§1 Review : limit theorems for iid rv

. $X_1 \dots X_N$ N sample of i.i.d rv

. μ : common distribution.

. $S_N = X_1 + \dots + X_N$ sum

. $M_N = \max(X_1, \dots X_N)$ max

. $X^{(1)} \geq \dots \geq X^{(N)}$ ~~ex~~ order statistics

$(X^{(1)} = M_N)$

§1.a Behavior of S_N :

1) If, for some normalization constants $a_N \in \mathbb{R}$ and $b_N > 0$, $\frac{S_N - a_N}{b_N}$ converges in distribution then the limit law is a stable law, $\alpha \in [0,2]$ ($\alpha = 2$: Gaussian)

2) The domain of attraction of stable laws $\alpha \in (0,2]$ is well known as well as the possible normalizing constants a_N, b_N.

μ is in the domain of attraction of an α-stable law $(0<\alpha<2)$ iff:

$$\mu(x,\infty) \approx \frac{L(x)}{x^\alpha}$$

L slowly varying

then one can choose:

$$a_N = Nm \quad \text{if} \quad 1<\alpha<2 \quad . \quad m = \int x \, d\mu(x)$$

$$a_N = 0 \quad \text{if} \quad \alpha<1$$

and

$$b_N = \inf\left(x, \ \mu(x,\infty) \geq \tfrac{1}{N}\right)$$

$$b_N \sim \cancel{} L_0(N) \, N^{1/\alpha} \quad L_0 \text{ slowly varying.}$$

—

Rk: μ is in the domain of normal attraction

if $\mu(x,\infty) \sim \dfrac{K}{x^\alpha}$ as $x \to \infty$

then $b_N \sim c N^{1/\alpha}$.

—

3) Convergence to a subordinator:

Fix $0<\alpha<1$, and μ in the domain of (normal) attraction of α-stable law; then

$$S_N = \frac{1}{b_N} \sum_{1}^{N} X_k \xrightarrow{(d)} \alpha\text{-stable}$$

The process:

$$S_N(t) = \frac{1}{b_N} S_{\lfloor Nt \rfloor}$$

converges in distribution (in $D([0,\infty), \mathbb{R})$)

to an α-stable subordinator $V_\alpha(t)$.

- V_α is a Levy process: independent increments
- V_α is ~~non decreasing~~ increasing
- Laplace transform: $E(e^{-\lambda V_\alpha(t)}) = e^{-ct\lambda^\alpha}$

An important property of stable subordinators

The Arc-sine law.

Theorem: Let $T_\alpha(x) = \inf(t, V_\alpha(t) > x)$

be the inverse of the α-stable subordinator

then $V_\alpha(T_\alpha(x)-)/x$ has the ~~distribution~~.

generalized arc-sine distribution

$$\mathbb{P}(V_\alpha(T_\alpha(x)-)/x \le u) = F_\alpha(u)$$

$$F_\alpha(u) = \frac{\sin \alpha\pi}{\pi} \int_0^u u^{\alpha-1}(1-u)^{-\alpha} du$$

Corollary: $0 < a < b < \infty$

Let $\mathcal{R}_\alpha$ be the range of V_α

$$\mathcal{R}_\alpha = V_\alpha([0,\infty))$$

$$\mathbb{P}\left([a,b] \cap \mathcal{R}_\alpha = \emptyset \right) = F_\alpha\left(\frac{a}{b}\right)$$

Pf: $\mathbb{P}\left([a,b] \cap \mathcal{R}_\alpha = \emptyset \right) = \mathbb{P}\left(\frac{V_\alpha(T_a(b)-)}{b} < \frac{a}{b} \right)$

$$= F_\alpha\left(\frac{a}{b}\right)$$

Rk: 1) $\dim \mathcal{R}_\alpha = \alpha \quad \in (0,1)$

2) Important fact: this probability is a function of the Ratio $\frac{a}{b}$.

3) T_α is continuous (even Hölder) with longer & longer "plateaux".

§ 1.b : Extreme values

$X_1 \dots X_N$ i.i.d rv's (≥ 0)

μ common distribution

$M_N = \max(X_1 \dots X_N)$

If, for some $a_N, b_N > 0$, $\frac{M_N - a_N}{b_N}$ converges in distribution then the limit law is of one of three types (extreme value distributions: Gumbel, Frechet, Weibull). Moreover the max-domains of attraction are well known as well as the normalizing constants a_N, b_N.

When $\mu(x, \infty) \sim \frac{L(x)}{x^\alpha}$ $\quad 0 < \alpha < 1$

$$\frac{M_N}{b_N} \overset{(d)}{\Longrightarrow} \text{Weibull.} \qquad b_N = L_0(N)\, N^{1/\alpha}$$

~~So~~ Moreover: $\frac{M_N}{S_N}$ converges in distribution to a non-degenerate rv.

In fact: if $X^{(1)} \geq \dots \geq X^{(N)}$

if $u_N := \left(\frac{X^{(1)}}{S_N}, \dots, \frac{X^{(N)}}{S_N}, 0 \dots \right)$

$$u_N \in \mathcal{S} = \left\{ u = (x^i)_{i \geq 1, \dots} ;\ x^i \geq x^{i+1},\ \Sigma x^i \leq 1 \right\}$$

u_N converges in distribution in $\mathcal{S}$ to the Poisson-Dirichlet process. $PD(\alpha)$.

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§2: Fractional Kinetics:

Definition:

Let $B_d(t)$ d-dimensional BM

& $V_\alpha(t)$ α-stable subordinator $0<\alpha<1$

$T_\alpha(s) = \inf(t, V_\alpha(t) > s)$ its inverse

B_d & V_α independent.

Fractional-Kinetics process $FK_{d,\alpha}$

$$Z_{d\alpha}(s) = B_d(T_\alpha(s))$$

Properties:

1) $Z_{d,\alpha}$ has continuous paths (Hölder)

2) $Z_{d\alpha}$ not Markov

3) Fixed-time marginals are smooth.

$$E\left(e^{i\xi Z_{d\alpha}(t)}\right) = E_\alpha(-|\xi|^2 t^\alpha)$$

E_α Mittag Leffler

$$E_\alpha(z) = \sum_{m=0}^{\infty} z^m / \Gamma(1+m\alpha)$$

4) $Z_{d\alpha}$ is self-similar

$$Z_{d\alpha}(t) \overset{(d)}{=} \lambda^{-\alpha/2} Z_{d,\alpha}(\lambda t)$$

Rk: Why Fractional - Kinetics ?

