_0^t\left[\varphi_t(X_s,s)+\langle D\varphi(X_s,s),b(X_s,s)\rangle+\Delta\varphi(X_s,s)\right]\ ds+\int_0^t\langle D\varphi(X_s,s),dB_s\rangle\ .\end{array}$$

Taking the expectation on both sides of the equality, we have, since

$$\mathbb{E}\left[\int_0^t\langle D\varphi(X_s,s),dB_s\rangle\right]=0$$

because $t \to \int_0^t \langle D\varphi(X_s,s), dB_s\rangle$ is a martingale,

$$\mathbb{E}\left[\varphi(X_t,t)\right] = \mathbb{E}\left[\varphi(Z_0,0) + \int_0^t \left[\varphi_t(X_s,s) + \langle D\varphi(X_s,s), b(X_s,s)\rangle + \Delta\varphi(X_s,s)\right]\, ds\right] .$$

So by definition of $m(t)$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x,t)dm(t)(x) &= \int_{\mathbb{R}^d} \varphi(x,0)dm_0(x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \left[\varphi_t(x,s) + \langle D\varphi(x,s), b(x,s)\rangle + \Delta\varphi(x,s)\right] dm(s)(x)\ ds ,\end{aligned}$$

i.e., m is a weak solution to (4.14). □

This above interpretation of the continuity equation allows to get very easily some estimates on the map $t \to m(t)$ in $\mathcal{P}_2$.

Lemma 4.2.4. *Let m be defined as above. There is a constant $c_0 = c_0(T)$, such that*

$$\mathbf{d}_1(m(t), m(s)) \le c_0(1 + \|b\|_\infty)|t-s|^{\frac{1}{2}} \qquad \forall s,t \in [0,T] .$$

Proof : Recalling the definition of $\mathbf{d}_1$ we note that the law γ of the pair (X_t, X_s) belongs to $\Pi(m(t), m(s))$, so that

$$\mathbf{d}_1(m(t), m(s)) \le \int_{\mathbb{R}^{2d}} |x-y| d\gamma(x,y) = \mathbb{E}\left[|X_t - X_s|\right] .$$

Therefore, if for instance $s < t$,

$$\begin{aligned}\mathbb{E}\left[|X_t - X_s|\right] &\le \mathbb{E}\left[\int_s^t |b(X_\tau,\tau)|\, d\tau + \sqrt{2}\,|B_t - B_s|\right] \\ &\le \|b\|_\infty (t-s) + \sqrt{2(t-s)}\end{aligned}$$

□

Moreover we also obtain some integral estimates:

Lemma 4.2.5. *There is a constant $c_0 = c_0(T)$ such that*

$$\int_{\mathbb{R}^d} |x|^2 dm(t)(x) \le c_0\Big(\int_{\mathbb{R}^d} |x|^2 dm_0(x) + 1 + \|b\|_\infty^2\Big) \qquad \forall t \in [0,T] .$$

Proof : Indeed:

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2 dm(t)(x) = \mathbb{E}\left[|X_t|^2\right] &\le 2\mathbb{E}\left[|X_0|^2 + \left|\int_0^t b(X_\tau,\tau)d\tau\right|^2 + 2\,|B_t|^2\right] \\ &\le 2\left[\int_{\mathbb{R}^d} |x|^2 dm_0(x) + t^2\|b\|_\infty^2 + 2t\right]\end{aligned}$$

□

4.2.2 Proof of the existence Theorem

Before starting the proof of Theorem 4.2.1, let us recall some basic existence/uniqueness result for the heat equation

$$\text{(4.16)} \qquad \left\{\begin{array}{l} w_t - \Delta w + \langle a(x,t), Dw\rangle + b(x,t)w = f(x,t) \qquad \text{in } \mathbb{R}^d \times [0,T] \\ w(x,0) = w_0(x) \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

For this it will be convenient to denote by $\mathcal{C}^{s+\alpha}$ (for an integer $s \geq 0$ and $\alpha \in (0,1)$) the set of maps $z : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ such that the derivatives $\partial_t^k D_x^l z$ exist for any pair (k,l) with $2k+l \leq s$ and such that these derivatives are bounded and $\alpha-$Hölder continuous in space and $(\alpha/2)-$Hölder continuous in time. If we assume that, for some $\alpha \in (0,1)$, $a : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, $b, f : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ and $w_0 : \mathbb{R}^d \to \mathbb{R}$ belong to $\mathcal{C}^\alpha$, then the above heat equation is has a unique weak solution. Furthermore this solution belongs to $\mathcal{C}^{2+\alpha}$ (Theorem 5.1 p. 320 of [22]).

We will also need the following interior estimate (Theorem 11.1 p. 211 of [22]): if $a = b = 0$ and f is continuous and bounded, any classical, bounded solution w of (4.16) satisfies, for any compact set $K \subset \mathbb{R}^d \times (0,T)$,

$$\sup_{(x,t),(y,s)\in K} \frac{|Dw(x,t) - Dw(y,s)|}{|x-y|^\beta + |t-s|^{\beta/2}} \leq C(K, \|w\|_\infty)\|f\|_\infty \ , \tag{4.17}$$

where $\beta \in (0,1)$ depends only on the dimension d while $C(K, \|w\|_\infty)$ depends on the compact set K, on $\|w\|_\infty$ and on d.

Let C_1 be a large constant to be chosen below and $\mathcal{C}$ be the set of maps $\mu \in \mathcal{C}^0([0,T], \mathcal{P}_1)$ such that

$$\sup_{s \neq t} \frac{\mathbf{d}_1(\mu(s), \mu(t))}{|t-s|^{\frac{1}{2}}} \leq C_1 \tag{4.18}$$

and

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 dm(t)(x) \leq C_1 \ .$$

Then $\mathcal{C}$ is a convex closed subset of $\mathcal{C}^0([0,T], \mathcal{P}_1)$. It is actually compact, because the set of probability measures m for which $\int_{\mathbb{R}^d} |x|^2 dm(x) \leq C_1$ is finite, is compact in $\mathcal{P}_1$ (see Lemma 3.2.4).

To any $\mu \in \mathcal{C}$ we associate $m = \Psi(\mu) \in \mathcal{C}$ in the following way: Let u be the unique solution to

$$\left\{ \begin{array}{l} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, \mu(t)) \qquad \text{in } \mathbb{R}^d \times (0,T) \\ u(x,T) = G(x, \mu(T)) \qquad \text{in } \mathbb{R}^d \end{array} \right. \tag{4.19}$$

Then we define $m = \Psi(\mu)$ as the solution of the Fokker-Planck equation

$$\left\{ \begin{array}{l} \partial_t m - \Delta m - \operatorname{div}(m\ Du) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \\ m(0) = m_0 \qquad \text{in } \mathbb{R}^d \end{array} \right. \tag{4.20}$$

Let us check that Ψ is well-defined and continuous. To see that a solution to (4.19) exists and is unique, we use the Hopf-Cole transform: setting $w = e^{u/2}$ we easily check that u is a solution of (4.19) if and only if w is a solution of the linear (backward) equation

$$\left\{ \begin{array}{l} -\partial_t w - \Delta w = wF(x, \mu(t)) \qquad \text{in } \mathbb{R}^d \times (0,T) \\ w(x,T) = e^{G(x,\mu(T))/2} \qquad \text{in } \mathbb{R}^d \end{array} \right.$$

Note that the maps $(x,t) \to F(x, m(t))$ and $x \to e^{G(x,\mu(T))/2}$ belong to $\mathcal{C}^{1/2}$, because μ satisfies (4.18) and from our assumptions on F and G. Therefore the above equation is uniquely solvable and the solution belongs to $\mathcal{C}^{2+\alpha}$ with $\alpha = \frac{1}{2}$, which in turn implies the unique solvability of (4.19) with a solution u which belongs to $\mathcal{C}^{2+\alpha}$. Recall that the maps $x \to F(x, \mu(t))$ and $x \to G(x, \mu(T))$ are bounded by C_0, so that a straightforward application of the comparison principle implies that u is bounded by $(1+T)C_0$. In the same way, since moreover the maps $x \to F(x, \mu(t))$ and $x \to G(x, \mu(T))$ are C_0-Lipschitz continuous (again by our assumptions on F and G), u is also C_0-Lipschitz continous. Hence Du is bounded by C_0.

