The high temperature Ising model on the triangular lattice is a critical percolation model

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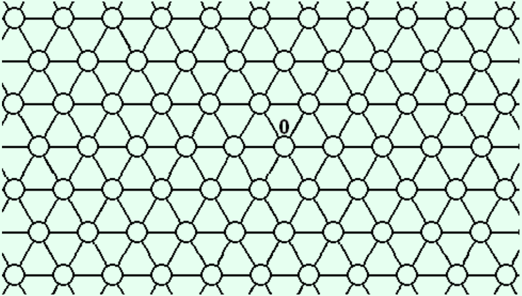
1. The Ising model on $\mathbb{T}$
   - The lattice $\mathbb{T}$ and the model
   - Random-cluster representation
2. Our model
   - Definition
   - Results
   - Related results, open questions

The triangular lattice $\mathbb{T}$



Vertex set $\mathcal{V}_{\mathbb{T}}$, edge set $\mathcal{E}_{\mathbb{T}}$,
special vertex: $\mathbf{0}$, called *origin*

The Ising model on $\mathbb{T}$

- Dependent site percolation model, configuration space: $\Omega := \{-1, +1\}^{\mathcal{V}_{\mathbb{T}}}$.
- Two parameters: inverse temperature $\beta \geq 0$, external field h; we only consider $h = 0$.
- Standard: $\exists \beta_c \in (0, \infty)$ s.t. for $\beta < \beta_c$, there exists a unique *Ising Gibbs measure* on Ω, which we denote by μ_β.
- A way to obtain μ_β (through the *random-cluster representation* of the Ising model) will follow.

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Random-cluster measures on finite graphs

$G = (\mathcal{V}, \mathcal{E})$ a finite graph. For $\eta \in \{0,1\}^{\mathcal{E}}$, we define

- the set of open edges in η: $\mathcal{E}_{\text{open}}(\eta) = \{e \in \mathcal{E} : \eta(e) = 1\}$,
- $k(\eta)$ as the number of components in the graph $G_{\text{open}}(\eta) = (\mathcal{V}, \mathcal{E}_{\text{open}}(\eta))$.

For $p \in (0,1), q > 0$, we define the *random-cluster measure* $\Phi^G_{p,q}$ so that

$$\Phi^G_{p,q}(\eta) \propto \prod_{e \in \mathcal{E}} p^{\eta(e)}(1-p)^{1-\eta(e)} q^{k(\eta)}.$$

Components in $G_{\text{open}}(\eta)$ are called *(open) FK clusters (in η)*.

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Example

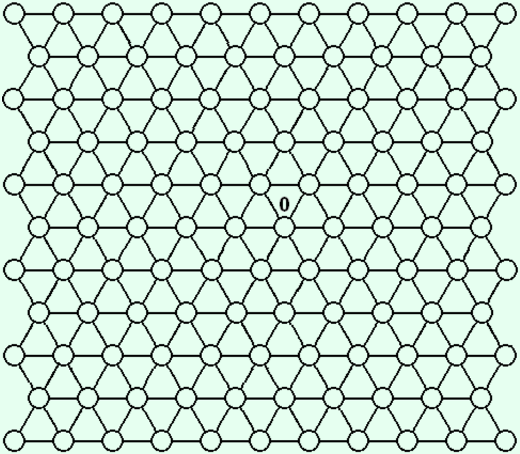

Figure: The graph

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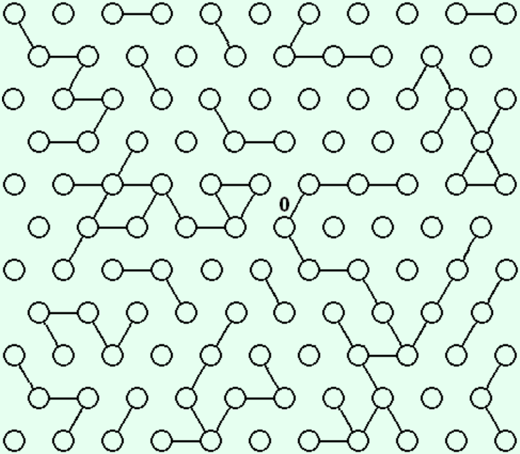

Figure: $p = 1/4, q = 1$

Properties of the random-cluster measure $\Phi_{p,2}$

A random-cluster measure $\Phi_{p,2}$ on $\mathbb{T}$ is obtained as a limit of $\Phi^G_{p,2}$ as $G \uparrow \mathbb{T}$.

Theorem

There exists a critical value $0 < p_c < 1$ such that for a random-cluster measure $\Phi_{p,2}$,

$$\begin{cases} \Phi_{p,2}(\eta : \exists\, \infty \text{ FK cluster in } \eta) = 0 & \text{if } p < p_c, \\ \Phi_{p,2}(\eta : \exists\, \infty \text{ FK cluster in } \eta) = 1 & \text{if } p > p_c. \end{cases}$$

If $p < p_c$, then $\Phi_{p,2}$ is unique.

Connection between models

μ_β: Ising measure, β_c: critical inverse temperature in the Ising model on $\mathbb{T}$.

Theorem (The random-cluster representation of the Ising model)

For $\beta < \beta_c$, a configuration in Ω distributed according to μ_β can be obtained by the following procedure.