The probability density of $Z_{d,\alpha}(t)$ satisfies the Fractional Kinetic equation.

$$\frac{\partial^\alpha}{\partial t^\alpha} p(t,x) = \frac{1}{2} \Delta p(t,x) + \delta(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

Here: $\frac{\partial^\alpha p}{\partial t^\alpha}$ is the inverse Laplace transform of $s^\alpha \hat{p}(s,x)$ where $\hat{p}(s,x) = \int_0^\infty e^{-st} p(t,x)\,dt$

—

Question: Probability that $Z_{d,\alpha}$ does not ~~jump~~ move during the interval $[a,b]$?

Answer: $F_\alpha\left(\frac{a}{b}\right)$!

(Arcsine law for the α-stable subordinator)

§2 A first model of trapped RW's
the CTRW of Montroll-Weiss '60 s.

On $\mathbb{Z}^d$, $X(t)$ moves as a SRW

but waits at each site a time μ-distributed.

(Resample at each visit !) (= annealed)

Assume $\mu(x,\infty) \sim \frac{K}{x^\alpha}$ $0<\alpha<1$

Theorem: For any $d \geq 1$

$$\frac{1}{n^{\alpha/2}} X(nt) \overset{(d)}{\Longrightarrow} c . Z_{d,\alpha}(t)$$

Rk: • c constant

• Convergence at process level.

(Meerschaert- Scheffler J. Appl. Proba 04)

Proof: 1) CLOCK PROCESS. Converges to subordinator

$$S_k = \text{time of the } k^{th} \text{ jump}$$
$$= \sum_1^{k-1} s_i \qquad s_i \text{ i.i.d } (\mu)$$

$$S_N(t) := S_{\lfloor Nt \rfloor}$$

$$\frac{1}{N^{1/\alpha}} S_N(t) \overset{(d)}{\Longrightarrow} c . V_\alpha(t)$$

2) The CTRW $X(t)$ is a SRW time-changed by the clock process.

$$X(t) = Y(k) \quad \text{if} \quad S_k \leq t < S_{k+1}$$

Y is a (discrete-time) SRW on $\mathbb{Z}^d$

3) Y and S are independent.

and both have a scaling limit (resp. B_d & V_α).

—

Conclusion: $FK_{d,\alpha}$ is the scaling limit of "completely annealed" trapping mechanisms.

here: "trapping independent of path."

We will see that this trivial limit is widely universal, even when

- Quenched disorder & trapping
- Heavy correlations
- No heavy tails !

§3 The Bouchaud Trap Model

Definition of the BTM on a general graph.

- $G = (V, E)$ graph
- ~~Zdeka~~ τ $V \to (0, \infty)$

  τ_x : "depth of trap" at $x \in V$

  τ_x measure on V

- $X(t)$ Continuous Time Markov Chain

  Jump rates:

$$c(x,y) = \tau_x^{a-1} \tau_y^{a} \qquad a \in [0,1]$$

  Detailed balance:

$$\tau_x c(x,y) = \tau_x^a \tau_y^a = \tau_y c(y,x)$$

so: τ reversible ($\forall a \in [0,1]$).

- Simplest case: $a = 0$ RHT

  (Random Hopping Times).

$$c(x,y) \cancel{\tau_x} = \frac{1}{\tau_x} \quad \text{ind}^t \text{ of } y.$$

$X(t)$ is a time-change of ~~a~~ SRW

- Clock process

S_k : time of k^{th} jump

$$\begin{cases} X(t) = Y_k & S_k \le t < S_{k+1} \\ Y_k \text{ SRW on } G. \end{cases}$$

Now: S_k sums of the τ_x's along the path of the SRW Y_k.

- Interesting case: τ_x's very heterogeneous

τ_x's i.i.d rv in the domain of attraction of α-stable $0<\alpha<1$.

Theorem: $\mathbb{Z}^d$, $d \geq 2$, Quenched disorder, $a=0$.
(τ_x's fixed)

1) The scaling limit of the clock process is α-stable subordinator

2) The scaling limit of $X(t)$ is Fractional Kinetics

$d \geq 3$
$$\frac{C_d(\alpha)}{n^{\alpha/2}} X(n\cdot) \overset{(d)}{\Longrightarrow} FK_{d,\alpha}$$
$$\frac{1}{n} S(n^\alpha s) \overset{(d)}{\Longrightarrow} V_\alpha(s)$$

$d = 2$
$$\frac{C_2(\alpha)}{n^{\alpha/2}(\log n)^{(1-\alpha)/2}} X(n\cdot) \overset{(d)}{\Longrightarrow} FK_{2,\alpha}$$
$$\frac{1}{n} S(n^\alpha (\log n)^{1-\alpha} s) \overset{(d)}{\Longrightarrow} V_\alpha(s)$$

Proof: . BA - Mountford - Cerny PTRF 06
. BA - Cerny Annals Proba '07

Corollary : Aging for Bouchaud Trap Model
($d \geq 2$, $a = 0$, $0 < \alpha < 1$)

1) $R(t, t+s) := \mathbb{P}(X(t) = X(t+s))$

$$\lim_{t \to \infty} R(t, t + \theta t) = F_\alpha\left(\frac{1}{1+\theta}\right).$$

—

~~Proof: [illegible]~~

~~[illegible]~~

2) $\Pi(t, t+s) = \mathbb{P}(X(t) = X(t+u) \quad \forall\, 0 \leq u \leq s)$

$d \geq 3$ $\quad \lim_{t \to \infty} \Pi(t, t + \theta t) = \Pi_{d,\alpha}(\theta)$

$d = 2$ $\quad \lim_{t \to \infty} \Pi\left(t, t + \theta \frac{t}{\log t}\right) = \Pi_{2,\alpha}(\theta)$

& $\lim \Pi_{d,\alpha}(\theta) = F_\alpha(1/1+\theta)$

Pf: Coarse-Graining of the path of the SRW Y_E.

$d=2$.

- Disk $D(n)$ area $m2^n n^{1-\alpha}$ centered at 0

  Radius $R_n = \sqrt{\frac{m}{\pi} 2^n n^{1-\alpha}}$

- Y makes $O(R_n^2)$ steps in $D(n)$ before exiting and visits $O(R_n^2/\log R_n^2)$ sites (traps)

- So deepest trap <u>visited</u> has depth

$$O\left(\left(\frac{R_n^2}{\ln R_n^2}\right)^{1/\alpha}\right) = O(2^{n/\alpha} n^{-1})$$

Call deep traps:

$$T_\varepsilon^M = \{ x \in D(n), \ \varepsilon 2^{n/\alpha} n^{-1} \leq \tau_x < M 2^{n/\alpha} n^{-1} \}$$

- Cut trajectory of Y is small pieces. Each piece ends when Y exits for the first time the disk of area $2^n n^\gamma$ around its initial point, $\gamma < 1-\alpha$

- ~~Proc~~ "Score" of a piece = time spent in the deep traps.

