The high temperature Ising model on the triangular lattice is a critical percolation model

András Bálint

Joint work with Federico Camia and Ronald Meester, Vrije Universiteit Amsterdam
1 The Ising model on $\mathbb{T}$
   - The lattice $\mathbb{T}$ and the model
   - Random-cluster representation

2 Our model
   - Definition
   - Results
   - Related results, open questions
The triangular lattice $\mathbb{T}$

Vertex set $\mathcal{V}_\mathbb{T}$, edge set $\mathcal{E}_\mathbb{T}$,
special vertex: $0$, called origin
The Ising model on $\mathbb{T}$

- Dependent site percolation model, configuration space: $\Omega := \{-1, +1\}^\mathbb{T}$.
- Two parameters: inverse temperature $\beta \geq 0$, external field $h$; we only consider $h = 0$.
- Standard: $\exists \beta_c \in (0, \infty)$ s.t. for $\beta < \beta_c$, there exists a unique Ising Gibbs measure on $\Omega$, which we denote by $\mu_\beta$.
- A way to obtain $\mu_\beta$ (through the random-cluster representation of the Ising model) will follow.
The Ising model on $\mathbb{T}$

- Dependent site percolation model, configuration space: $\Omega := \{-1, +1\}^{\mathbb{T}}$.
- Two parameters: inverse temperature $\beta \geq 0$, external field $h$; we only consider $h = 0$.
- Standard: $\exists \beta_c \in (0, \infty)$ s.t. for $\beta < \beta_c$, there exists a unique Ising Gibbs measure on $\Omega$, which we denote by $\mu_\beta$.
- A way to obtain $\mu_\beta$ (through the random-cluster representation of the Ising model) will follow.
The Ising model on $\mathbb{T}$

- Dependent site percolation model, configuration space: $\Omega := \{-1, +1\}^\mathbb{V}$.
- Two parameters: inverse temperature $\beta \geq 0$, external field $h$; we only consider $h = 0$.
- Standard: $\exists \beta_c \in (0, \infty)$ s.t. for $\beta < \beta_c$, there exists a unique Ising Gibbs measure on $\Omega$, which we denote by $\mu_\beta$.
- A way to obtain $\mu_\beta$ (through the random-cluster representation of the Ising model) will follow.
Dependent site percolation model, configuration space:
\[ \Omega := \{-1, +1\}^\mathbb{V}_T. \]

Two parameters: inverse temperature \( \beta \geq 0 \), external field \( h \); we only consider \( h = 0 \).

Standard: \( \exists \beta_c \in (0, \infty) \) s.t. for \( \beta < \beta_c \), there exists a unique *Ising Gibbs measure* on \( \Omega \), which we denote by \( \mu_\beta \).

A way to obtain \( \mu_\beta \) (through the *random-cluster representation* of the Ising model) will follow.
Random-cluster measures on finite graphs

$G = (\mathcal{V}, \mathcal{E})$ a finite graph. For $\eta \in \{0, 1\}^\mathcal{E}$, we define
- the set of open edges in $\eta$: $\mathcal{E}_{\text{open}}(\eta) = \{ e \in \mathcal{E} : \eta(e) = 1 \}$,
- $k(\eta)$ as the number of components in the graph $G_{\text{open}}(\eta) = (\mathcal{V}, \mathcal{E}_{\text{open}}(\eta))$.

For $p \in (0, 1), q > 0$, we define the random-cluster measure $\Phi^G_{p,q}$ so that
\[ \Phi^G_{p,q}(\eta) \propto \prod_{e \in \mathcal{E}} p^{\eta(e)}(1 - p)^{1 - \eta(e)} q^{k(\eta)}. \]

Components in $G_{\text{open}}(\eta)$ are called (open) FK clusters (in $\eta$).
Random-cluster measures on finite graphs

Let $G = (\mathcal{V}, \mathcal{E})$ be a finite graph. For $\eta \in \{0, 1\}^\mathcal{E}$, we define

- the set of open edges in $\eta$: $\mathcal{E}_{\text{open}}(\eta) = \{e \in \mathcal{E} : \eta(e) = 1\}$,
- $k(\eta)$ as the number of components in the graph $G_{\text{open}}(\eta) = (\mathcal{V}, \mathcal{E}_{\text{open}}(\eta))$.