Next we turn to the Fokker-Planck equation (4.20), that we write into the form

$$\partial_t m - \Delta m - \langle Dm, Du(x,t) \rangle - m\Delta u(x,t) = 0 \ .$$

Since $u \in \mathcal{C}^{2+\alpha}$, the maps $(x,t) \to Du(x,t)$ and $(x,t) \to \Delta u(x,t)$ belong to $\mathcal{C}^{\alpha}$, so that this equation is uniquely solvable and the solution m belongs to $\mathcal{C}^{2+\alpha}$. Moreover, in view of the discussion of subsection 4.2.1, we have the following estimates on m:

$$\mathbf{d}_1(m(t), m(s)) \leq c_0(1+C_0)|t-s|^{\frac{1}{2}} \qquad \forall s,t \in [0,T]$$

and

$$\int_{\mathbb{R}^d} |x|^2 dm(t)(x) \leq c_0(1+C_0^2) \qquad \forall t \in [0,T] ,$$

where c_0 depends only on T. So if we choose $C_1 = \max\{c_0(1+C_0), c_0(1+C_0^2)\}$, m belongs to $\mathcal{C}$.

We have just proved that the mapping $\Psi : \mu \to m = \Psi(\mu)$ is well-defined. Let us check that it is continuous. Let $\mu_n \in \mathcal{C}$ converge to some μ. Let (u_n, m_n) and (u, m) be the corresponding solutions. Note that $(x,t) \to F(x, \mu_n(t))$ and $x \to G(x, \mu_n(T))$ locally uniformly converge to $(x,t) \to F(x, \mu(t))$ and $x \to G(x, \mu(T))$. Then one gets the local uniform convergence of (u_n) to u by standard arguments (of viscosity solutions for instance). Since the (Du_n) are uniformly bounded, the (u_n) solve an equation of the form

$$\partial_t u_n - \Delta u_n = f_n$$

where $f_n = \frac{1}{2}|Du_n|^2 - F(x, m_n)$ is uniformly bounded in x and n. Then the interior regularity result (4.17) implies that (Du_n) is locally uniformly Hölder continuous and therefore locally uniformly converges to Du. This easily implies that any converging subsequence of the relatively compact sequence m_n is a weak solution of (4.20). But m is the unique weak solution of (4.20), which proves that (m_n) converges to m.

We conclude by Schauder fixed point Theorem that the continuous map $\mu \to m = \Psi(\mu)$ has a fixed point in $\mathcal{C}$. Then this fixed point is a solution of our system (4.13). $\square$

4.2.3 Uniqueness

Let us assume that, besides the assumptions given at the beginning of the section, the following conditions hold:

$$(4.21) \qquad \int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \geq 0 \qquad \forall m_1, m_2 \in \mathcal{P}_1,\ m_1 \neq m_2$$

and

$$(4.22) \qquad \int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2)) d(m_1 - m_2)(x) \geq 0 \qquad \forall m_1, m_2 \in \mathcal{P}_1 .$$

Theorem 4.2.6. *Under the above conditions, there is a unique classical solution to the mean field equation (4.13).*

Proof. Before starting the proof, let us notice that we can use as a test function for m any map w which is of class $\mathcal{C}^2$: indeed, the result follows easily by regularizing and truncating w by cut-off functions of the form $\phi_\epsilon(t)\psi_R(x)$ where

$$\phi_\epsilon(t) = \begin{cases} 1 & \text{if } t \leq T-\epsilon \\ 1 + (T-\epsilon-t)/\epsilon & \text{if } T-\epsilon \leq t \leq T \\ 0 & \text{if } t \geq T \end{cases} \quad \text{and } \psi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R \\ R+1-|x| & \text{if } R \leq |x| \leq R+1 \\ 0 & \text{if } |x| \geq R+1 \end{cases}$$

Let now consider (u_1, m_1) and (u_2, m_2) two classical solutions of (4.13). We set $\bar{u} = u_1 - u_2$ and $\bar{m} = m_1 - m_2$. Then

$$(4.23) \qquad -\partial_t \bar{u} - \Delta \bar{u} + \frac{1}{2}\left(|Du_1|^2 - |Du_2|^2\right) - (F(x, m_1) - F(x, m_2)) = 0$$

while

$$\partial_t \bar m - \Delta \bar m - \operatorname{div}(m_1 Du_1 - m_2 Du_2) = 0 \ .$$

Let us use $\bar u$ as a test function in the second equation. Since $\bar u$ is $\mathcal{C}^2$ we have (recall the remark at the begining of the proof)

$$-\int_{\mathbb{R}^d} (\bar m \bar u)(T) + \int_{\mathbb{R}^d} m_0 \bar u(0) + \int_0^T \int_{\mathbb{R}^d} (\partial_t \bar u + \Delta \bar u)\, d\bar m - \int_0^T \int_{\mathbb{R}^d} \langle D\bar u, m_1 Du_1 - m_2 Du_2\rangle = 0$$

Let us multiply equality (4.23) by $\bar m$, integrate over $\mathbb{R}^d \times (0,T)$ and add to the previous equality. We get, after simplification and using that $\bar m(0) = 0$,

$$\begin{aligned} &-\int_{\mathbb{R}^d} (m_1(T) - m_2(T))\,(G(m_1(T)) - G(m_2(T))) \\ &+ \int_0^T \int_{\mathbb{R}^d} \Big(\frac{\bar m}{2}\left(|Du_1|^2 - |Du_2|^2\right) - \bar m\,(F(x,m_1) - F(x,m_2)) - \langle D\bar u, m_1 Du_1 - m_2 Du_2\rangle \Big) \ = 0 \ . \end{aligned}$$

Let us recall that

$$\int_{\mathbb{R}^d} (m_1(T) - m_2(T))\,(G(m_1(T)) - G(m_2(T))) \geq 0 \ ,$$

and also note that

$$\frac{\bar m}{2}\left(|Du_1|^2 - |Du_2|^2\right) - \langle D\bar u, m_1 Du_1 - m_2 Du_2\rangle = -\frac{m_1 + m_2}{2}|Du_1 - Du_2|^2.$$

So we find

$$\int_0^T \int_{\mathbb{R}^d} \frac{m_1 + m_2}{2}|Du_1 - Du_2|^2 \leq 0,$$

which shows that $Du_1 = Du_2$ m_1- and m_2-a.s. Thus m_1 and m_2 solve the same Kolmogorov equation, which, by uniqueness, shows that $m_1 = m_2$. As u_1 and u_2 solve the same Hamilton-Jacobi equation, we infer that $u_1 = u_2$ as well. □

4.2.4 Potential MFGs

In this part we consider a specific class of MFGs where the equilibria come from a minimization problem. Namely we assume that there exist continuous maps $\mathcal{F}, \mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that

$$\mathcal{F}(m') - \mathcal{F}(m) = \int_0^1 \int_{\mathbb{R}^d} F(x, (1-t)m + tm')(m - m')(dx) \qquad \forall m, m' \in \mathcal{P}_2(\mathbb{R}^d)$$

and the symmetric equality holds for $\mathcal{G}$ and G. The above equality means that F is "the derivative" of $\mathcal{F}$. We will discuss this idea in details in the next chapter.

For instance, if $F(x,m) = \int_{\mathbb{R}^d} \phi(x,y) m(dy)$, where ϕ is continuous, bounded and symmetric, it is not difficult to show that one can take:

$$\mathcal{F}(m) = \frac{1}{2}\int_{\mathbb{R}^d} \phi(x,y) m(dy) m(dx).$$

We consider the problem

$$\inf_{m,v} \mathcal{J}(m,v) \text{ where } \mathcal{J}(m,v) := \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |v(t,x)|^2 m(t,x)\,dxdt + \int_0^T \mathcal{F}(m(t))dt + \mathcal{G}(m(T)),$$

the minimum being taken over the set of smooth maps $(m,v) : [0,T] \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ such that

$$\partial_t m - \Delta m + \operatorname{div}(mv) = 0 \text{ in } (0,T) \times \mathbb{R}^d, \qquad m(0,x) = m_0(x). \tag{4.24}$$

Our aim is to show the following result:

Proposition 4.2.7. *Assume that $(\bar m, \bar v)$ is a smooth minimizer of the above problem with $m > 0$. Then there exists a map $\bar u = \bar u(t,x)$ such that the pair $(\bar u, \bar m)$ is a solution to the MFG system (4.13).*