- Prove : Scores are "approximately" i.i.d rv's , in the domain of attraction of α- stable.
- Then : Clock process $\approx$ sum of scores
- So Clock process converges to subordinator
- Remains : independence between clock process & location of SRW !

Reason: Many deep traps to chose from.

$X(\cdot)$ Does not care about the deepest traps.

$d=2$. After 2^{2n} steps, $X(\cdot)$ gets to the distance 2^n and visits $O\left(\frac{2^{2n}}{\log 2^{2n}}\right)$

$= O(2^{2n}/n)$ sites

So the <u>deepest</u> trap visited in first 2^{2n} steps has a depth $O\left(\frac{2^{2n/\alpha}}{n^{1/\alpha}}\right) \ll 2^{2n/\alpha}$ (= depth of the deepest trap at distance $\leq 2^n$).

Eventually X will visit the deepest trap in the Disk of radius 2^n, but it will then be too late for this trap to be relevant, since much deeper traps will then have been found.

Bouchaud trap model on large tori
& multiple time scales / depth scales.

Consider the BTM, $a=0$, on G_n = 2^n-torus in $\mathbb{Z}^d$.

Again: τ_x's are i.i.d & in domain of attraction of an α-stable $0<\alpha<1$

$$P(\tau_x \geq u) \sim \frac{K}{u^\alpha}$$

(GBA-Cerny CPAM 08)

Theorem: $d=2$, $0<\gamma<1/6$

Let $t_\gamma(n) = 2^{2n/\alpha}\, n^{1-\frac{\gamma}{\alpha}}$

then for a.e τ's:

$$\lim R(t_\gamma(n), (1+\theta)t_\gamma(n)) = F_\alpha\left(\frac{1}{1+\theta}\right)$$

Rk: For each γ, i.e each time scale $t_\gamma(n)$, a different depth of traps is relevant, i.e

$$\text{depth}_\gamma(n) = 2^{2n/\alpha}\, n^{-\gamma/\alpha}$$

$\gamma \to 0$ longest possible time scales before equilibrium

$\gamma > 1$ would follow from results on $\mathbb{Z}^2$: time scale too short to see the difference

$\frac{1}{6} < \gamma < 1$: theorem must be true. (not yet done)

The Bouchaud Trap Model on $\mathbb{Z}$

Here $G = \mathbb{Z}$, n.n

the τ_x's are i.i.d

$$P(\tau_x \geq u) \sim \frac{K}{u^\alpha} \qquad 0 < \alpha < 1$$

$X(t)$ CTMC, with jump rates

$$c(x,y) = \tau_x^{a-1} \tau_y^{a} \qquad 0 \leq a \leq 1$$

Theorem: Scaling limit of BTM in $d=1$. (Averaged).

$$\frac{1}{N^{\alpha/\alpha+1}} X(N\cdot) \xrightarrow{(d)} FIN_\alpha(\cdot)$$

(Fontes Isopi Newman '02 Ann. Proba, $a=0$
GBA - Černý '05, Ann Appl. Proba, $a \neq 0$)

Definition of the limit process $FIN_\alpha(\cdot)$

1) $B(t)$ BM $d=1$; $\ell(t,x)$ its local time

2) ρ random measure given by $\rho = \sum_i a_i \delta_{b_i}$

where $(a_i, b_i)_i$ is a Poisson Point Process on

$\mathbb{R} \times (0,\infty)$ with intensity $dx \otimes \alpha \frac{dv}{v^{\alpha+1}}$

3) $\varphi(u) = \int \ell(u,y)\, \rho(dy) = \sum_i a_i \ell(u, b_i)$

$\psi(t) = \inf(u, \varphi(u) > t)$ inverse of φ

4) $FIN_\alpha(t) = B(\psi(t))$.

1/ Note that the scaling limit is again a time change of BM, but now the clock φ depends on the path of the BM.

2/ Note the different scaling exponent $\alpha/_{\alpha+1}$ vs $\alpha/2$ ($\alpha/_{\alpha+1} > \alpha/2$).

Properties of FIN_α :

- Atoms of ρ are dense
- Conditionally on ρ : $FIN_\alpha(t)$ has a discrete law

$$\mu_t^\rho = \sum w_i(t)\, \delta_{a_i}$$

- FIN_α is a (singular) diffusion : Markov + continuous sample paths
- Scaling : $FIN_\alpha(t) \overset{(d)}{=} \frac{1}{\lambda} FIN_\alpha(\lambda^{\frac{1+\alpha}{\alpha}} t)$
- Anomalous diffusion:

$$P(FIN_\alpha(t) \geq x) \leq C \exp - c\left(\frac{x}{t^{\alpha/1+\alpha}}\right)^{1+\alpha}$$

FIN_α vs FK_α

- Markov & Singular vs Non Markov & Smooth
- FIN_α keeps random environment, feels <u>deepest</u> traps

  FK_α averages random environment, does not feel deepest traps.

§4 : The RTRW

(Randomly Trapped Random Walk)

Model introduced in R. Royfman's thesis ('08) to understand the transition FIN / FK & the scaling limit of RW on critical trees.

The Model: • $G = (V, E)$ Graph

• $Z(k)$, $k = 0, 1 \ldots$ SRW on G

• ν map from V to the set of probability measures on $(0, \infty)$ (or $(1, \infty)$ to avoid technicality)

$$\nu : V \to M_1(0, \infty)$$
$$x \mapsto \nu_x$$

Call $\tau_x = \int_0^\infty t \, \nu_x(dt)$ the mean of ν_x

(assume $\tau_x < \infty$, for every $x \in V$).

We now define the Trapped Random Walk TRW(G, ν) as the continuous time process $X(t)$, which waits at site x a random time ~~drawn~~ sampled from ν_x (independently at each visit) and then jumps to a nearest neighbor chosen u.a.r.

Other possible description of RTW (G, ν):

Time change of the SRW :

1) $\forall z \in V$ $\quad s_z(i)$ $\quad i = 1, 2 \dots$ independent distribution ν_z

2) Clock process:

$$S(k) := \sum_{i=1}^{k} s_{Z(i-1)}(i)$$

$S(k)$ is the time of the k^{th} jump of X

i.e : $X(t) = Z(k)$ for $S(k) \leq t < S(k+1)$

—

Definition: RTRW (G, Q)

Let $Q \in M_1(M_1(0,\infty))$ and choose the ν_x's i.i.d with distribution Q, then consider $X^{\nu}(t)$ the RTRW (G, ν).

This model contains the two previously introduced models: the trivial CTRW & the BTM.