For $p \in (0, 1), q > 0$, we define the random-cluster measure $\Phi^G_{p,q}$ so that

$$\Phi^G_{p,q}(\eta) \propto \prod_{e \in \mathcal{E}} p^{\eta(e)}(1 - p)^{1-\eta(e)} q^{k(\eta)}.$$

Components in $G_{\text{open}}(\eta)$ are called (open) FK clusters (in $\eta$).
The high temperature Ising model on the triangular lattice is a critical percolation model.
The high temperature Ising model on the triangular lattice is a critical percolation model.
A random-cluster measure $\Phi_{p,2}$ on $\mathbb{T}$ is obtained as a limit of $\Phi_{p,2}^G$ as $G \uparrow \mathbb{T}$.

**Theorem**

There exists a critical value $0 < p_c < 1$ such that for a random-cluster measure $\Phi_{p,2}$,

$$
\begin{align*}
\Phi_{p,2}(\eta : \exists \infty \text{ FK cluster in } \eta) &= 0 & \text{if } p < p_c, \\
\Phi_{p,2}(\eta : \exists \infty \text{ FK cluster in } \eta) &= 1 & \text{if } p > p_c.
\end{align*}
$$

If $p < p_c$, then $\Phi_{p,2}$ is unique.
Connection between models

\(\mu_\beta\): Ising measure, \(\beta_c\): critical inverse temperature in the Ising model on \(\mathbb{T}\).

**Theorem (The random-cluster representation of the Ising model)**

For \(\beta < \beta_c\), a configuration in \(\Omega\) distributed according to \(\mu_\beta\) can be obtained by the following procedure.

- Set \(p = 1 - e^{-\beta}\).
- Step 1: Draw a bond configuration \(\eta\) with distribution \(\Phi_{p,2}\).
- Step 2: For each FK cluster in \(\eta\), assign to all vertices in the cluster spin \(+\) with probability \(1/2\) and spin \(-\) with probability \(1/2\), independently for different FK clusters.
Connection between models

$\mu_\beta$: Ising measure, $\beta_c$: critical inverse temperature in the Ising model on $\mathbb{T}$.

**Theorem (The random-cluster representation of the Ising model)**

*For $\beta < \beta_c$, a configuration in $\Omega$ distributed according to $\mu_\beta$ can be obtained by the following procedure.*

- **Set** $p = 1 - e^{-\beta}$.
- **Step 1:** Draw a bond configuration $\eta$ with distribution $\Phi_{p,2}$.
- **Step 2:** For each FK cluster in $\eta$, assign to all vertices in the cluster spin $+$ with probability $1/2$ and spin $-$ with probability $1/2$, independently for different FK clusters.
Connection between models

\( \mu_\beta \): Ising measure, \( \beta_c \): critical inverse temperature in the Ising model on \( \mathbb{T} \).

**Theorem (The random-cluster representation of the Ising model)**

*For* \( \beta < \beta_c \), *a configuration in* \( \Omega \) *distributed according to* \( \mu_\beta \) *can be obtained by the following procedure.*

- **Step 1:** Draw a bond configuration \( \eta \) with distribution \( \Phi_{p,2} \).
- **Step 2:** For each FK cluster in \( \eta \), assign to all vertices in the cluster spin \( + \) with probability \( \frac{1}{2} \) and spin \( - \) with probability \( \frac{1}{2} \), independently for different FK clusters.
Connection between models

\( \mu_\beta \): Ising measure, \( \beta_c \): critical inverse temperature in the Ising model on \( \mathbb{T} \).

**Theorem (The random-cluster representation of the Ising model)**

For \( \beta < \beta_c \), a configuration in \( \Omega \) distributed according to \( \mu_\beta \) can be obtained by the following procedure.

- Set \( p = 1 - e^{-\beta} \).
- **Step 1:** Draw a bond configuration \( \eta \) with distribution \( \Phi_{p,2} \).
- **Step 2:** For each FK cluster in \( \eta \), assign to all vertices in the cluster spin + with probability \( 1/2 \) and spin − with probability \( 1/2 \), independently for different FK clusters.
Definition of the model

- Fix $\beta < \beta_c$, $r \in [0, 1]$, and set $p = 1 - e^{-\beta}$.
- Step 1: Draw a bond configuration $\eta$ with distribution $\Phi_{p,2}$.
- Step 2: For each FK cluster in $\eta$, assign to all vertices in the cluster spin $+$ with probability $r$, and spin $-$ with probability $1 - r$, independently for different FK clusters.
- Denote the resulting joint measure on $\{0, 1\}^E_T \times \Omega$ by $\mathbb{P}_{\beta,r}$.