Sketch of proof. Let (m, v) satisfy the constraint (4.24). Then, for any $\lambda > 0$,

$$(m_\lambda, v_\lambda) := ((1-\lambda)\bar m + \lambda m, \frac{(1-\lambda)\bar m \bar v + \lambda m v}{(1-\lambda)\bar m + \lambda m})$$

satisfies the constraint (4.24). By optimality of $(\bar m, \bar v)$ we have

$$\begin{aligned} 0 \le \mathcal{J}(m_\lambda, v_\lambda) - \mathcal{J}(\bar m, \bar v) &= \int_0^T \int_{\mathbb{R}^d} (\frac{|(1-\lambda)\bar m \bar v + \lambda m v|^2}{2((1-\lambda)\bar m + \lambda m)} - |\bar v|^2 \bar m) dxdt \\ &\quad + \int_0^T (\mathcal{F}(m_\lambda(t)) - \mathcal{F}(\bar m(t)))dt + \mathcal{G}(m_\lambda(T)) - \mathcal{G}(\bar m(T)) \\ &= \int_0^T \int_{\mathbb{R}^d} (\frac{|(1-\lambda)\bar m \bar v + \lambda m v|^2}{2((1-\lambda)\bar m + \lambda m)} - |\bar v|^2 \bar m) dxdt \\ &\quad + \int_0^T \int_0^1 \int_{\mathbb{R}^d} F(x, (1-s)\bar m(t) + s m_\lambda(t))(m_\lambda - \bar m)(t,x) dxdsdt \\ &+ \int_0^1 \int_{\mathbb{R}^d} G(x, (1-s)\bar m(T) + s m_\lambda(T))(m_\lambda - \bar m)(T,x) dxds \end{aligned}$$

Note that m_λ tends to $\bar m$ as $\lambda \to 0^+$. So, dividing by λ and letting $\lambda \to 0^+$ gives

$$\begin{aligned} 0 \le & \int_0^T \int_{\mathbb{R}^d} ((v - \bar v) \cdot \bar v \bar m + \frac{1}{2}|\bar v|^2 (m - \bar m)) dxdt \\ & + \int_0^T \int_{\mathbb{R}^d} F(x, \bar m(t))(m - \bar m)(t,x) dxdsdt + \int_0^1 \int_{\mathbb{R}^d} G(x, \bar m(T))(m - \bar m)(T,x) dxds. \end{aligned} \tag{4.25}$$

Let u be the solution to the backward equation

$$\begin{cases} -\partial u - \Delta u - v \cdot Du = F(x, \bar m(t)) + \frac{1}{2}|\bar v(t,x)|^2 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(T,x) = G(x, \bar m(T)) & \text{in } \mathbb{R}^d. \end{cases} \tag{4.26}$$

Plugging this relation into (4.25), we have

$$\begin{aligned} 0 \le & \int_0^T \int_{\mathbb{R}^d} ((v - \bar v) \cdot \bar v \bar m + (-\partial u - \Delta u - v \cdot Du)(m - \bar m)) dxdt \\ & + \int_0^1 \int_{\mathbb{R}^d} G(x, \bar m(T))(m - \bar m)(T,x) dxds \\ = & \int_0^T \int_{\mathbb{R}^d} ((v - \bar v) \cdot \bar v \bar m + u(\partial_t (m - \bar m) - \Delta(m - \bar m)) - v \cdot Du(m - \bar m)) dxdt \\ = & \int_0^T \int_{\mathbb{R}^d} ((v - \bar v) \cdot \bar v \bar m + Du \cdot (mv - \bar m \bar v) - v \cdot Du(m - \bar m)) dxdt \\ = & \int_0^T \int_{\mathbb{R}^d} (v - \bar v) \cdot \bar v \bar m + \bar m Du \cdot (v - \bar v) dxdt. \end{aligned}$$

Now we choose $v = \bar v + hz$ for $h > 0$ and z smooth and let u_h be the associated solution to the HJ equation (4.26). As $h \to 0^+$, u_h converges to $\bar u$ solution to

$$\begin{cases} -\partial \bar u - \Delta \bar u - \bar v \cdot D\bar u = F(x, \bar m(t)) + \frac{1}{2}|\bar v(t,x)|^2 & \text{in } (0,T) \times \mathbb{R}^d, \\ \bar u(T,x) = G(x, \bar m(T)) & \text{in } \mathbb{R}^d. \end{cases}$$

Then

$$0 = \int_0^T \int_{\mathbb{R}^d} z \cdot \bar v \bar m + \bar m D\bar u \cdot z dxdt.$$

As z is arbitrary, (meaning that, for any $h > 0$, there exists a unique classical solution to (4.26) associated with the drift $v = \bar{v} + hz$) and $\bar{m} > 0$, we obtain that $D\bar{u} = -\bar{v}$. This proves that $\bar{u}$ actually solves

$$\begin{cases} -\partial \bar{u} - \Delta \bar{u} + \frac{1}{2}|D\bar{u}(t,x)|^2 = F(x, \bar{m}(t)) & \text{in } (0,T)\times\mathbb{R}^d, \\ \bar{u}(T,x) = G(x, \bar{m}(T)) & \text{in } \mathbb{R}^d. \end{cases}$$

and $\bar{m}$ solves

$$\partial_t \bar{m} - \Delta \bar{m} - \operatorname{div}(mD\bar{u}) = 0 \text{ in } (0,T)\times\mathbb{R}^d, \qquad m(0,x) = m_0(x).$$

So the pair $(\bar{u}, \bar{m})$ is a solution to (4.13). □

4.2.5 Application to games with finitely many players

Before starting the discussion of games with a large number of players, let us fix a solution (u, m) of the mean field equation (4.13) and investigate the optimal strategy of a generic player who considers the density m "of the other players" as given. He faces the following minimization problem

$$\inf_{\alpha} \mathcal{J}(\alpha) \qquad \text{where} \qquad \mathcal{J}(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_s|^2 + F\left(X_s, m(s)\right)\, ds + G\left(X_T, m(T)\right)\right] .$$

In the above formula, $X_t = X_0 + \int_0^t \alpha_s ds + \sqrt{2}B_s$, X_0 is a fixed random intial condition with law m_0 and the control α is adapted to some filtration $(\mathcal{F}_t)$. We assume that (B_t) is an $d-$dimensional Brownian motion adapted to $(\mathcal{F}_t)$ and that X_0 and (B_t) are independent. We claim that the feedback strategy $\bar{\alpha}(x,t) := -Du(x,t)$ is optimal for this optimal stochastic control problem.

Lemma 4.2.8. *Let $(\bar{X}_t)$ be the solution of the stochastic differential equation*

$$\begin{cases} d\bar{X}_t = \bar{\alpha}(\bar{X}_t, t)dt + \sqrt{2}dB_t \\ \bar{X}_0 = X_0 \end{cases}$$

and $\tilde{\alpha}(t) = \bar{\alpha}(X_t, t)$. Then

$$\inf_{\alpha} \mathcal{J}(\alpha) = \mathcal{J}(\tilde{\alpha}) = \int_{\mathbb{R}^N} u(x,0)\ dm_0(x)\ .$$

Proof : This kind of result is known as a verification Theorem: one has a good candidate for an optimal control, and one checks, using the equation satisfied by the value function u, that this is indeed the minimum. Let α be an adapted control. We have, from Itô's formula,

$$\begin{aligned} &\mathbb{E}[G(X_T, m(T))] = \mathbb{E}[u(X_T, T)] \\ &\quad = \mathbb{E}\left[u(X_0,0) + \int_0^T (\partial_t u(X_s,s) + \langle \alpha_s, Du(X_s,s)\rangle + \Delta u(X_s,s))\ ds\right] \\ &\quad = \mathbb{E}\left[u(X_0,0) + \int_0^T (\frac{1}{2}|Du(X_s,s)|^2 + \langle \alpha_s, Du(X_s,s)\rangle - F(X_s, m(s)))\ ds\right] \\ &\quad \geq \mathbb{E}\left[u(X_0,0) + \int_0^T (-\frac{1}{2}|\alpha_s|^2 - F(X_s, m(s)))\ ds\right] \end{aligned}$$

This shows that $\mathbb{E}\left[u(X_0, 0)\right] \leq \mathcal{J}(\alpha)$ for any adapted control α. If we replace α by $\tilde{\alpha}$ in the above computations, then, since the process (X_t) becomes $(\bar{X}_t)$, the above inequalities are all equalities. So $\mathbb{E}\left[u(X_0, 0)\right] = \mathcal{J}(\bar{\alpha})$ and the result is proved. □

We now consider a differential game with N players which consists in a kind of discrete version of the mean field game. In this game player i ($i = 1, \dots, N$) is controlling through his control α^i a dynamics of the form

$$dX_t^i = \alpha_t^i dt + \sqrt{2}dB_t^i \tag{4.27}$$

where (B^i_t) is a $d-$dimensional brownian motion. The initial condition X^i_0 for this system is also random and has for law m_0. We assume that the all X^i_0 and all the brownian motions (B^i_t) $(i=1,\dots,N)$ are independent. However player i can choose his control α^i adapted to the filtration $(\mathcal{F}_t=\sigma(X^j_0,\ B^j_s,\ s\leq t,\ j=1,\dots,N\})$. His payoff is then given by

$$\begin{aligned}&\mathcal{J}^N_i(\alpha^1,\dots,\alpha^N)\\ &\quad=\mathbb{E}\left[\int_0^T\frac{1}{2}|\alpha^i_s|^2+F\left(X^i_s,\frac{1}{N-1}\sum_{j\neq i}\delta_{X^j_s}\right)\,ds+G\left(X^i_T,\frac{1}{N-1}\sum_{j\neq i}\delta_{X^j_T}\right)\right]\end{aligned}$$

