Large Deviations for Itô Diffusions

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Let X be a Polish space. Let $\{I_\varepsilon\}$ be a sequence of functionals $I_\varepsilon : X \to [0, +\infty]$. We define two functionals $\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon$, $\Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon : X \to [0, +\infty]$ as

$$\big(\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon\big)(x) := \inf\Big\{\underline{\lim}_{\varepsilon\to 0} I_\varepsilon(x^\varepsilon),\ x^\varepsilon \to x\Big\}$$
$$\big(\Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon\big)(x) := \inf\Big\{\overline{\lim}_{\varepsilon\to 0} I_\varepsilon(x^\varepsilon),\ x^\varepsilon \to x\Big\}$$

Whenever $\Gamma\text{–}\underline{\lim}_\varepsilon I_\varepsilon(x) = \Gamma\text{–}\overline{\lim}_\varepsilon I_\varepsilon(x) = I(x)$ for al $x \in X$, we say that I_ε Γ-converges to I. The sequence $\{I_\varepsilon\}$ is called *equicoercive* iff for each $M > 0$ there exists an $\varepsilon_M > 0$ and a compact $K_M \subset X$ such that $\cup_{\varepsilon \le \varepsilon_M}\{x \in X \,:\, I_\varepsilon(x) \le M\} \subset K_M$.

Let $\{a_\varepsilon\} \subset [0,1]$ be such that $\lim_{\varepsilon\to 0} a_\varepsilon = 0$. Let $I : X \to [0,+\infty]$ be a lower semicontinuous functional on X. A sequence $\{\mathbb{P}^\varepsilon\} \subset \mathcal{P}(X)$ satisfies

- a *large deviations weak upper bound* with speed $\{a_\varepsilon^{-1}\}$ and rate I iff for each compact set $K \subset X$

$$\overline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(K) \le -\inf_{v\in K} I(v) \tag{1}$$

- $\{\mathbb{P}^\varepsilon\}$ satisfies a *large deviations (full) upper bound* with speed $\{a_\varepsilon^{-1}\}$ and rate I iff for each closed set $\mathcal{C} \subset X$

$$\overline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(\mathcal{C}) \le -\inf_{v\in \mathcal{C}} I(v) \tag{2}$$

- $\{\mathbb{P}^\varepsilon\}$ satisfies a *large deviations lower bound* with speed $\{a_\varepsilon^{-1}\}$ and rate I iff for each open set $\mathcal{O} \subset X$

$$\underline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(\mathcal{O}) \ge -\inf_{v\in \mathcal{O}} I(v) \tag{3}$$

$\{\mathbb{P}^\varepsilon\}$ satisfies a *large deviation principle* iff an upper bound and a lower bound hold with the same speeds and rates. $\{\mathbb{P}^\varepsilon\}$ is called *exponentially tight* iff there exists a sequence $\{K_\ell\}$ of compact subsets of X such that

$$\overline{\lim_{\ell}}\,\overline{\lim_{\varepsilon}}\, a_\varepsilon \log \mathbb{P}^\varepsilon(K_\ell) = -\infty \tag{4}$$

Note that an exponentially tight family of probability measures satisfies a large deviations upper bound iff it satisfies a large deviations weak upper bound.

A classical example: the vanishing noise limit

Consider the SDE

$$\begin{aligned} \dot{x} &= b(x) + \sqrt{\varepsilon}\sigma(x)\dot{W} \\ x(0) &= x_0 \end{aligned}$$

and let $\mathbb{P}^\varepsilon$ be the law of its solution. A classical result by Freidlin and Wentcell states that $\mathbb{P}^\varepsilon$ satisfies a large deviations principle with speed ε^{-1} and rate

$$I(x) = \frac{1}{2}\int_0^T dt \, \frac{|\, \dot{x}(t) - b(x(t)) \,|^2}{\sigma(x(t))\sigma(x(t))^\dagger}$$

Consider the more general case

$$\begin{aligned} \dot{x} &= b^{\varepsilon}(x) + \sqrt{\varepsilon}\sigma^{\varepsilon}(x)\dot{W} \\ x(0) &= x_0 \end{aligned}$$

and for $\gamma < 1$ introduce the functional

$$I_{\varepsilon}(x) = \frac{\varepsilon^{-\gamma}}{2}\int_0^T dt\, \frac{|\ \dot{x}(t) - b^{\varepsilon}(x(t))\ |^2}{\sigma^{\varepsilon}(x(t))\sigma^{\varepsilon}(x(t))^{\dagger}}$$

Large Deviations for Itô processes

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Then, w.r.t. to the speed $\varepsilon^{-1+\gamma}$

- If I_{ε} is equicoercive then $\mathbb{P}^{\varepsilon}$ is exponentially tight.
- $\mathbb{P}^{\varepsilon}$ satisfies a weak large deviations upper bound with rate $\Gamma\text{–}\underline{\lim}\, I_{\varepsilon}$.
- $\mathbb{P}^{\varepsilon}$ satisfies a large deviations lower bound with rate $\Gamma\text{–}\overline{\lim}\, I_{\varepsilon}$.