Note:
- the marginal of $\mathbb{P}_{\beta,1/2}$ on $\Omega$ is $\mu_\beta$.
- $\beta = 0 \leftrightarrow$ Bernoulli site percolation with parameter $r$. 

András Bálint  The high temperature Ising model on the triangular lattice is a critical percolation model
Definition of the model

- Fix $\beta < \beta_c$, $r \in [0, 1]$, and set $p = 1 - e^{-\beta}$.
- Step 1: Draw a bond configuration $\eta$ with distribution $\Phi_{p,2}$.
- Step 2: For each FK cluster in $\eta$, assign to all vertices in the cluster spin $+$ with probability $r$, and spin $-$ with probability $1 - r$, independently for different FK clusters.
- Denote the resulting joint measure on $\{0, 1\}^{E_T} \times \Omega$ by $\mathbb{P}_{\beta,r}$.

Note:
- the marginal of $\mathbb{P}_{\beta,1/2}$ on $\Omega$ is $\mu_\beta$.
- $\beta = 0 \leftrightarrow$ Bernoulli site percolation with parameter $r$. 
Definition of the model

- Fix $\beta < \beta_c$, $r \in [0, 1]$, and set $p = 1 - e^{-\beta}$.
- Step 1: Draw a bond configuration $\eta$ with distribution $\Phi_{p,2}$.
- Step 2: For each FK cluster in $\eta$, assign to all vertices in the cluster spin $+$ with probability $r$, and spin $-$ with probability $1 - r$, independently for different FK clusters.
- Denote the resulting joint measure on $\{0, 1\}^E \times \Omega$ by $P_{\beta,r}$.

Note:
- the marginal of $P_{\beta,1/2}$ on $\Omega$ is $\mu_{\beta}$.
- $\beta = 0 \leftrightarrow$ Bernoulli site percolation with parameter $r$. 
Definition of the model

- Fix $\beta < \beta_c$, $r \in [0, 1]$, and set $p = 1 - e^{-\beta}$.
- Step 1: Draw a bond configuration $\eta$ with distribution $\Phi_{p,2}$.
- Step 2: For each FK cluster in $\eta$, assign to all vertices in the cluster spin $+$ with probability $r$, and spin $-$ with probability $1 - r$, independently for different FK clusters.
- Denote the resulting joint measure on $\{0, 1\}^{E_T} \times \Omega$ by $\mathbb{P}_{\beta,r}$.

Note:
- the marginal of $\mathbb{P}_{\beta,1/2}$ on $\Omega$ is $\mu_{\beta}$.
- $\beta = 0 \leftrightarrow$ Bernoulli site percolation with parameter $r$. 

András Bálint
The high temperature Ising model on the triangular lattice is
Definition of the model

- Fix $\beta < \beta_c$, $r \in [0, 1]$, and set $p = 1 - e^{-\beta}$.
- Step 1: Draw a bond configuration $\eta$ with distribution $\Phi_{p,2}$.
- Step 2: For each FK cluster in $\eta$, assign to all vertices in the cluster spin $+$ with probability $r$, and spin $-$ with probability $1 - r$, independently for different FK clusters.
- Denote the resulting joint measure on $\{0, 1\}^{E_T} \times \Omega$ by $P_{\beta,r}$.

Note:
- the marginal of $P_{\beta,1/2}$ on $\Omega$ is $\mu_\beta$.
- $\beta = 0 \leftrightarrow$ Bernoulli site percolation with parameter $r$. 
The high temperature Ising model on the triangular lattice is a critical percolation model.
The high temperature Ising model on the triangular lattice is a critical percolation model.
András Bálint

The high temperature Ising model on the triangular lattice is a critical percolation model.
Question: Is the origin in an infinite (+)-cluster?

Θ(β, r) := P_{β,r}(0 is in an infinite (+)-cluster).

Clear: For all β, Θ(β, 0) = 0, Θ(β, 1) = 1, Θ(β, r) is increasing in r.

Critical value: \( r_c(β) := \sup \{ r ∈ [0, 1] : Θ(β, r) = 0 \} \).
Question: Is the origin in an infinite (+)-cluster?

\( \Theta(\beta, r) := \mathbb{P}_{\beta,r}(0 \text{ is in an infinite (+)-cluster}). \)

Clear: For all \( \beta \), \( \Theta(\beta, 0) = 0 \), \( \Theta(\beta, 1) = 1 \), \( \Theta(\beta, r) \) is increasing in \( r \).