In deed:

1) Choose $Q = \delta_\nu$ for a fixed $\nu \in M_1(0,\infty)$

i.e $\forall x \in V \quad \nu_x \equiv \nu$

This is the CTRW discussed above.

So if ν is in the domain of attraction of an α-stable, the scaling limit is $FK_{d,\alpha}$ (for $G = \mathbb{Z}^d$ }, $d \geq 1$)

2) Choose: ν_x = exponential dist. with mean τ_x

and τ_x's i.i.d rv.

This is the Bouchaud Trap Model $(a=0)$ discussed above.

So if the distribution of the τ_x's is in the domain of attraction of an α-stable $0<\alpha<1$ the scaling limit is $FK_{d,\alpha}$ for $G = \mathbb{Z}^d$ $d \geq 2$ and FIN_α for $G = \mathbb{Z}$, $d=1$.

We will now concentrate on scaling limits of RTRW in $d=1$ and show that $FK_{1,\alpha}$ & FIN_α are not the only possible such limits.

Randomly Trapped Brownian Motions:

Let $\mathcal{P} = (a_i, b_i)_i \in \mathbb{R} \times \mathbb{R}_+$ countable

& $j_i \quad [0,\infty) \to [0,\infty)$ non-decreasing

$$j_i(0) = 0 \qquad j_i(\infty) = \infty \qquad (\cancel{j_i =})$$

Define for $\gamma > 0$:

$$\varphi_{\mathcal{P},\underline{j}}(u) = \sum_i j_i\left(a_i^{-\gamma}\, \ell(u, b_i)\right) a_i^{1+\gamma}$$

where $\ell(u,x)$ is local time of a BM : $B(t)$.

Define the inverse:

$$\psi_{\mathcal{P},\underline{j}}(t) = \inf(u, \varphi_{\mathcal{P},\underline{j}}(u) > t)$$

Consider now the process defined by the time-change of BM :

$$Z_{\mathcal{P},\underline{j}}(t) := B(\psi_{\mathcal{P},\underline{j}}(t))$$

Call $\mathbb{M}_{\mathcal{P},\underline{j}}$ its distribution.

We will choose $\mathcal{P}$ & $\underline{j}$ randomly :

1) $\mathcal{P}$ is PPP with intensity $db\, \frac{da}{a^{\gamma+1}}$

2) Let F be the set of Levy-Kintchine exponents f of subordinators:

$$f(\lambda) = v\lambda + w \int_0^\infty (1 - e^{-\lambda t})\, d\mu(t)$$

with $v, w \geq 0$, $v + w \leq 1$, μ proba on $(0,\infty)$

Finally consider $\mathbb{F}$ a probability measure on F.
and f_i, $i=1,\dots$ i.i.d sample of $\mathbb{F}$

Let j_i : independent Levy subordinators
with Levy-Kintchine exponent f_i

$$E\left(e^{-\lambda j_i(t)}\right) = e^{-t f_i(\lambda)}$$

We now ~~build~~ consider the law $\mathbb{M}_{\beta, \underline{j}}$ of the

process $Z_{\beta, j}(t) = B(\psi_{\beta, j}(t))$

(RTBM $(\gamma, \mathbb{F})$).

Ex 1 : If $\mathbb{F} = \delta_{Id}$ then RTBM$(\gamma, \mathbb{F})$ = FIN_γ

Indeed: $f_i(\lambda) = \lambda \quad \forall i$

So $\quad \varphi(u) = \sum_i a_i^{-\gamma} \ell(u, b_i) a_i^{1+\gamma} = \sum a_i \ell(u, b_i)$

as in FIN_γ.

The class of RTBM's is the natural class
of scaling limits of RTRW's, (when
$P(\tau_x \geq u t) \sim \frac{K}{u^\alpha}$).

Theorem: Scaling limits of RTRW, $d=1$

Define: $\hat{\nu}(\lambda) := \int e^{-\lambda t}\, \nu(dt)$

$$\phi(\rho,\lambda,y) := E\left(e^{-\rho(1-\hat{\nu}(\lambda))} \mid \tau_\nu = y \right)$$

$$\phi_\varepsilon(\rho,\lambda) := \phi\left(\frac{\rho}{\varepsilon}, \lambda\, \varepsilon^{1+\frac{1}{\gamma}}, \varepsilon^{-1/\gamma} \right)$$

Assume: $\lim_{\varepsilon\to 0} \phi_\varepsilon$ exists pointwise $=: \phi(\rho,\lambda)$

Then: there exists a probability measure $\mathbb{F}$ on the set F s.t

$$\phi(\rho,\lambda) := \lim \phi_\varepsilon(\rho,\lambda) = \hat{\mathbb{F}}(\rho,\lambda) = \mathbb{F}\left(e^{-\rho f(\lambda)}\right)$$

and:

$$\frac{1}{N^{\gamma/\gamma+1}}\, X(N\,\cdot) \overset{d}{\Longrightarrow} \text{RTBM defined by } \mathbb{F}$$

—

~~(results ...)~~

Ex 1: BTM: $\hat{\nu}(\lambda) = \dfrac{1}{\lambda\tau+1}$

$$\lim \phi_\varepsilon(\rho,\lambda) = e^{-\rho\lambda}$$

i.e $\mathbb{F} = \delta_{Id}$: recover convergence to FIN_γ

$$P(\tau_x \geq u) \sim \frac{K}{u^\gamma}$$

τ_x mean of ν_x

Theorem : For any $\mathbb{F} \in M_1(F)$

there exists a Q such that : $\lim \phi_\varepsilon = \hat{\mathbb{F}}$

i.e such that the scaling limit of the RTRW(Q) is RTBM$(\mathbb{F})$.

—

$FK_{\gamma,1}$ as a scaling limit of RTRW?

If $\lim \phi_\varepsilon \equiv 1$, then $\mathbb{F} = \delta_0$, all f_i's $\equiv 0$

& φ vanishes identically. Scaling not appropriate!

Let $q_{FK}(\varepsilon) = \Gamma^{-1}(\varepsilon^2)$ where $\Gamma(\varepsilon) = 1 - \hat{U}(\varepsilon)$

with $U(A) = E(\nu(A))$ "average trap"

$\Gamma(0) = 0$, $\Gamma'(0) = E(\tau_\nu) = \infty$.

Hyp: $\Gamma(\varepsilon)$ regularly varying at 0, exponent κ

ex: $\Gamma(\varepsilon) \sim \varepsilon^\kappa$, then $q_{FK}(\varepsilon) = \varepsilon^{-2/\kappa}$.

