Scars on compact Quantum Graphs

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Quantum ergodicity:

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"We expect that In Systems in wich the classical dynamics is ergodic, the probability measure associated to the square-moduli of the wavefunction converges to the classical invariant measure as one approachs to the semi-classical limit (large eigenvalues limit) through almost all the sequences of eigenstates."
(Result proofed in special cases)

Conjecture (Universality in spectral statistics):

- Chaotic classical dynamic: Random Matrix Theory
- Regular Classical dynamic: Poisson process

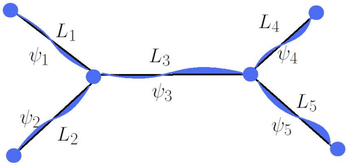
Scars

Def: Localization of the wave function for a certain sequence of states somewhere near an periodic orbit.

Even the classical dynamics is ergodic, existence of scars indicates that the semiclassical limit could be not quantum ergodic.

Scarring Phenomena is not completely understood.

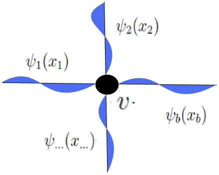
Quantum Graphs



- For each edge $e \mapsto [0, L_e] \subset \mathbb{R}$ & a Hilbert Space $L^2[0, L_e]$.
- $\Psi = (\psi_1(x_1), \psi_2(x_2), ..., \psi_b(x_b))$, **wave function** on the graph, ia the solution to the equation with eigenvalue k_n^2:

$$\Delta\Psi^{(n)} = \left(-\frac{d^2\psi_1^{(n)}}{dx_1^2}, ..., -\frac{d^2\psi_b^{(n)}}{dx_b^2} \right) = k_n^2 \Psi^{(n)}$$

And contour conditions at the vertices:



- Continuity at the vertices :

$$\psi_1^{(n)}(v) = \psi_2^{(n)}(v) = ... = \psi_b^{(n)}(v)$$

- Null flow of probability at the vertices :

$$\frac{d\psi_1^{(n)}}{dx_1}|_v + ... + \frac{d\psi_b^{(n)}}{dx_b}|_v = 0$$

Quantum probability on a graph Γ

Remember that the square-moduli of the wave function is density of probablility:

$$\langle \Psi^{(n)}, \Psi^{(n)} \rangle = \sum_{elos} \int_0^{L_{elo}} |\psi_{elo}^{(n)}|^2 = 1$$

Definition:

$$\mathbb{P}_n : \Gamma \times \{k_n^2\}$$

$$\mathbb{P}_n(\Lambda \subset \Gamma) = \sum_{edges \in \Lambda} \int_0^{L_{edge}} |\psi_{edge}^{(n)}|^2 dx_{edge}$$

Scars on Graphs

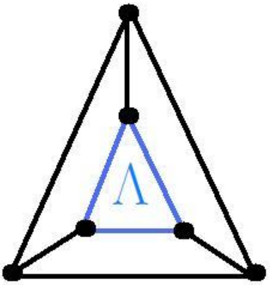
Definition:

An scar is a subgraph $\Lambda \subset \Gamma$ with the property that :

$$\exists (k_n^2) \subset \sigma(\Delta) \ \textit{such that}$$

$$\lim_{n\to\infty} \mathbb{P}_n(\Lambda) = 1 \ \ \& \ \ \forall \textit{edge} \in \Lambda$$

$$\liminf_{n\to\infty} \mathbb{P}_n(\textit{edge}) > 0$$



Definition: $x_1, ..., x_n \in \mathbb{R}$ are **rationaly independent** if

$$\sum_{i=1}^{n} q_i x_i = 0, \text{ com } q_1, ..., q_n \in \mathbb{Q} \Rightarrow$$

$q_{i's} = 0.$

example of scars

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Example of scars

Assume Γ star graph whose lenght of the edges are rationaly independent. Then any subraph consting of 2 or more edges is an scar.

Question

If we see the star graph as subgraph of another graph this properties holds?

Definition: *The degree of a vertice =$\#$ edges + 2x $\#$ loops.*

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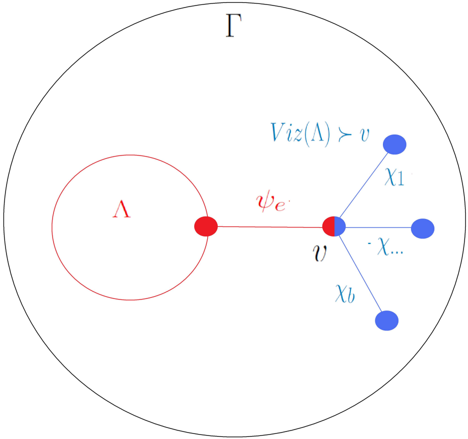
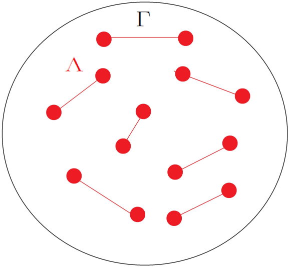
Theorem:
An scar $\Lambda \subset \Gamma$ can not have a end-edge that is interior with respect to Γ.