Critical value: \( r_c(\beta) := \sup\{ r \in [0, 1] : \Theta(\beta, r) = 0 \} \).
Question: Is the origin in an infinite (+)-cluster?

\[ \Theta(\beta, r) := \mathbb{P}_{\beta,r}(0 \text{ is in an infinite (+)-cluster}). \]

Clear: For all \( \beta \), \( \Theta(\beta, 0) = 0 \), \( \Theta(\beta, 1) = 1 \), \( \Theta(\beta, r) \) is increasing in \( r \).

Critical value: \( r_c(\beta) := \sup\{r \in [0, 1] : \Theta(\beta, r) = 0\} \).

András Bálint

The high temperature Ising model on the triangular lattice...
Question: Is the origin in an infinite (+)-cluster?

$\Theta(\beta, r) := \mathbb{P}_{\beta, r}(0 \text{ is in an infinite } (+)-\text{cluster}).$

Clear: For all $\beta$, $\Theta(\beta, 0) = 0$, $\Theta(\beta, 1) = 1$, $\Theta(\beta, r)$ is increasing in $r$.

Critical value: $r_c(\beta) := \sup \{ r \in [0, 1] : \Theta(\beta, r) = 0 \}$. 
The high temperature Ising model on the triangular lattice is a critical percolation model.

**Main results**

Theorem (Bálint, Camia, Meester, 2008)

For all $\beta < \beta_c$,

$$r_c(\beta) = 1/2.$$  

Moreover, the phase transition at $r = 1/2$ is sharp, i.e.,

- If $r < 1/2$, the distribution of the size of the $(+)$-cluster of the origin has an exponentially decaying tail.
- If $r = 1/2$, $\Theta(\beta, 1/2) = 0$ and the mean size of the $(+)$-cluster of the origin is infinite.
- If $r > 1/2$, there exists a.s. a unique infinite $(+)$-cluster.

Theorem (Bálint, Camia, Meester, 2008)

For each $\beta < \beta_c$, the function $\Theta(\beta, r)$ is continuous in $r$. 
Conjecture

The high temperature ($\beta < \beta_c$) Ising model on the *triangular lattice* $\mathbb{T}$ with no external field shows critical behaviour and is in the universality class of Bernoulli (independent) percolation.

- Main result (i.e., $r_c(\beta) = 1/2$) confirms criticality.
- Choice of $\mathbb{T}$ is important. High temperature Ising model on $\mathbb{Z}^2$ is subcritical:
  - for $\beta = 0$, known: $r_c^{\mathbb{Z}^2}(0) = r_c^{\mathbb{Z}^2, \text{Bernoulli}} > 1/2$;
  - for all $\beta < \beta_c$, the distribution of the size of the (+)-cluster of the origin has an exponentially decaying tail (Higuchi (1993), van den Berg (2007)), and this implies $r_c(\beta) > 1/2$. 
Conjecture

The high temperature ($\beta < \beta_c$) Ising model on the *triangular lattice* $\mathbb{T}$ with no external field shows critical behaviour and is in the universality class of Bernoulli (independent) percolation.

- Main result (i.e., $r_c(\beta) = 1/2$) confirms criticality.
- Choice of $\mathbb{T}$ is important. High temperature Ising model on $\mathbb{Z}^2$ is subcritical:
  - for $\beta = 0$, known: $r_c^{\mathbb{Z}^2}(0) = r_c^{\mathbb{Z}^2, \text{Bernoulli}} > 1/2$;
  - for all $\beta < \beta_c$, the distribution of the size of the (+)-cluster of the origin has an exponentially decaying tail (Higuchi (1993), van den Berg (2007)), and this implies $r_c(\beta) > 1/2$. 
Conjecture

The high temperature \((\beta < \beta_c)\) Ising model on the \textit{triangular lattice} \(\mathbb{T}\) with no external field shows critical behaviour and is in the universality class of Bernoulli (independent) percolation.

- Main result (i.e., \(r_c(\beta) = 1/2\)) confirms criticality.
- Choice of \(\mathbb{T}\) is important. High temperature Ising model on \(\mathbb{Z}^2\) is subcritical:
  - for \(\beta = 0\), known: \(r_c(\mathbb{Z}^2) = r_c^{\mathbb{Z}^2, \text{Bernoulli}} > 1/2\);
  - for all \(\beta < \beta_c\), the distribution of the size of the (+)-cluster of the origin has an exponentially decaying tail (Higuchi (1993), van den Berg (2007)), and this implies \(r_c(\beta) > 1/2\).
Conjecture

The high temperature \((\beta < \beta_c)\) Ising model on the triangular lattice \(\mathbb{T}\) with no external field shows critical behaviour and is in the universality class of Bernoulli (independent) percolation.