Theorem : If Γ is reg varying then

$$\frac{1}{N^{\kappa/2}} X(N\cdot) \Rightarrow FK_\kappa$$

under the variance condition:

$$\lim \varepsilon^{-3}\, \mathbb{E}(1 - \hat{\nu}(\varepsilon^{2/\kappa}))^2 = 0.$$

The transition FIN / FK in a simple model:

On $\mathbb{Z}$: $\nu_x = (1-\pi_x)\delta_1 + \pi_x \exp(\bar{\tau}_x)$

with $\pi_x = \frac{1}{\bar{\tau}_x^{\beta}}$, & $P(\bar{\tau}_x > u) \sim \frac{K}{u^{\alpha}}$

$0 < \alpha < 1$

So if $\beta = 0$ this model is BTM

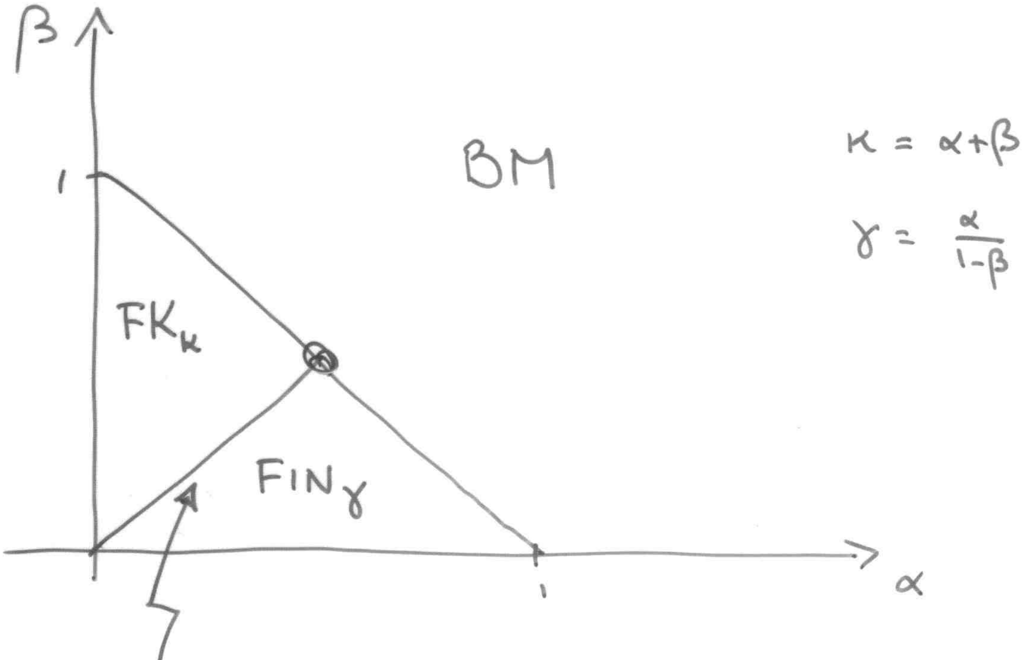
The mean of ν_x: $\tau_x = (1-\pi_x) + \pi_x \bar{\tau}_x$

$$\tau_x \sim \bar{\tau}_x^{\,1-\beta}$$

$$P(\tau_x > u) \sim \frac{K}{u^{\gamma}} \qquad \gamma = \frac{\alpha}{1-\beta}$$

$$\gamma < 1 \iff \alpha + \beta < 1$$

Scaling limit:



what about the scaling limit on the critical line $\alpha = \beta$? RTBM!

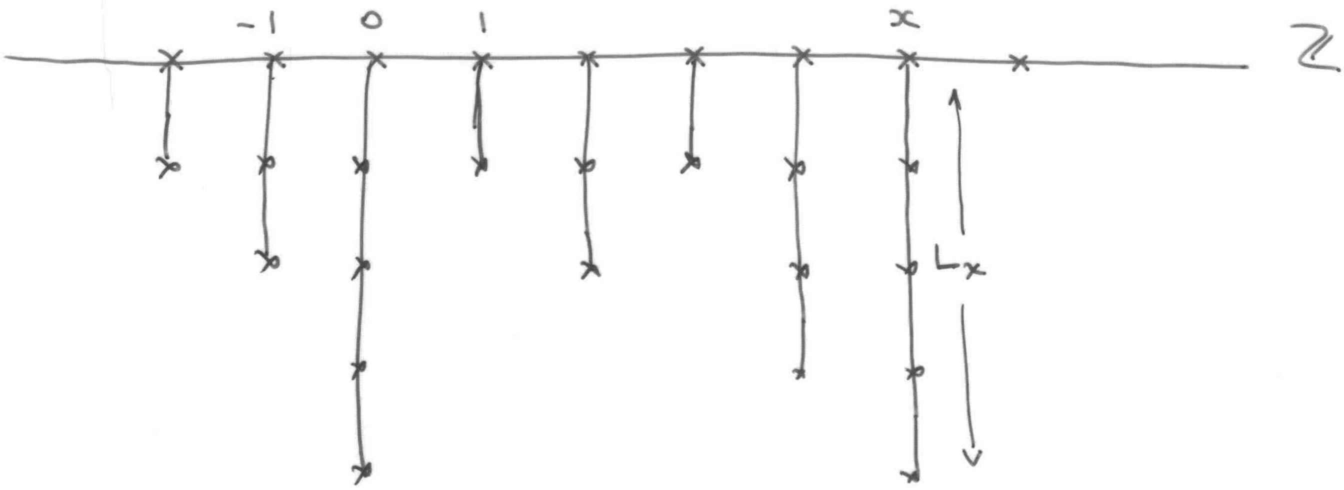
When : β large enough $\alpha < \beta$ $(< 1-\alpha)$
the probability to feel a deep trap $\left(\frac{1}{\tau^{\beta}} \right)$
is small enough to allow the process to
avoid the deepest traps (~~[illegible]~~, and thus
the scaling limit is FK.

When β is too small the process feels the
deepest traps enough to "remember" the random
environment & the scaling limit is FIN.

When β is critical, the process feels "some
of the deepest traps".

Another example: Comb Model

(First "geometric" model)



At each $x \in \mathbb{Z}$: "tooth" of length L_x

L_x heavy tails: $P(L_x = n) = \frac{c}{n^{1+\alpha}}$

$Y(t)$ RW on this "comb".

$X(t)$ = projection of $Y(t)$ on "backbone" $\mathbb{Z}$

This is a RTRW, here

ν_x = Law of $T_{L(x)}$

T_L time for a SRW on $[0, L]$, started at 1 reflected at L, to reach 0.