Our aim is to explain that the strategy given by the mean field game is suitable for this problem. More precisely, let (u,m) be one classical solution to (4.13) and let us set $\bar{\alpha}(x,t)=-Du(x,t)$. With the closed loop strategy $\bar{\alpha}$ one can associate the open-loop control $\tilde{\alpha}^i$ obtained by solving the SDE

$$d\bar{X}^i_t=\bar{\alpha}(\bar{X}^i_t,t)dt+\sqrt{2}dB^i_t \tag{4.28}$$

with random initial condition X^i_0 and setting $\tilde{\alpha}^i_t=\bar{\alpha}(\bar{X}^i_t,t)$. Note that this control is just adapted to the filtration $(\mathcal{F}^i_t=\sigma(X^i_0,\ B^i_s,\ s\leq t\})$, and not to the full filtration $(\mathcal{F}_t)$ defined above.

Theorem 4.2.9. *For any $\epsilon>0$, there is some N_0 such that, if $N\geq N_0$, then the symmetric strategy $(\tilde{\alpha}^1,\dots,\tilde{\alpha}^N)$ is an $\epsilon-$Nash equilibrium in the game $\mathcal{J}^N_1,\dots,\mathcal{J}^N_N$: Namely*

$$\mathcal{J}^N_i(\tilde{\alpha}^1,\dots,\tilde{\alpha}^N)\leq\mathcal{J}^N_i((\tilde{\alpha}^j)_{j\neq i},\alpha)+\epsilon$$

for any control α adapted to the filtration $(\mathcal{F}_t)$ and any $i\in\{1,\dots,N\}$.

Remark 4.2.10. This result is very close to one-shot games and its proof is mainly based on the stability property of the mean field equation. In some sense it is rather "cheap": the difficult question is in what extend Nash equilibria for differential games in *feedback strategies* give rise to a mean field equation. This question is for a large extent open.

Proof : Fix $\epsilon>0$. Since the problem is symmetrical, it is enough to show that

$$\mathcal{J}^N_1(\tilde{\alpha}^1,\dots,\tilde{\alpha}^N)\leq\mathcal{J}^N_1((\tilde{\alpha}^j)_{j\neq 1},\alpha)+\epsilon \tag{4.29}$$

for any control α, as soon as N is large enough. Let us denote by $\bar{X}^j_t$ the solution of the stochastic differential equation (4.28) with initial condition X^j_0. We note that the $(\bar{X}^j_t)$ are independent and identically distributed with law $m(t)$ (the law comes from Lemma 4.2.3). Therefore, using (as in subsection 2.4) the law of large numbers 3.2.5, there is some N_0 such that, if $N\geq N_0$,

$$\mathbb{E}\left[\sup_{|y|\leq 1/\sqrt{\epsilon}}\left|F\left(y,\frac{1}{N-1}\sum_{j\geq 2}\delta_{\bar{X}^j_s}\right)-F(y,m(t))\right|\right]\leq\epsilon \tag{4.30}$$

for any $t\in[0,T]$ and

$$\mathbb{E}\left[\sup_{|y|\leq 1/\sqrt{\epsilon}}\left|G\left(y,\frac{1}{N-1}\sum_{j\geq 2}\delta_{\bar{X}^j_T}\right)-G(y,m(T))\right|\right]\leq\epsilon\ . \tag{4.31}$$

For the first inequality, one can indeed choose N_0 independent of t because, F being C_0-Lipschitz continuous with respect to m, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{|y|\leq\sqrt{\epsilon}}\left|F\left(y,\frac{1}{N-1}\sum_{j\geq 2}\delta_{\bar{X}^j_t}\right)-F\left(y,\frac{1}{N-1}\sum_{j\geq 2}\delta_{\bar{X}^j_s}\right)\right|\right]\\ &\qquad\leq\ \mathbb{E}\left[C_0\mathbf{d}_1\left(\frac{1}{N-1}\sum_{j\geq 2}\delta_{\bar{X}^j_t},\frac{1}{N-1}\sum_{j\geq 2}\delta_{\bar{X}^j_s}\right)\right]\\ &\qquad\leq\ \frac{1}{N-1}\sum_{j\geq 2}\mathbb{E}\left[\left|\bar{X}^j_t-\bar{X}^j_s\right|\right]\leq c_0(1+\|\bar{\alpha}\|_\infty)(t-s)^{1/2}\ ,\end{aligned}$$

where the last inequality easily comes from computations similar to that for Lemma 4.2.4.

Let now α be a control adapted to the filtration $(\mathcal{F}_t)$ and X_t be the solution to

$$dX_t = \alpha_t dt + \sqrt{2} dB_t^1$$

with random initial condition X_0^1. Let us set $K = 2(T\|F\|_\infty + \|G\|_\infty) + \mathbb{E}[\int_0^T \frac{1}{2}|\bar{\alpha}_s^1|^2 ds]$. Note that, if $\mathbb{E}[\int_0^T \frac{1}{2}|\alpha_s|^2 ds] \geq K$, then (4.29) holds.

Let us now assume that $\mathbb{E}[\int_0^T \frac{1}{2}|\alpha_s|^2 ds] \leq K$. We first estimate $\mathbb{E}[\sup_{t\in[0,T]} |X_t|^2]$:

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] \leq 2\mathbb{E}\left[|X_0^1|^2 + \int_0^T \frac{1}{2}|\alpha_s|^2 ds + 2 \sup_{t\in[0,T]} |B_t|^2\right] \leq 2\mathbb{E}[|X_0^1|^2] + 2K + 4T$$

where the last estimates comes from the Burkholder-Davis-Gundy inequality. Denoting by K_1 the right-hand side of the above inequality we obtain therefore that

$$\mathbb{P}\left[\sup_{t\in[0,T]} |X_t| \geq 1/\sqrt{\epsilon}\right] \leq K_1 \epsilon\ .$$

Let us now fix $N \geq N_0$ and estimate $\mathcal{J}_1^N((\tilde{\alpha}^j)_{j\neq 1}, \alpha)$ by separating the expectation for the F and G terms according to the fact that $\sup_{t\in[0,T]} |X_t| \geq 1/\sqrt{\epsilon}$ or not. Taking into account (4.30) and (4.31) we have

$$\begin{aligned}
\mathcal{J}_1^N((\tilde{\alpha}^j)_{j\neq 2}, \alpha) \ \geq\ & \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_s|^2 + F\left(X_s^i, m(t)\right) \ + G\left(X_T^i, m(T)\right)\right] - (1+T)\epsilon \\
& -2\mathbb{P}\left[\sup_{t\in[0,T]} |X_t| \geq 1/\sqrt{\epsilon}\right] (T\|F\|_\infty + \|G\|_\infty) \\
\geq\ & \mathcal{J}_1^N((\tilde{\alpha}^j)_{j\neq 1}) - C\epsilon
\end{aligned}$$

for some constant C independent of N and α, where the last inequality comes from the optimality of $\bar{\alpha}$ in Lemma 4.2.8. □

4.3 Comments

Existence: Existence of solutions for the MFG system can be achieved either by Banach fixed point Theorem (as in the papers by Caines, Huang and Malhamé [20], under a smallness assumption on the coefficients or on the time interval) or by Schauder arguments (we used this argument in Theorem 4.2.1), as in Lasry and Lions [24, 25]). For MFG systems with local coupling functions, i.e., when $F = F(x, m(t,x))$, one can also use variational methods (see Subsection 4.2.4) similar to the techniques for optimal transport problems (see [25]).

Uniqueness: Concerning the uniqueness of the solution, one can distinguish two kinds of regimes. Of course the Banach fixed point argument provides directly uniqueness of the solution of the MFG system. However, as explained above, it mostly concerns local in time results. For the large time uniqueness, one can rely on the monotonicity conditions (4.21) and (4.22). These conditions first appear in [24, 25].

Nash equilibria for the $N-$player games: the use of the MFG system to obtain $\epsilon-$*Nash equilibria* (Theorem 4.2.9) has been initiated—in a slightly different framework—in a series of papers due to Caines, Huang and Malhamé: see in particular [17] (for linear dynamics) and [20] (for nonlinear dynamics). In these papers, the dependence with respect of the empirical measure of dynamics and payoff occurs through an average, so that the CTL implies that the error term is a order $N^{-1/2}$ (instead of $N^{-1/(d+4)}$ as in Theorem 4.2.9). The genuinely non linear version of the result given above is a variation on a result by Carmona and Delarue [7].