The statement also holds in infinite dimensions. Consider

$$\begin{aligned} \dot{x} &= b^{\varepsilon}(x) + \sqrt{\varepsilon}\,\text{Noise}^{\varepsilon} \\ x(0) &= x_0 \end{aligned}$$

where the noise term is a martingale with values in some Banach space B. For each $\varphi \in B^*$, the action of the martingale on φ define a real valued martingale with quadratic variation

$$\big[\langle \text{Noise}^{\varepsilon}, \varphi \rangle\big](T) = A^{\varepsilon}(x; \varphi, \varphi) =: \| \varphi \|^2_{\mathcal{H}^{\varepsilon}_x}$$

For $\mathcal{D}^{\varepsilon}_x$ the dual of $\mathcal{H}^{\varepsilon}_x$ and $\gamma < 1$ introduce the functional

$$I_{\varepsilon}(x) = \frac{\varepsilon^{-\gamma}}{2} \| \dot{x} - b^{\varepsilon}(x) \|^2_{\mathcal{D}^{\varepsilon}_x}$$

Then large deviations and Γ-convergence enjoy the same equivalence of the finite dimensional case.

Consider the SDE

$$\begin{aligned} \dot{x} &= \sqrt{\varepsilon}\sigma\big(\frac{x}{\varepsilon^\alpha}\big)\dot{W} \\ x(0) &= x_0 \end{aligned}$$

for some periodic continuous σ. The law of the solution satisfies a large deviations principle with speed ε^{-1} and rate

$$I(x) = \frac{1}{2}\int_0^T dt\,\bar{A} \mid \dot{x}(t) \mid^2$$

where $\bar{A} = \frac{1}{\int dz\,\sigma^2(z)}$.

Applications II: singular limits

One can consider stochastic perturbations of PDEs and investigate a joint vanishing noise - singular limit asymptotic. This may have some physical motivations. Consider in particular

$$\begin{aligned}
&\partial_t u + f(u)_x = \frac{\varepsilon}{2} \left(D(u)u_x\right)_x + \varepsilon^\gamma \left(\sigma(u)\dot{W}^\varepsilon\right)_x \\
&u(0,x) = u_0(x).
\end{aligned}$$

As well known, the limiting equation ($\varepsilon = 0$) does not admit smooth solutions, but admits infinitely many weak solutions. By adapting the argument above, one can prove a large deviations principle with speed $\varepsilon^{-2\gamma}$ on a space of Young measures. The corresponding rate functional vanishes on "measure valued solutions" to the limiting equation.

Applications II: singular limits

$$\partial_t u + f(u)_x = \frac{\varepsilon}{2}\left(D(u)u_x\right)_x + \varepsilon^\gamma\left(\sigma(u)\dot{W}^\varepsilon\right)_x$$
$$u(0,x) = u_0(x).$$

Since there are infinitely many measure valued solutions to the limiting equation, one can investigate large deviations at the scale $\varepsilon^{-2\gamma+1}$. The corresponding (candidate) rate functional is finite only on weak solutions of the limiting equation, and quantifies how a weak solution violates the entropic condition. In the case $f(u) = \sigma^2(u)$ and $D(u) = 1$ such a functional is consistent with the (candidate) rate functional for TASEP.

- Consider

$$\begin{aligned} \dot{x} &= b(x) + \sqrt{\varepsilon}\dot{W} \\ x(0) &= x_0 \end{aligned}$$

  with non-Lipschitz b. Is it easy to see that $\mathbb{P}^\varepsilon$ satisfies a large deviations principle with speed ε^{-1} and the Freidlin-Wentcell rate. But this functional has many zeros (the solutions of the deterministic limiting equation). This principles may perhaps be investigated approximating b with a suitable b^ε and studying the Γ-limit of the ε dependent Freidlin-Wentcell functional
- Investigate large deviations for multiscaled diffusions, as like

$$\begin{aligned} \dot{x} &= b^\varepsilon(x) + \sqrt{\varepsilon}\sigma^\varepsilon(x,y)\dot{W} \\ \dot{y} &= \frac{1}{\varepsilon}\beta(y) + \frac{1}{\sqrt{\varepsilon}}s(y)\dot{W} \end{aligned}$$