- *Set $p = 1 - e^{-\beta}$.*
- *Step 1: Draw a bond configuration η with distribution $\Phi_{p,2}$.*
- *Step 2: For each FK cluster in η, assign to all vertices in the cluster spin $+$ with probability $1/2$ and spin $-$ with probability $1/2$, independently for different FK clusters.*

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Definition of the model

- Fix $\beta < \beta_c$, $r \in [0,1]$, and set $p = 1 - e^{-\beta}$.
- Step 1: Draw a bond configuration η with distribution $\Phi_{p,2}$.
- Step 2: For each FK cluster in η, assign to all vertices in the cluster spin $+$ with probability r, and spin $-$ with probability $1 - r$, independently for different FK clusters.
- Denote the resulting joint measure on $\{0,1\}^{\mathcal{E}_{\mathbb{T}}} \times \Omega$ by $\mathbb{P}_{\beta,r}$. Note:
  - the marginal of $\mathbb{P}_{\beta,1/2}$ on Ω is μ_β,
  - $\beta = 0 \leftrightarrow$ Bernoulli site percolation with parameter r.

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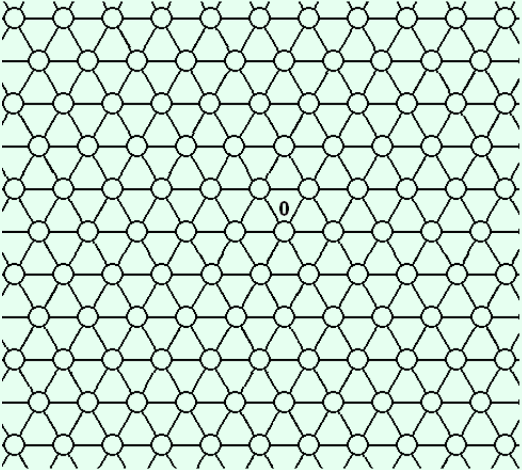

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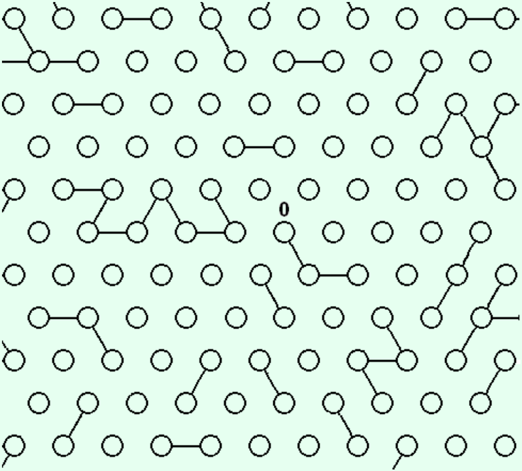

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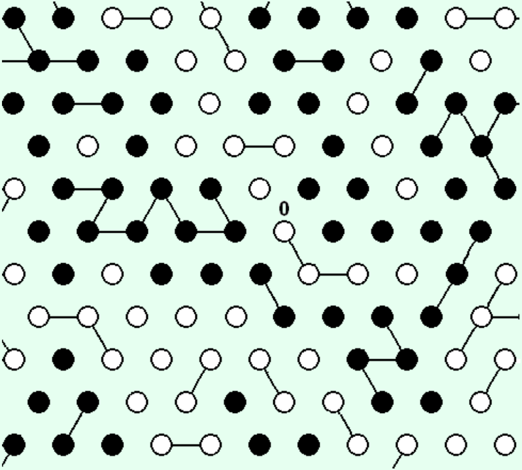

- Question: Is the origin in an infinite (+)-cluster?
- $\Theta(\beta, r) := \mathbb{P}_{\beta,r}(\mathbf{0}$ is in an infinite (+)-cluster).
- Clear: For all β, $\Theta(\beta, 0) = 0$, $\Theta(\beta, 1) = 1$, $\Theta(\beta, r)$ is increasing in r.
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Main results

Theorem (Bálint, Camia, Meester, 2008)

For all $\beta < \beta_c$,

$$r_c(\beta) = 1/2.$$

Moreover, the phase transition at $r = 1/2$ is sharp, i.e.,

- *If $r < 1/2$, the distribution of the size of the $(+)$-cluster of the origin has an exponentially decaying tail.*
- *If $r = 1/2$, $\Theta(\beta, 1/2) = 0$ and the mean size of the $(+)$-cluster of the origin is infinite.*
- *If $r > 1/2$, there exists a.s. a unique infinite $(+)$-cluster.*

Theorem (Bálint, Camia, Meester, 2008)

For each $\beta < \beta_c$, the function $\Theta(\beta, r)$ is continuous in r.

Related results, conjectures

Conjecture

The high temperature ($\beta < \beta_c$) Ising model on the *triangular lattice* $\mathbb{T}$ with no external field shows critical behaviour and is in the universality class of Bernoulli (independent) percolation.

- Main result (i.e., $r_c(\beta) = 1/2$) confirms criticality.
- Choice of $\mathbb{T}$ is important. High temperature Ising model on $\mathbb{Z}^2$ is subcritical:
  - for $\beta = 0$, known: $r_c^{\mathbb{Z}^2}(0) = r_c^{\mathbb{Z}^2,\mathrm{Bernoulli}} > 1/2$;
  - for all $\beta < \beta_c$, the distribution of the size of the (+)-cluster of the origin has an exponentially decaying tail (Higuchi (1993), van den Berg (2007)), and this implies $r_c(\beta) > 1/2$.

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Open questions

- What else can be proved on the square lattice?
  (We expect $r_c(\beta) + r_c^*(\beta) = 1$,
  where $r_c^*(\beta)$ is the critical value for the matching lattice of $\mathbb{Z}^2$.)
- What happens if $\Phi_{p,2}$ is replaced by $\Phi_{p,q}$ in the definition of the model?
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