The reverse statement, namely in what extend the MFG system pops up as the limit of Nash equilibria, is much more difficult and investigated in [5].

Extensions: it is difficult to discuss all the extensions of the MFG systems since the number of papers on this subject has grown exponentially in the last years. We give here only a brief overview.

The ergodic MFG system has been introduced by Lasry and Lions in [23] as the limit, when the number of players tends to infinity, of Nash equilibria in ergodic differential games. As explained in Lions [26], this system also pops up as the limit, as the horizon tends to infinity, of the finite horizon MFG system.

The natural issue of boundary conditions has not been thoroughly investigated up to now. For the PDE approach, the authors have mostly worked with periodic data (as we did above), which completely eliminates this question. In the "probabilistic literature" (as in the work by Caines, Huang and Malhamé), the natural set-up is the full space. Beside these two extreme cases, little has been written (see however Cirant [10], for Neumann boundary condition in ergodic multi-population MFG systems).

The interesting MFG systems with several populations were introduced in the early paper by Caines, Huang and Malhamé [20] and revisited by Cirant [10] (for Neumann boundary conditions) and by Kolokoltsov, Li and Yang [21] (for very general diffusions, possibly with jumps).

A very general MFG model (the so-called extended MFG) for a single population is described in Gomes, Patrizi and Voskanyan [15] and Gomes and Voskanyan[16]. There the velocity of the population is a nonlocal function of the (repartition of) actions of the players. See also Chapter I.4.6 of the monograph by Carmona and Delarue [8]. The example in Subsection 4.1.2 is also of this form and borrowed from the paper [6].

Chapter 5

The master equation

In this chapter we consider a mean field game problem and investigate in what extend the optimal feedback for the players depends on the time, the current position of the player *and the current distribution* of the other players. Compared to the standard MFG system (where the action of a player depends on time and position only), having such a feedback would yield a more robust strategy: it would allow to take into account errors in the computation of the mean field.

This question can be answered by the "master equation", which subsume in a single function the two unknowns of the MFG system. The price to pay, however, is the fact that this equation is in infinite dimension: Actually it lives in the space of measures. Let us also point out that the master equation allows to handle MFG problems with common noise; it is also the good object to understand the limit, as the number of players tends to infinity, of Nash equilibria in differential games. Unfortunately, these two last points largely exceed the scope of these notes and won't be discussed.

As the master equation is an equation in the space of measures (which is not a vector space), one has first to understand the notion of derivative in this space. Then one will discuss some aspects of the master equation for classical MFG system (without common noise). We briefly present at the end a simple example of common noise, to stress out the difference with the classical setting.

5.1 Derivatives in the space of measures

5.1.1 Derivatives in the L^2 sense

Here we work in $\mathbb{R}^d$ and consider the space $\mathcal{P}_2$ of Borel probability measures with finite second order moment, endowed with the Wasserstein distance $\mathbf{d}_2$.

Definition 5.1.1. *We say that $U : \mathcal{P}_2 \to \mathbb{R}^k$ is C^1 in the L^2 sense if there exists a bounded continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}_2 \times \mathbb{T}^d \to \mathbb{R}^k$ such that, for any $m, m' \in \mathcal{P}_2$,*

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y)\ d(m' - m)(y).$$

We say that $\delta U/\delta m$ is the L^2-derivative of U.

Remark 5.1.2. 1. Another way to write the above relation is to require that, for any $m, m' \in \mathcal{P}_2$,

$$\lim_{t\to 0^+} \frac{U((1-t)m + tm') - U(m)}{t} = \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, y)\ d(m' - m)(y).$$

2. We have required $\delta U/\delta m$ to be bounded for simplicity. This restriction could be relaxed: as we work in $\mathcal{P}_2$, we could ask to $\delta U/\delta m$ to have at most a quadratic growth at infinity in the space variable.

3. Note that $\frac{\delta U}{\delta m}$ is defined only up to an additive constant. To fix the idea, it is often convenient to assume that
$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m,y)m(dy) = 0 \qquad \forall m \in \mathcal{P}_2.$$

5.1.2 The intrinsic derivative

Definition 5.1.3. *We say that $U : \mathcal{P}_2 \to \mathbb{R}^k$ is C^1 in the intrinsic sense U is C^1 in the L^2 sense and if $\delta U/\delta m$ is differentiable with respect to the space variable with $D_y \delta U/\delta m$ continuous and bounded on $\mathbb{R}^d \times \mathcal{P}_2$.*

Then we define the intrinsic derivative $D_m U : \mathcal{P}_2 \times \mathbb{R}^d \to \mathbb{R}^d$ as
$$D_m U(m,y) := D_y \frac{\delta U}{\delta m}(m,y).$$

There are several ways to understand the intrinsic derivative. The first one is through displacements in the space $\mathcal{P}_2$. Let us recall that, if $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable and m is a Borel probability measure on $\mathbb{R}^d$, then the image of m by Φ is the probability measure $\Phi\sharp m$ defined by
$$\int_{\mathbb{R}^d} \phi(x)\Phi\sharp m(dx) := \int_{\mathbb{R}^d} \phi(\Phi(x))m(dx) \qquad \forall \phi \in C^0_b(\mathbb{R}^d).$$
If Φ has at most a linear growth at infinity and $m \in \mathcal{P}_2$, then $\Phi\sharp m$ is also in $\mathcal{P}_2$.

Proposition 5.1.4. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be Borel measurable with at most a linear growth at infinity and assume that U is C^1 in the intrinsic sense. Then*
$$\lim_{h\to 0^+} \frac{U((id + h\Phi)\sharp m) - U(m)}{h} = \int_{\mathbb{R}^d} D_m U(m,y)\cdot\Phi(y)\ m(dy) \qquad \forall m \in \mathcal{P}_2.$$

Proof. We have, setting $m^h_t := (1-t)m + t(id + h\Phi)\sharp m$,
$$\begin{aligned} U((id + h\Phi)\sharp m) - U(m) &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m^h_t, y)((id + h\Phi)\sharp m - m)(dy)dt \\ &= \int_0^1 \int_{\mathbb{R}^d} (\frac{\delta U}{\delta m}(m^h_t, y + h\Phi(y)) - \frac{\delta U}{\delta m}(m^h_t, y))\ m(dy)dt \\ &= h\int_0^1\int_0^1 \int_{\mathbb{R}^d} D_y\frac{\delta U}{\delta m}(m^h_t, y + sh\Phi(y))\cdot\Phi(y)\ m(dy)dtds. \end{aligned}$$
As $h \to 0^+$, $m^h_t \to m$ in $\mathcal{P}_2$ and $y + sh\Phi(y) \to y$. Using the continuity of $D_y\delta U/\delta m = D_m U$ and dominated convergence theorem gives the result. □

Next we investigate another interpretation of the intrinsic derivative. It is the link with function of many variable. Let $U : \mathcal{P}_2 \to \mathbb{R}$ and, for a (large) integer N, let us introduce $U^N : (\mathbb{R}^d)^N \to \mathbb{R}$ defined by
$$U^N(\mathbf{x}) = U(m^N_{\mathbf{x}}) \qquad \forall \mathbf{x} = (x_1,\dots,x_N) \in (\mathbb{R}^d)^N,\ m^N_{\mathbf{x}} := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}.$$
The map U^N is nothing but the restriction of U to the space of empirical measures of size N.

