Invasion percolation in 2D

Bálint Vágvölgyi

Joint work with M. Damron and A. Sapozhnikov
Consider the square lattice $\mathbb{Z}^2$ with its set of nearest neighbour bonds $\mathbb{E}^2$.

Assign *uniform* $[0,1]$ random variables to each edge independently, denoted by $\tau(e)$ for an edge $e$.

For a given $p$ we say that an edge $e$ is $p$-open if $\tau(e) \leq p$.

This model is called Bernoulli percolation with parameter $p$.

The same model can be obtained by declaring each edge open with probability $p$ and closed otherwise, independently of each other.
Bernoulli bond percolation

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Some important results for Bernoulli percolation

- Let $\Theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty)$ be the percolation function.
- Let $p_c = \inf\{0 \leq p \leq 1 : \Theta(p) > 0\}$ be the critical probability.
- For all $p > p_c$ there is a unique infinite $p$-open cluster, denoted by $C_p$.
- $p_c = 1/2$ and $\Theta(p_c) = 0$.
- Russo, Seymour, Welsh theorem: For any $k > 0$ let $A_{n,k}$ be the event that the box $[0, kn] \times [0, n]$ contains a horizontal open crossing and let $p \geq p_c$. Then there exists a constant $\delta_k$, independent of $n$ and $p$ such that $\mathbb{P}_p(A_{n,k}) > \delta_k$.
- Consequence: For all $p \geq p_c$, the origin is surrounded by infinitely many open circuits with probability one.
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The invaded region $S(v)$ of a vertex $v$ is defined as the increasing union of subgraphs $S_n(v)$, where

- $S_0(v) = \{v\}$
- $S_{n+1}(v)$ is $S_n(v)$ together with the lowest edge not in $S_n(v)$ but incident to some vertex in $S_n$

In this talk we always consider the invaded region of the origin and we write $S = S(0)$. 
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For all $p > p_c$ we have that $S \cap C_p \neq \emptyset$, since the origin is surrounded by infinitely many $p$-open circuit.

It is clear from the invasion mechanism that if $S_n \cap C_p \neq \emptyset$ for some $n > 0$ then $S \setminus S_n \subset C_p$.

By other words: If for any $p$ the invasion hits the infinite $p$-open cluster, it will never leave this cluster again.

If $e_i$ is the edge invaded at time $i$ then $\limsup_{i \to \infty} \tau(e_i) = p_c$.

Since there is no percolation at $p_c$ we get that $\hat{\tau} = \max_{e \in E_\infty} \tau(e)$ exists and is greater than $p_c$. 

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Definition of the first pond

- Let $\hat{e}$ be the edge where the maximum value of $\tau$ is taken, namely $\tau(\hat{e}) = \hat{\tau}$.
- $\hat{e}$ exists and it is well-defined with probability 1.
- Suppose that $\hat{e}$ is added to the invasion at time $i + 1$, then the graph $S_i = \hat{V}_1$ is called the first pond of the invasion or the first pond of the origin.
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Further ponds

- Assume that the first pond is $S_{i_1}$ for some $i_1$.
- Then $\max_{e_i \in E_\infty, i > i_1} \tau(e)$ exists and greater than $p_c$.
- Let $\hat{e}_2$ be the edge where this value is taken.
- If $\hat{e}_2$ is invaded at time $i_2 + 1$, than the graph $S_{i_2} \setminus S_{i_1}$ is the second pond of the invasion.
- The other ponds of the invasion can be defined in similar way.
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