- Main result (i.e., \(r_c(\beta) = 1/2\)) confirms criticality.
- Choice of \(\mathbb{T}\) is important. High temperature Ising model on \(\mathbb{Z}^2\) is subcritical:
  - for \(\beta = 0\), known: \(r^\mathbb{Z}^2_c(0) = r^\mathbb{Z}^2_c,\text{Bernoulli} > 1/2\);
  - for all \(\beta < \beta_c\), the distribution of the size of the (+)-cluster of the origin has an exponentially decaying tail (Higuchi (1993), van den Berg (2007)), and this implies \(r_c(\beta) > 1/2\).
Related results, conjectures

Conjecture

The high temperature \((\beta < \beta_c)\) Ising model on the \textit{triangular lattice} \(\mathbb{T}\) with no external field shows critical behaviour and is in the universality class of Bernoulli (independent) percolation.

- Main result (i.e., \(r_c(\beta) = 1/2\)) confirms criticality.
- Choice of \(\mathbb{T}\) is important. High temperature Ising model on \(\mathbb{Z}^2\) is subcritical:
  - for \(\beta = 0\), known: \(r_{c,\mathbb{Z}^2}(0) = r_{c,\text{Bernoulli}} > 1/2\);
  - for all \(\beta < \beta_c\), the distribution of the size of the (+)-cluster of the origin has an exponentially decaying tail (Higuchi (1993), van den Berg (2007)), and this implies \(r_{c}(\beta) > 1/2\).
Open questions

- What else can be proved on the square lattice?
  (We expect $r_c(\beta) + r_c^*(\beta) = 1$, where $r_c^*(\beta)$ is the critical value for the matching lattice of $\mathbb{Z}^2$.)

- What happens if $\Phi_{p,2}$ is replaced by $\Phi_{p,q}$ in the definition of the model?
  For $q = 1$, we have:
  - $r_c = 1/2$ on $\mathbb{T}$,
  - $r_c + r_c^* = 1$ on $\mathbb{Z}^2$.

Thank you for your attention!
Open questions

- What else can be proved on the square lattice?
  (We expect $r_c(\beta) + r_c^*(\beta) = 1$, where $r_c^*(\beta)$ is the critical value for the matching lattice of $\mathbb{Z}^2$.)

- What happens if $\Phi_{p,2}$ is replaced by $\Phi_{p,q}$ in the definition of the model?
  For $q = 1$, we have:
  - $r_c = 1/2$ on $\mathbb{T}$,
  - $r_c + r_c^* = 1$ on $\mathbb{Z}^2$.

Thank you for your attention!
Open questions

- What else can be proved on the square lattice?
  
  (We expect $r_c(\beta) + r_c^*(\beta) = 1$, where $r_c^*(\beta)$ is the critical value for the matching lattice of $\mathbb{Z}^2$.)

- What happens if $\Phi_{p,2}$ is replaced by $\Phi_{p,q}$ in the definition of the model?
  
  For $q = 1$, we have:
  
  - $r_c = 1/2$ on $\mathbb{T}$,
  - $r_c + r_c^* = 1$ on $\mathbb{Z}^2$.

Thank you for your attention!
Open questions

- What else can be proved on the square lattice?
  (We expect $r_c(\beta) + r_c^*(\beta) = 1$, where $r_c^*(\beta)$ is the critical value for the matching lattice of $\mathbb{Z}^2$.)

- What happens if $\Phi_{p,2}$ is replaced by $\Phi_{p,q}$ in the definition of the model?
  For $q = 1$, we have:
  - $r_c = 1/2$ on $\mathbb{T}$,
  - $r_c + r_c^* = 1$ on $\mathbb{Z}^2$.

Thank you for your attention!
Open questions

- What else can be proved on the square lattice?
  (We expect $r_c(\beta) + r_c^*(\beta) = 1$, where $r_c^*(\beta)$ is the critical value for the matching lattice of $\mathbb{Z}^2$.)

- What happens if $\Phi_{p,2}$ is replaced by $\Phi_{p,q}$ in the definition of the model?
  For $q = 1$, we have:
  - $r_c = 1/2$ on $\mathbb{T}$,
  - $r_c + r_c^* = 1$ on $\mathbb{Z}^2$.

Thank you for your attention!