Invasion percolation in 2D

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Joint work with M. Damron and A. Sapozhnikov

Bernoulli bond percolation

- Consider the square lattice $\mathbb{Z}^2$ with its set of nearest neighbour bonds $\mathbb{E}^2$.
- Assign *uniform*[0,1] random variables to each edge independently, denoted by $\tau(e)$ for an edge e.
- For a given p we say that an edge e is p-open if $\tau(e) \leq p$.
- This model is called Bernoulli percolation with parameter p.
- The same model can be obtained by declaring each edge open with probability p and closed otherwise, independently of each other.

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Some important results for Bernoulli percolation

- Let $\Theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty)$ be the percolation function.
- Let $p_c = \inf\{0 \leq p \leq 1 : \Theta(p) > 0\}$ be the critical probability.
- For all $p > p_c$ there is a unique infinite p-open cluster, denoted by $\mathcal{C}_p$.
- $p_c = 1/2$ and $\Theta(p_c) = 0$.
- Russo, Seymour, Welsh theorem: For any $k > 0$ let $A_{n,k}$ be the event that the box $[0, kn] \times [0, n]$ contains a horizontal open crossing and let $p \geq p_c$. Then there exists a constant δ_k, independent of n and p such that $\mathbb{P}_p(A_{n,k}) > \delta_k$.
- Consequence: For all $p \geq p_c$, the origin is surrounded by infinitely many open circuits with probability one.

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Invasion percolation

- Assign again *uniform*[0,1] random variables to each edge independently, denoted by $\tau(e)$ for an edge e.
- The invaded region $\mathcal{S}(v)$ of a vertex v is defined as the increasing union of subgraphs $\mathcal{S}_n(v)$, where
  - $\mathcal{S}_0(v) = \{v\}$
  - $\mathcal{S}_{n+1}(v)$ is $\mathcal{S}_n(v)$ together with the lowest edge not in $\mathcal{S}_n(v)$ but incident to some vertex in $\mathcal{S}_n$
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Results for invasion percolation

- For all $p > p_c$ we have that $\mathcal{S} \cap \mathcal{C}_p \neq \emptyset$, since the origin is surrounded by infinitely many p-open circuit.
- It is clear from the invasion mechanism that if $\mathcal{S}_n \cap \mathcal{C}_p \neq \emptyset$ for some $n > 0$ then $\mathcal{S} \setminus \mathcal{S}_n \subset \mathcal{C}_p$
- By other words: If for any p the invasion hits the infinite p-open cluster, it will never leave this cluster again.
- If e_i is the edge invaded at time i then $\limsup_{i\to\infty} \tau(e_i) = p_c$.
- Since there is no percolation at p_c we get that $\hat{\tau} = \max_{e \in E_\infty} \tau(e)$ exists and is greater than p_c.

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Definition of the first pond

- Let $\hat{e}$ be the edge where the maximum value of τ is taken, namely $\tau(\hat{e}) = \hat{\tau}$.
- $\hat{e}$ exists and it is well-defined with probability 1.
- Suppose that $\hat{e}$ is added to the invasion at time $i+1$, then the graph $\mathcal{S}_i = \hat{V}_1$ is called the first pond of the invasion or the first pond of the origin.

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Further ponds

- Assume that the first pond is $\mathcal{S}_{i_1}$ for some i_1.
- Then $\max_{e_i \in E_\infty, i > i_1} \tau(e)$ exists and greater than p_c.
- Let $\hat{e}_2$ be the edge where this value is taken.
- If $\hat{e}_2$ is invaded at time $i_2 + 1$, than the graph $\mathcal{S}_{i_2} \setminus \mathcal{S}_{i_1}$ is the second pond of the invasion.
- The other ponds of the invasion can be defined in similar way.

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