Proposition 5.1.5. *Assume that U is C^1 in the intrinsic sense. Then U^N is C^1 on $(\mathbb{R}^d)^N$ and*
$$D_{x_i} U^N(\mathbf{x}) = \frac{1}{N} D_m U(m^N_{\mathbf{x}}, x_i) \qquad \forall \mathbf{x} = (x_1,\dots,x_N) \in (\mathbb{R}^d)^N.$$

Proof. We have, for any $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$,

$$
\begin{aligned}
U^N(\mathbf{y}) - U^N(\mathbf{x}) = U(m^N_{\mathbf{y}}) - U(m^N_{\mathbf{x}}) &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-t)m^N_{\mathbf{x}} + tm^N_{\mathbf{y}}, z)(m^N_{\mathbf{y}} - m^N_{\mathbf{x}})(dz)dt \\
&= \frac{1}{N}\sum_{i=1}^N \int_0^1 (\frac{\delta U}{\delta m}((1-t)m^N_{\mathbf{x}} + tm^N_{\mathbf{y}}, y_i) - \frac{\delta U}{\delta m}((1-t)m^N_{\mathbf{x}} + tm^N_{\mathbf{y}}, x_i))dt \\
&= \frac{1}{N}\sum_{i=1}^N \int_0^1 \int_0^1 D_m U((1-t)m^N_{\mathbf{x}} + tm^N_{\mathbf{y}}, (1-s)x_i + sy_i) \cdot (y_i - x_i)dtds \\
&= \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^d} D_m U(m^N_{\mathbf{x}}, x_i) \cdot (y_i - x_i) + R(\mathbf{y}),
\end{aligned}
$$

where

$$
\begin{aligned}
|R(\mathbf{y})| &= |\frac{1}{N}\sum_{i=1}^N \int_0^1 \int_0^1 (D_m U((1-t)m^N_{\mathbf{x}} + tm^N_{\mathbf{y}}, (1-s)x_i + sy_i) - D_m U(m^N_{\mathbf{x}}, x_i)) \cdot (y_i - x_i)dtds| \\
&\le \frac{1}{N}\sum_{i=1}^N \int_0^1 \int_0^1 |D_m U((1-t)m^N_{\mathbf{x}} + tm^N_{\mathbf{y}}, (1-s)x_i + sy_i) - D_m U(m^N_{\mathbf{x}}, x_i)||y_i - x_i|dtds.
\end{aligned}
$$

As $D_m U$ is continuous, we can find, for any $\epsilon > 0$, some $\eta > 0$ such that

$$
|D_m U(m, z) - D_m U(m^N_{\mathbf{x}}, x_i)| \le \epsilon, \qquad \forall |z - x_i| \le \eta,\ \mathbf{d}_2(m, m^N_{\mathbf{x}}) \le \eta,\ i = 1, \dots, N.
$$

So, if $|\mathbf{y} - \mathbf{x}| := \max_i |y_i - x_i| \le \eta$, we obtain $\mathbf{d}_2(m^N_{\mathbf{y}}, m^N_{\mathbf{x}}) \le \eta$ and $|R(\mathbf{y})| \le |\mathbf{y} - \mathbf{x}|\epsilon$. This proves that U^N is differentiable with $D_{x_i} U^N(\mathbf{x}) = D_m U(m^N_{\mathbf{x}}, x_i)$. This latter map being continuous, U^N is also C^1. □

The intrinsic derivative allows to quantify the Lipschitz regularity of U.

Proposition 5.1.6. *Assume that U is C^1 in the intrinsic sense and that there exists $C_0 > 0$ such that*

$$
\int_{\mathbb{R}^d} |D_m U(m, y)|^2 \, m(dy) \le C_0^2 \qquad \forall m \in \mathcal{P}_2.
$$

Then U is C_0-Lipschitz continuous.

Proof. By density, we just need to prove the result for empirical densities. Let N be a large integer, $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$ and let us define U^N as in Proposition 5.1.5. From Lemma 5.1.7 below, there exists a permutation σ on $\{1, \dots, N\}$ such that

$$
\mathbf{d}_2(m^N_{\mathbf{y}}, m^N_{\mathbf{x}}) = \left(\frac{1}{N}\sum_{i=1}^N |y_{\sigma(i)} - x_i|^2 \right)^{1/2}
$$

and we set $\tilde{\mathbf{y}} = (y_{\sigma(1)}, \dots, y_{\sigma(N)})$. Let now $\mathbf{z}_t = (z_{1,t}, \dots, z_{N,t}) = (1-t)\mathbf{x} + t\tilde{\mathbf{y}}$. We have, by Proposition 5.1.5,

$$
\begin{aligned}
\left| \frac{d}{dt} U^N(\mathbf{z}_t) \right| = \left| \nabla U^N(\mathbf{z}_t) \cdot (\tilde{\mathbf{y}} - \mathbf{x}) \right| &= \left| \frac{1}{N}\sum_{i=1}^N D_m U(m^N_{\mathbf{z}_t}, z_{i,t}) \cdot (\tilde{y}_i - x_i) \right| \\
&\le \left(\frac{1}{N}\sum_{i=1}^N |D_m U(m^N_{\mathbf{z}_t}, z_{i,t})|^2 \right)^{1/2} \left(\frac{1}{N}\sum_{i=1}^N |\tilde{y}_i - x_i|^2 \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^d} |D_m U(m^N_{\mathbf{z}_t}, z)|^2 m^N_{\mathbf{z}_t}(dz) \right)^{1/2} \mathbf{d}_2(m^N_{\mathbf{y}}, m^N_{\mathbf{x}}) \ \le\ C_0 \mathbf{d}_2(m^N_{\mathbf{y}}, m^N_{\mathbf{x}}).
\end{aligned}
$$

Therefore

$$|U(m_{\mathbf{y}}^N) - U(m_{\mathbf{x}}^N)| = |U^N(\mathbf{y}) - U^N(\mathbf{x})| \le \int_0^1 |\frac{d}{dt}U^N(\mathbf{z}_t)|dt \le C_0 \mathbf{d}_2(m_{\mathbf{y}}^N, m_{\mathbf{x}}^N),$$

which proves the Lipschitz continuity of U with constant C_0. □

Lemma 5.1.7. *Let N be a positive integer, $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$ and let $m_{\mathbf{x}}^N$ and $m_{\mathbf{y}}^N$ be the associated empirical measures. Then there exists a permutation σ on $\{1, \dots, N\}$ such that*

$$\mathbf{d}_2(m_{\mathbf{y}}^N, m_{\mathbf{x}}^N) = \left(\frac{1}{N} \sum_{i=1}^N |y_{\sigma(i)} - x_i|^2 \right)^{1/2}.$$

Proof. Let π be a transport plan from $m_{\mathbf{x}}^N$ to $m_{\mathbf{y}}^N$. Note that π is simply characterized by the π_{ij} where $\pi_{ij} := \pi(x_i, y_j)$. The (π_{ij}) are nonnegative and satisfy

$$\sum_{k=1}^N \pi_{ik} = \sum_{k=1}^N \pi_{kj} = \frac{1}{N}.$$

This means that $(N\pi_{ij})$ is a bi-stochastic matrix. Conversely, given a bi-stochastic matrix (a_{ij}), the plan

$$\pi := \frac{1}{N} \sum_{i,j=1}^N a_{ij} \delta_{(x_i, y_j)}$$

is a transport plan from $m_{\mathbf{x}}^N$ to $m_{\mathbf{y}}^N$.

We denote by $\mathcal{B}_N$ the set of such matrices. Let us recall that this is a compact convex set and that the Birkhoff-von Neumann theorem states that its extremal points are the permutation matrices, i.e., the matrices in $\mathcal{B}_N$ with entries in $\{0, 1\}$. Then

$$\mathbf{d}_2^2(m_{\mathbf{x}}^N, m_{\mathbf{y}}^N) = \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) = \inf_{(a_{ij}) \in \mathcal{B}_N} \frac{1}{N} \sum_{i,j=1}^N a_{ij} |x_i - y_j|^2,$$

which is a linear optimization problem. Therefore the minimum is reached at an extremal point of $\mathcal{B}_N$ and thus at a permutation matrix. □

A last way to understand the intrinsic derivative is through the lifting of the map to a space of random variables. More precisely, let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ sufficiently rich (a "standard probability space"). Note that $L^2(\Omega)$ is then a Hilbert space. Given a map $U : \mathcal{P}_2 \to \mathbb{R}$, we define its lifting $\tilde{U} : L^2(\Omega) \to \mathbb{R}$ by

$$\tilde{U}(X) := U([X]) \qquad \forall X \in L^2(\Omega),$$

where $[X]$ denotes the law of X.

Proposition 5.1.8. *Assume that U is C^1 in the intrinsic sense. Then $\tilde{U}$ is C^1 (i.e., Fréchet differentiable with a continuous Fréchet derivative) in $L^2(\Omega)$ with*

$$\nabla \tilde{U}(X) = D_m U([X], X) \qquad \forall X \in L^2(\Omega).$$

Conversely, if the map $\tilde{U}$ is C^1 in $L^2(\Omega)$, then U is C^1 in the intrinsic sense.

(Partial) proof of Proposition 5.1.8. We only prove first statement. The converse is more difficult and exceeds the scope of these notes. See, for instance, the presentation in Carmona and Delarue's monograph [8].