Theorem: GBA Royfman

1) If $\alpha > 1$

$$\frac{1}{\sqrt{N}} X(N\cdot) \Longrightarrow BM$$

2) If $\alpha < 1$

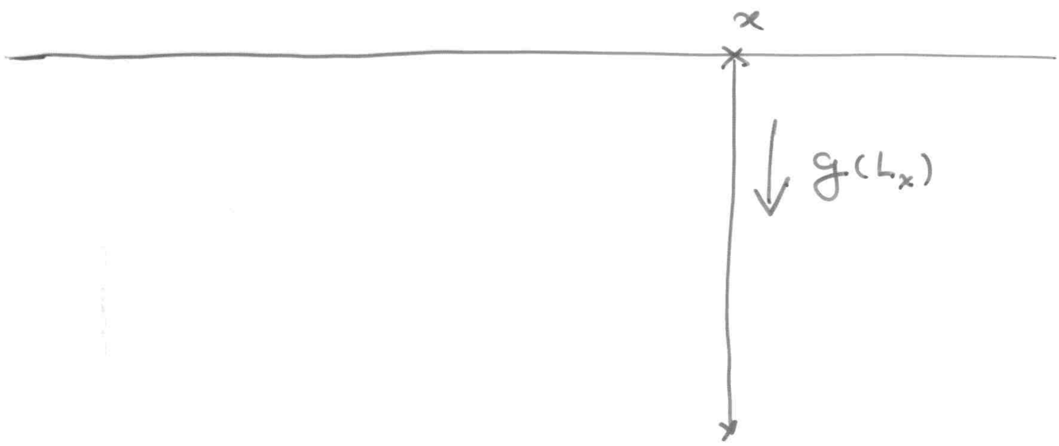
$$\frac{1}{N^{\alpha/2}} X(N\cdot) \Longrightarrow FK_\kappa \qquad \kappa = \frac{1+\alpha}{2}$$

So here, when the traps (="teeth") are sufficiently trapping ($\alpha<1$) the limit is FK

Why: X avoids the largest traps

<u>Short excursions</u> in a very long tooth do not "feel" the length of the tooth

Modify the model: Add a drift $g(L)$ in the teeth, pointing to the end of the teeth

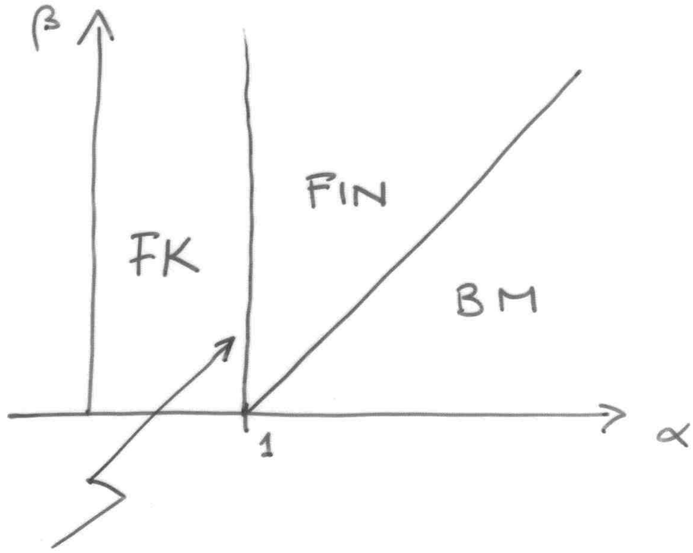


For weak drift, as before, X does not feel deepest traps : FK ?

For large drift : FIN ?

Critical value for $g(L) = \beta \frac{\log L}{L}$

Result :



On the critical line FK / FIN : RTBM ?

No ! can still be FK .

But ~~yes~~ the answer depends on details of the law of the length of teeth L_x .

$$P(L_x \geq u) \sim \frac{L(u)}{u^\alpha}$$

L slowly varying

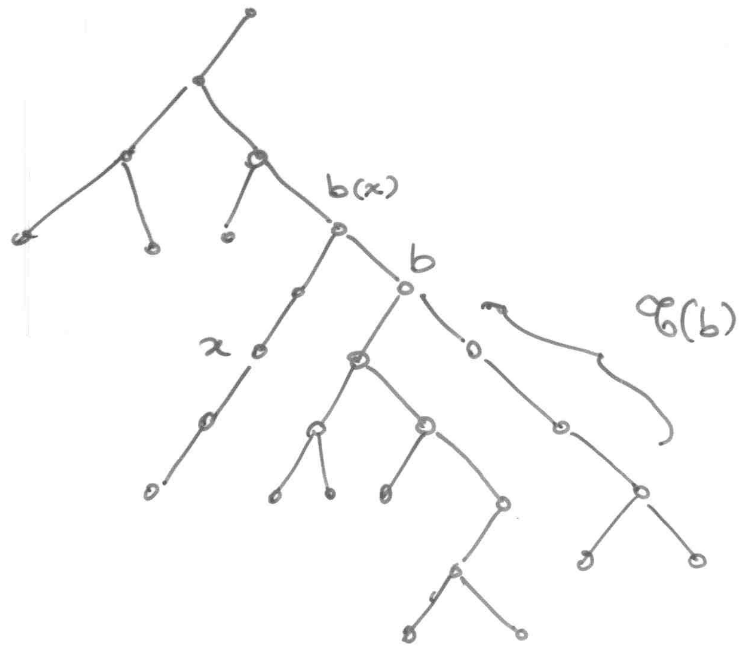
Depends on the function L !

for $L \equiv cst$: FK ; for $L \gg (\ln u)$ RTBM .

§ 5 RW on critical trees.

Consider an Incipient Critical Galton-Watson tree. (Assume offspring distribution has finite variance).

Backbone : unique path

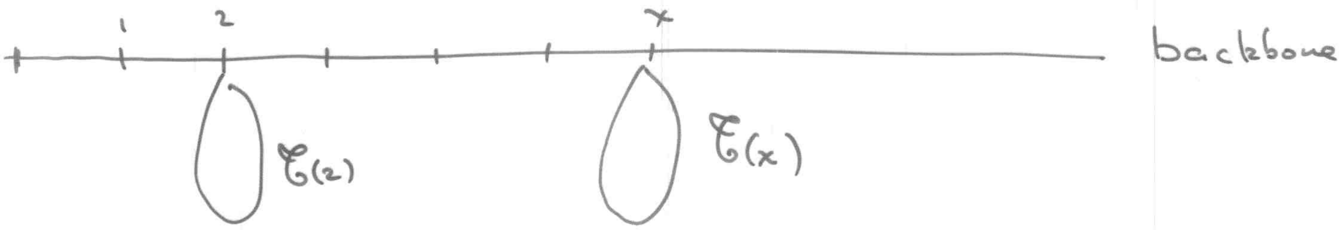


Attached to the backbone: Critical trees. (Finite).