The situation

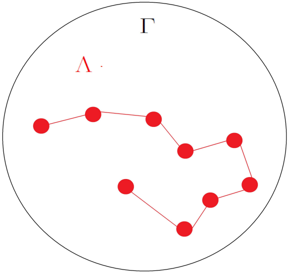


Consequences

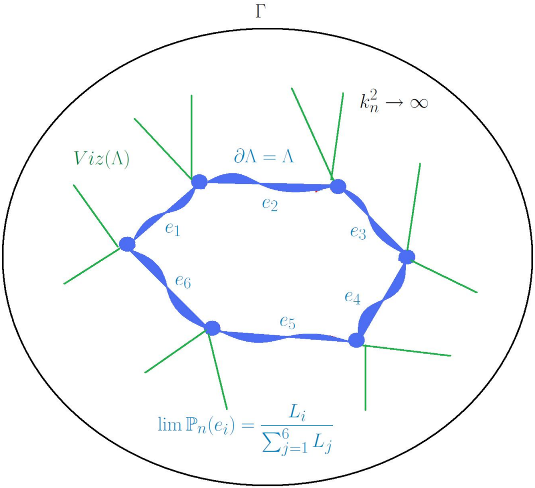


In other words, if disconected edges are concetrating the probablity, then must exist a path whose edges have probability bounded away form 0 conecting the disconected edges.

Consequences



Closed paths and the spectrum



Necessary conditions on the spectrum for scars

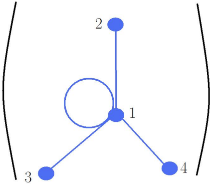
For a closed path be a scar is necessary a subsquence (k_n^2) such that:
For all edge $f \in \Lambda$ has $k_n L_f$ **mod** $\pi \to 0$.

Conectivity Matrix

Definition:

$\mathcal{C}(\Gamma) = [\mathcal{C}_{i,j}] = b + 2l$ if v_i and v_j are b edges and l loops

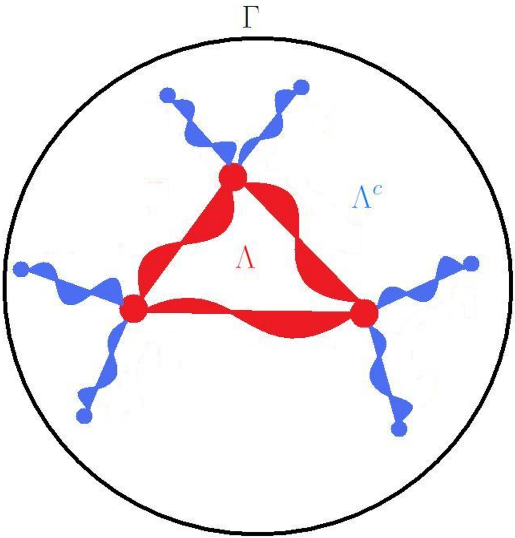
or 0 otherwise.



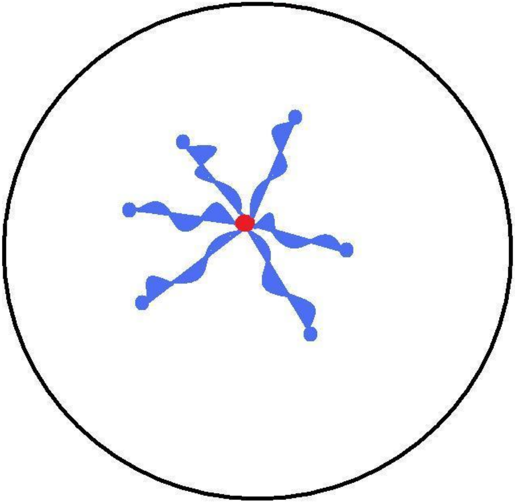
$$= \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Collapse of subgraphs

Definition: The collapse of a subgraph Λ denoted by $\Gamma \ominus \Lambda$:



$\Gamma \ominus \Lambda$



Sufficient Conditions

Theorem:
Assume that Λ is a closed path of a graph Γ such that

$$\det \mathcal{D}(\Gamma \ominus \Lambda) \neq 0$$

For Λ be a scar are enough conditions:

$$\exists (k_n^2) \subset \sigma(\Delta);$$

1. $\forall e \in \Lambda$, $k_n L_e \textbf{ mod } 2\pi \to 0$
2. $\forall e \in \Lambda^c$, $k_n L_e \textbf{ mod } 2\pi \to \frac{\pi}{2}$

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