Let $X, Y \in L^2(\Omega)$. Then

$$\begin{aligned}\tilde U(Y) - \tilde U(X) = U([Y]) - U([X]) &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-t)[X]+t[Y],x)([Y]-[X])(dx)dt \\ &= \int_0^1 \mathbb{E}\left[\frac{\delta U}{\delta m}((1-t)[X]+t[Y],Y) - \frac{\delta U}{\delta m}((1-t)[X]+t[Y],X)\right] dt \\ &= \int_0^1\int_0^1 \mathbb{E}\left[D_mU((1-t)[X]+t[Y],(1-s)X+sY)\cdot(Y-X)\right]dsdt \\ &= \mathbb{E}\left[D_mU([X],X)\cdot(Y-X)\right] + R(Y),\end{aligned}$$

where

$$\begin{aligned}|R(Y)| &= \left|\int_0^1\int_0^1 \mathbb{E}\left[(D_mU((1-t)[X]+t[Y],(1-s)X+sY) - D_mU([X],X))\cdot(Y-X)\right]dsdt\right| \\ &\le \int_0^1\int_0^1 \mathbb{E}^{1/2}\left[|D_mU((1-t)[X]+t[Y],(1-s)X+sY) - D_mU([X],X)|^2\right]\mathbb{E}^{1/2}\left[|Y-X|^2\right]dsdt.\end{aligned}$$

By the dominated convergence theorem the term

$$\int_0^1\int_0^1 \mathbb{E}^{1/2}\left[|D_mU((1-t)[X]+t[Y],(1-s)X+sY) - D_mU([X],X)|^2\right]dsdt$$

tends to 0 as $Y \to X$ in $L^2(\Omega)$, which proves the Fréchet differentiability of $\tilde U$ at X. The continuity of the derivative is then straightforward. □

We complete this section by an Itô's formula. We consider an Itô process of the form

$$dX_t = b_t dt + \sigma_t dB_t,$$

where B is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, with associated filtration $(\mathcal{F}_t)$, (b_t) and (σ_t) are progressively measurable processes, such that, for any $T > 0$,

$$\mathbb{E}\left[\int_0^T |b_s|^2 + |\sigma_s|^2 \; ds\right] < +\infty.$$

Under the above assumption, the map $t \to [X_t]$ is continuous in $\mathcal{P}_2$ because

$$\begin{aligned}\mathbf{d}_2^2([X_s],[X_t]) \le \mathbb{E}[|X_t - X_s|^2] \le \mathbb{E}\left[\left|\int_s^t b_r dr + \sigma_r dB_r\right|^2\right] &\le 2\mathbb{E}\left[\left|\int_s^t b_r dr\right|^2\right] + 2\mathbb{E}\left[\left|\int_s^t \sigma_r dB_r\right|^2\right] \\ &\le 2(t-s)\mathbb{E}\left[\int_s^t |b_r|^2 dr\right] + 2\mathbb{E}\left[\int_s^t |\sigma_r|^2 dr\right].\end{aligned}$$

Proposition 5.1.9. *Assume that $U : \mathcal{P}_2 \to \mathbb{R}$ is C^1 in the intrinsic sense and assume that D_yD_mU exists and is continuous and bounded. Then*

$$U([X_t]) = U([X_0]) + \int_0^t \mathbb{E}\left[D_mU([X_s],X_s)\cdot b_s + \frac{1}{2}\mathrm{Tr}(\sigma_s\sigma_s^* D_yD_mU([X_s],X_s))\right]ds.$$

Remark 5.1.10. The formula can be easily generalized to a time dependent map $U = U(t,m)$. In this case the partial derivative $\partial_t U$ also appears in the integral.

Proof. Fix $s, t > 0$ and set $m^{s,t}(r) := (1-r)[X_s] + r[X_t]$ for $r \in [0,1]$. Then

$$\begin{aligned}U([X_t]) &= U([X_s]) + \int_0^1 \frac{\delta U}{\delta m}(m^{s,t}(r),x)([X_t]-[X_s])(dx)dr \\ &= U([X_s]) + \int_0^1 \mathbb{E}\left[\frac{\delta U}{\delta m}(m^{s,t}(r),X_t) - \frac{\delta U}{\delta m}(m^{s,t}(r),X_s))(dx)\right]dr.\end{aligned}$$

As $\delta U/\delta m$ is C^2 with bounded derivatives in the space variable, we obtain by Itô's formula:

$$U([X_t]) = U([X_s]) + \int_0^1 \int_s^t \mathbb{E}\left[D_m U(m^{s,t}(r), X_l) + \frac{1}{2}\mathrm{Tr}(\sigma_l \sigma_l^* D_y D_m U(m^{s,t}(r), X_l))\right] dl dr.$$

To obtain the final expression, we decompose the interval $[0,t]$ through a grid $t_k = kt/n$ (where n is large and $k \in \{0, \dots, n\}$) and write

$$U([X_t]) = U([X_0]) + \sum_{k=0}^{n-1} \int_0^1 \int_{t_k}^{t_{k+1}} \mathbb{E}\left[D_m U(m^k(r), X_l) + \frac{1}{2}\mathrm{Tr}(\sigma_l \sigma_l^* D_y D_m U(m^k(r), X_l))\right] dl dr,$$

where we have set $m_r^k := m_r^{t_k, t_{k+1}}$ to simplify the expression. As $n \to +\infty$ and $t_k \to l$, $m_r^{t_k,t_{k+1}}$ converges to $[X_l]$ uniformly on $r \in [0,1]$, so that

$$U([X_t]) = U([X_0]) + \int_0^t \mathbb{E}\left[D_m U([X_l], X_l) + \frac{1}{2}\mathrm{Tr}(\sigma_l \sigma_l^* D_y D_m U([X_l], X_l))\right] dl.$$

□

5.2 The first order Master equation

5.2.1 The MFG problem

To fix the ideas (and the notations), we consider here the MFG system of Section 4.2. We consider the second order MFG system with a quadratic Hamiltonian:

$$\text{(5.1)} \qquad \left\{ \begin{array}{lll} (i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x,m) & \text{in } (0,T) \times \mathbb{R}^d \\ (ii) & \partial_t m - \Delta m - \mathrm{div}\,(m\ Du) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ (iii) & m(0) = m_0\ ,\ u(x,T) = G(x, m(T)) & \text{in } \mathbb{R}^d \end{array} \right.$$

Let us recall that this system corresponds to a problem in which each agent has a dynamic of the form

$$dX_t = \alpha_t dt + \sqrt{2} dB_t$$

(where (α_t) is the control with values in $\mathbb{R}^d$ and B a Brownian motion) and a cost of the form

$$J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t|^2 + F(X_t, m(t))\ dt + G(X_T, m(T))\right].$$

As we explain below, the master equation associated with the above MFG problem is the following equation, where $U : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}$ is the unknown:

$$\text{(5.2)} \qquad \left\{ \begin{array}{l} -\partial_t U(t,x,m) - \Delta_x U(t,x,m) + \frac{1}{2}|D_x U(t,x,m)|^2 - \int_{\mathbb{R}^d} \mathrm{div}_y D_m U(t,x,m,y)\ m(dy) \\ \qquad + \int_{\mathbb{R}^d} D_m U(t,x,m,y) \cdot D_x U(t,y,m)\ m(dy) = F(x,m) \\ \qquad\qquad\qquad \text{in } (0,T) \times \mathbb{R}^d \times \mathcal{P}_2 \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}_2 \end{array} \right.$$

Note that this equation consists in two parts: the part involving only local terms (local derivatives), which are exactly the ones which appear in the Hamilton-Jacobi equation of the MFG system; and the part with nonlocal terms (the two integrals) which involve derivatives with respect to the measure. This later part corresponds to the Kolmogorov equation in the MFG system.

By a (classical) solution we mean a map U such that the function and all its derivatives involved in the equation exist, are Lipschitz continuous and bounded. Note for later use that these condition implies that $D_x U$ is Lipschitz continuous in (x,m) and bounded.

Proposition 5.2.1. *Assume that U is a classical solution to the master equation* (5.2)*. Let $m_0 \in \mathcal{P}_2$.*

(i) there exists a (weak) solution to the McKean-Vlasov equation

$$\partial_t m - \Delta m - \mathrm{div}(m D_x U(t,x,m(t))) = 0 \text{ in } (0,T)\times\mathbb{R}^d, \qquad m(0) = m_0,$$

(ii) if we set $u(t,x) = U(t,x,m(t))$, then the pair (u,m) solves the MFG system (5.1)*.*

Proof. Existence of a solution to the McKean-Vlasov equation can be achieved through the SDE

$$dX_t = X_0 - \int_0^t D_x U(t, X_t, [X_t])dt + \sqrt{2}dB_t,$$

where $[X_0] = m_0$. As the map $D_x U = D_x U(t,x,m)$ is bounded and Lipschitz continuous in (x,m), this solution exists and is unique. It is then easy to check that $m(t) = [X_t]$ is a weak solution to the equation. This proves (i).