At each $b \in$ Backbone : $\mathcal{T}(b)$ critical tree

Now consider Y RW on Incipient Critical tree, and $X = b(Y)$ its projection on the backbone

This can be seen as a RTRW on $\mathbb{Z}^{(+)}$



The law ν_x is now defined through the law of the time it takes for a RW on a critical tree to reach the root when starting from a child of the root.

It is easy to see that the mean of ~~the~~ this time is (proportional to) the volume of the tree, i.e the total progeny.

It is well known that this is heavy tailed $\alpha = 1/2$.

So scaling limit of X could be FK or FIN or RTBM, depending on the nature of the law ν_x.

~~the~~ For large trees, ν_x "starts as a power law & ends as an exponential" as in the comb model.

~~Theorem:~~

If one wants to compare with the first model

$$P(\text{touch the end of critical tree before getting back to root}) \sim \frac{1}{L} = \frac{1}{\sqrt{\operatorname{vol} \mathcal{C}}} = \frac{1}{\tau^{1/2}}$$

So here β would be $1/2$, $\alpha = 1/2$.

Falls on critical line.

Theorem:

$$\frac{1}{N^{1/3}} X(N\cdot) \Rightarrow RTBM(\mathcal{F}) \ (\neq \text{FIN})$$

Rk: a) The exponent $1/3$ is well known.

Kesten '86 shows that the distance to the root $d(\phi, Y(Nt))/N^{1/3}$ converges in dist.

b) Very sharp heat kernel estimates available for Y : Barlow-Kumagai '05

c) Theorem relies on a recent result by D. Croydon '07. ~~as [illegible]~~

Proof: Apply criterium of convergence for RTRWs to RTBMs.

Let $U_N(\lambda) := N^{1/2}\left(1 - E\left(e^{-\lambda N^{3/2}\sigma_N} \,\middle|\, \#\text{vert} = N\right)\right.$

here: $\#$vert = size of critical tree.

σ_N : return time to root, starting from a child of root.

To Prove: $\lim\limits_{N\to\infty} \cancel{} U_N(\lambda)$ exists $=: U(\lambda)$

Indeed we had to prove $\lim\limits_{\varepsilon\to 0} \phi_\varepsilon(\ell,\lambda)$ exists

$$\phi_\varepsilon(\ell,\lambda) = E\left(e^{-\frac{\ell}{\varepsilon}(1 - \hat{\nu}(\lambda\varepsilon^{1+1/\gamma}))} \,\middle|\, \tau_\nu = \varepsilon^{-1/\gamma}\right)$$

here : $N = \varepsilon^{-2}$ or $\varepsilon = N^{-1/2}$; $\gamma = 1/2$.

$$= E\left(e^{-\ell N^{1/2}(1 - \hat{\nu}(\lambda N^{-3/2}))} \,\middle|\, \tau_\nu = N\right)$$

$$\sim 1 - \ell E\left(N^{1/2}(1 - \hat{\nu}(\lambda N^{-3/2})) \,\middle|\, \tau_N = N\right)$$

$$= 1 - \ell U_N(\lambda).$$

Lemma: $\lim U_N(\lambda) = U(\lambda) = \sum_1^\infty (-1)^{n-1} q_n \lambda^n$

Explicit expression for U (or the q_n's):

$$q_n = \int_{\mathcal{T}^n} \prod_{i=2}^n d_{\mathcal{T}}(\rho, x_{i-1} \wedge x_i) \prod_1^n d\mu(x_i)$$

Here:

$\mathcal{T}$: Continuous Random Tree

μ : measure on $\mathcal{T}$.

$d_{\mathcal{T}}$: distance on $\mathcal{T}$.

ρ : root

This implies convergence to a RTBM (⊞)

+ FIN

Consequence of Croydon's result:

Scaling limit of critical tree = CRT

Scaling limit of (critical tree, RW on it)

is ((CRT, BM on CRT)).

$\mathcal{W}$: excursions $\mathbb{R}_+ \to \mathbb{R}_+$

ω $\mathbb{R}_+ \to \mathbb{R}_+$

$\omega(0) = 0$, $\omega(t) = 0$ for $t \geq \zeta(\omega)$.

$\mathcal{W}^{(1)}$ normalized excursions

$$m_\omega(s,t) = \inf(\omega(r),\ r \in [s \wedge t, s \vee t])$$

$$d_\omega(s,t) = \omega(s) + \omega(t) - 2\, m_\omega(s,t)$$

$$s \sim t \iff d_\omega(s,t) = 0$$

$$\mathcal{D}_\omega := [0,1]/\sim \ = \{ [s],\ s \in [0,1] \}$$

$$d_{\mathcal{D}_\omega}([s],[t]) = d_\omega(s,t)$$

$\mathcal{D}_\omega$ is a compact dendrite, root $= [0]$

(i.e arcwise connected & no "loop" homeo to circle)

$(\mathcal{D}_\omega, d_{\mathcal{D}_\omega}, [0])$ rooted real tree.

Mesure on $\mathcal{D}_\omega$: push. of Lebesgue

$$\mu_\omega(A) = \lambda(s \in [0,1],\ [s] \in A)$$

Natural construction of BM on $\mathcal{D}_\omega$

by Dirichlet forms (see Croydon, Krebs '95 '07)

§6 Biased RWs on supercritical trees.

(jt wk: G.B.A. - A. FRIBERGH - N. GANTERT - A. HAMMOND)

Consider a supercritical Galton–Watson tree

Generating function $f(z) = \sum_0^\infty p_k z^k$

Assume $p_0 > 0$. (extinction possible)

$$m = f'(1) > 1 \quad \text{(supercritical)}$$

Let q = extinction probability

$$f(q) = q \quad , \quad 0 < q < 1$$

Condition on non-extinction: infinite tree.

Consider a β-biased RW on it.

outward drift. (away from root)

$$P(X_{n+1} = \overleftarrow{u} \mid X_n = u) = \frac{1}{1+\beta k}$$

$$P(X_{n+1} = v_i \mid X_n = u) = \frac{\beta}{1+\beta k} \qquad 1 \le i \le k$$

Reversible Markov Chain

Conductances : $c(\overleftarrow{x}, x) = \beta^{|x|-1}$

Transient, Limiting speed v exists and is constant a.s (Lyons-Pemantle-Peres '96

$\beta < \beta_c = 1/f'(q) \Rightarrow v > 0$

$\beta \geq \beta_c \Rightarrow v = 0$

Large drift $\Rightarrow$ slow motion !

Reason: Traps !

Harris decomposition of supercritical tree.