To prove (ii), we use Itô's formula in Proposition 5.1.9: For a fixed $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
U(T,x,[X_T]) = U(t,x,[X_t]) &+ \int_t^T \mathbb{E}\Big[\partial_t U(s,x,[X_s]) - D_m U(s,x,[X_s],X_s)\cdot D_x U(t,X_s,[X_t]) \\
&+ \mathrm{Tr}(D_y D_m U(s,x,[X_s],X_s))\Big]ds \\
= U(t,x,m(t)) &+ \int_t^T \partial_t U(s,x,m(s)) \\
&- \int_0^t \int_{\mathbb{R}^d} D_m U(s,x,m(s),y)\cdot D_x U(t,y,m(s)) m(s,dy)ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \mathrm{div}_y D_m U(s,x,m(s),y))m(s,dy))ds.
\end{aligned}$$

So using the equation satisfied by U, we obtain:

$$G(x,[X_T]) = U(t,x,m(t)) + \int_t^T -\Delta U(s,x,m(s)) + \frac{1}{2}|D_x U(s,x,m(s))|^2 - F(x,m(s)) \ ds.$$

Recalling the definition of u and m, we conclude that the pair (u,m) solves the MFG system. $\square$

Let us now state an existence result for the solution to the master equation:

Theorem 5.2.2. *Assume that F and G are monotone and have smooth derivatives in space and measure (up to order 3 in space and to order 2 in measure). Then there exists a solution to the master equation.*

Rough ideas of proof. In view of the discussion of Proposition 5.2.1, it is natural to state, for any $(t_0, x, m_0) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2$,

$$U(t_0,x,m_0) = u(t_0,x)$$

where (u,m) is the solution to the MFG system (5.1) on the time interval $[t_0,T]$ and with the initial condition $m(t_0) = m_0$. Note that, if (u,m) is such a solution, then, calling $(\tilde{u},\tilde{m})$ the restriction of (u,m) to the time interval $[t_0+h,T]$ (for $h>0$ small) gives a solution of MFG (5.1) on $[t_0+h,T]$ with initial condition $\tilde{m}(t_0+h) = m(t_0+h)$. This implies that $U(t_0+h,x,m(t_0+h)) = \tilde{u}(t_0+h,x) = u(t_0+h,x)$. Assuming that U is smooth enough and taking the derivative of this equality at $h=0$ yields:

$$\begin{aligned}
\partial_t u(t_0,x) &= \partial_t U(t_0,x,m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(t_0,x,m_0,y)\partial_t m(t_0,y)dy \\
&= \partial_t U(t_0,x,m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(t_0,x,m_0,y)(\Delta m(t_0,y) + \mathrm{div}(m(t_0,y)Du(t_0,y)))dy \\
&= \partial_t U(t_0,x,m_0) + \int_{\mathbb{R}^d} \mathrm{div}_y D_m U(t_0,x,m_0,y)m_0(y)dy - \int_{\mathbb{R}^d} D_m U(t_0,x,m_0,y)\cdot Du(t_0,y)m_0(y)dy,
\end{aligned}$$

where we used the equation of m in the second line and integrations by part in the last one. Note that $D_xU = Du$ and $D^2_{xx}U = D^2u$. So, by the equation satisfied by u and the above equality we obtain:

$$\begin{aligned}\partial_t u(t_0,x) &= -\Delta U(t_0,x,m_0) + \frac{1}{2}|D_xU(t_0,x,m_0)|^2 - F(x,m_0)\\ &= \partial_t U(t_0,x,m_0) + \int_{\mathbb{R}^d} \mathrm{div}_y D_mU(t_0,x,m_0,y)m_0(y)dy - \int_{\mathbb{R}^d} D_mU(t_0,x,m_0,y)\cdot Du(t_0,y)))m_0(y)dy.\end{aligned}$$

This shows that U is a solution to the master equation (5.2).

The (relatively) difficult part of the proof, that we will not present here, is the fact that the map U is smooth enough to justify the above computation. The interested reader can find the argument (for a probability approach) in [9] and [8] and in [5] for a PDE approach. □

5.3 Common noise

In many applications, the system is perturbed by randomness which affect all agents simultaneously. We call this a "common noise". This is the case for instance in the simple example of optimal trading of Subsection 4.1.2: Indeed, in this example, the fluctuation of the price of the tradable instrument affects all the traders. It turns out that, by the specific form of the cost, this common noise plays no role in the analysis. However, this is more an exception than a rule. Another very interesting example is the case of a crowd of small agents facing a "major agent": if the dynamics of the major agent is random, this randomness might affect all the small agents.

5.3.1 An elementary common noise problem

We consider here a problem in which the agents face a common noise which, in this elementary example, is a random variable Z on which the coupling cost F depends: $F = F(x,m,Z)$. The exact value of Z is revealed to the agents at time $T/2$ (to fix the ideas).

As before the agents directly control their drift: Let us recall that this system corresponds to a problem in which each agent has a dynamic of the form

$$dX_t = \alpha_t dt + \sqrt{2}dB_t$$

(where (α_t) is the control with values in $\mathbb{R}^d$ and B a Brownian motion). In contrast to the previous discussions, the control α_t is now adapted to the filtration generated by B *and to the noise* Z when $t \geq T/2$. The cost is now of the form

$$J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t|^2 + F(X_t, m(t), W)\ dt + G(X_T, m(T))\right].$$

As all the agents will probably choose their optimal control in function of the realization of Z (of course after time $T/2$), one expect the distribution of players to be random after $T/2$ and to depend on the noise Z.

On the time interval $[T/2, T]$, the agents have to solve a classical control problem (which depends on Z and on $(m(t))$):

$$u(t,x) := \inf_\alpha \mathbb{E}\left[\int_t^T \frac{1}{2}|\alpha_t|^2 + F(X_t, m(t), W)\ dt + G(X_T, m(T)) \mid Z\right]$$

which depends on the realization of Z and solves the HJ equation (with random coefficients):

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x,m(t),Z) \text{ in } (T/2,T)\times\mathbb{R}^d \\ u(T,x,Z) = G(x,m(T)) \text{ in } \mathbb{R}^d. \end{cases}$$

On the other hand, on the time interval $[0, T/2)$, the agent has no information on Z and, by dynamic programming, one has

$$u(t,x) := \inf_{\alpha} \mathbb{E}\left[\int_t^{T/2} \frac{1}{2}|\alpha_t|^2 + \bar{F}(X_t, m(t))\ dt + u(T/2^+, X_{T/2})\right],$$

where $\bar{F}(x,m) = \mathbb{E}\left[F(x,m,Z)\right]$ (recall that $m(t)$ is deterministic on $[0,T/2]$). Thus, on the time interval $[0,T/2]$, u solves

$$\begin{cases} -\partial_t u - \Delta u + \dfrac{1}{2}|Du|^2 = \bar{F}(x, m(t)) \text{ in } (0,T/2)\times\mathbb{R}^d \\ u(T/2^-, x) = \mathbb{E}\left[u(T/2^+, x)\right] \text{ in } \mathbb{R}^d \end{cases}$$

As for the associated Kolmogorov equation, on the time interval $[0,T/2]$ (where the optimal feedback $-Du$ is purely deterministic) we have as usual:

$$\partial_t m - \Delta m - \mathrm{div}\,(m\ Du(t,x)) = 0 \text{ in } (0,T/2)\times\mathbb{R}^d, \qquad m(0) = m_0.$$

while on the time interval $[T/2, T]$, m becomes random (as the control $-Du$) and solves

$$\partial_t m - \Delta m - \mathrm{div}\,(m\ Du(t,x,Z)) = 0 \text{ in } (T/2)\times\mathbb{R}^d, \qquad m(T/2^-) = m(T/2^+).$$

Note the relation: $m(T/2^-) = m(T/2^+)$, which means that the dynamics of the crowed is continuous in time.

Let us point out some remarquable features of the problem. First the pairs (u,m) are no longer deterministic, and are adapted to the filtration generated by the common noise (here this filtration is trivial up to time $T/2$ and is the $\sigma-$algebra generated by Z after $T/2$). Second the map u is discontinuous: this is due to the shock of information at time $T/2$.

5.4 Comment

Most formal properties of the Master equation have been introduced and presented by Lions in [26]. Lions discusses also the existence and uniqueness of the solution in several framework. The most general reference on this subject is the monograph of Carmona and Delarue [8] (see also the interesting but mostly formal approach in the monograph by Bensoussan, Frehse and Yam [3]). One of the main interests of the master equation is that it allows to understand in a rigorous way the limit of Nash equilibria in $N-$player differential games as $N \to +\infty$: see [5] and [8].

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