Backbone: vertices with infinite "line of descent"

= supercritical GW tree with $p_0 = 0$

* Attached to backbone = SUBCRITICAL trees.

Drift pushes strongly in small trees. and traps the RW.

Obviously the SRW on backbone is ballistic $\forall \beta \geq 0$.

But the trapping times are heavy-tailed if β large.

Let $\gamma := - \frac{\ln f'(q)}{\ln \beta} = \frac{\ln \beta_c}{\ln \beta}$

& Δ_n = first hitting time to reach level n

$$= \inf (k, |X_k| = n)$$

"Theorem": $\Delta_n \approx n^{1/\gamma}$ if $\beta > \beta_c$

Theorem: Assume finite variance for the offspring distribution, & $\beta > \beta_c$

For any $\lambda > 0$ consider ~~sub~~sequence

$$n_\lambda(k) = \lfloor \lambda f'(q)^{-k} \rfloor$$

then

$$\frac{\Delta_{n_\lambda(k)}}{n_\lambda(k)^{1/\gamma}} \xrightarrow{(d)} I_\lambda \text{ Infinitely divisible law}$$

- I_λ not stable. (if β large enough)
- No limit in distribution (if β large enough)

Theorem: $\beta > \beta_c$

1) $\dfrac{\Delta_n}{n^{1/\gamma}}$ is tight

2) $\dfrac{X_n}{n^{\gamma}}$ is tight

Reason for "non-stable" limit:

Lattice effects

Exercise:

1) If X is exponentially distributed β^X is in the domain of attraction of a stable (if β large enough)

2) If X is geometric β^X is not in the domain of attraction of a stable

$S_N = \sum_{i=1}^{N} \beta^{X_i}$ (where X_i i.i.d geometric) does not converge in distribution (after normalization)

If now one randomizes the drift β (i.i.d drifts with non arithmeticity condition on support of distribution of drifts) then the scaling limit of Δ_n ~~walk~~ is stable, ~~...~~ the scaling limit of the ~~...~~ $|X_n|$ is the inverse of a stable subordinator

(jt work GBA-A.Hammond).

Spin Glass dynamics
& REM Universality

Model: Mean-Field Spin Glasses

- $S_N = \{-1, 1\}^N$ discrete hypercube

On S_N consider a Gaussian centered process $(H_N(\sigma))_{\sigma \in S_N}$ with covariance:

$$\mathrm{cov}(H_N(\sigma), H_N(\sigma')) = \nu(R_N(\sigma, \sigma'))$$

$$R(\sigma, \sigma') = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma_i' \quad \text{overlap}$$

$$= 1 - \frac{2}{N} d_H(\sigma, \sigma') \quad d_H: \text{Hamming distance.}$$

Always assume $\nu(0) = 0$.

i.e "covariance vanishes at typical distance"

and $\nu(1) = 1$ for normalization

i.e $\mathrm{Var}\, H_N(\sigma) = 1$.

ex 1 : $\nu(r) = 1_{r=1}$

So $$\mathrm{cov}(H_N(\sigma), H_N(\sigma')) = 0 \quad \text{if } \sigma \neq \sigma'$$
$$= 1 \quad \text{if } \sigma = \sigma'$$

$(H_N(\sigma))_{\sigma \in S_N}$ are i.i.d $N(0,1)$

REM ~~model~~ : Random Energy Model

ex 2 : $\nu(r) = r$

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_1^N g_i \sigma_i$$

$g_i = N(0,1)$ i.i.d.

Number Portioning Problem NPP

"Find σ minimizing $|H_N(\sigma)|$

i.e closest to zero energy".

ex 3 : $\nu(r) = r^2$

$$H_N(\sigma) = \frac{1}{N} \sum_{1 \le i_1, i_2 \le N} g_{i_1 i_2} \sigma_{i_1} \sigma_{i_2}$$

$g_{i_1 i_2}$ $N(0,1)$ i.i.d

Sherrington-Kirkpatrick SK

ex4: p-spin models

$$\nu(r) = r^p \qquad p \geq 3$$

$$H_{N,p}(\sigma) = \frac{1}{\sqrt{N}^{\,p}} \sum_{i_1 \ldots i_p} g_{i_1, \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p}$$

$g_{i_1 \ldots i_p}$ $N(0,1)$ iid

ex5: A general case could be:

$$\nu(r) = \sum_{p=1}^{\infty} a_p r^p \qquad a_p \geq 0$$

$$H_N(\sigma) = \sum_{p=1}^{\infty} \sqrt{a_p} \, H_{N,p}(\sigma)$$

Equilibrium questions

1) Understand the large N behavior of ground state, i.e $\min_{\sigma \in S_N} H_N(\sigma)$ ($\approx \sqrt{N}$)

2) Understand the large N behavior of the partition function: $Z_N = \sum_\sigma e^{-\beta H_N(\sigma) \sqrt{N}}$

& free energy: $-\frac{1}{N} \log Z_N$

3) Understand the structure of Gibbs measure $\mu_N(\sigma) = e^{-\beta H_N(\sigma)\sqrt{N}} / Z_N$

Dynamical questions
(out-of-equilibrium)

Let $c(\sigma,\sigma')$ (for $\sigma \sim \sigma'$ on S_N) be jump rates satisfying detailed balance

$$\mu_N(\sigma)\, c(\sigma,\sigma') = \mu_N(\sigma')\, c(\sigma',\sigma)$$

& consider $X_N(t)$ the continuous time Markov Chain on S_N defined by these jump rates.

μ_N is ~~as~~ invariant, and as $t \to \infty$ if N fixed $X_N(\cdot) \xrightarrow{d} \mu_N$.

Question: Understand the large time behavior of $X_N(\cdot)$, the approach to equilibrium. What are relevant time scales $t(N)$?

Mixing time $t(N) \approx \cancel{\ } e^{cN}$?

Natural jump rates :

1) $c(\sigma, \sigma') = e^{-\beta\sqrt{N}\,(H(\sigma') - H(\sigma))_+}$ Metropolis

2) $c(\sigma, \sigma') = \dfrac{1}{1 + e^{\beta\sqrt{N}(H(\sigma') - H(\sigma))}}$ Heat Bath

3) Bouchaud : $\left(\tau(\sigma) = \mu_N(\sigma)\right)$

- $c(\sigma, \sigma') = e^{+\beta\sqrt{N}H(\sigma)}$ $(a = 0)$

  Mean time at σ : $\mu_N(\sigma) = e^{-\beta\sqrt{N}H(\sigma)}$

- $c(\sigma, \sigma') = e^{\beta\sqrt{N}(H(\sigma)(1-a) - aH(\sigma'))}$ $a \neq 0$

The REM (equilibrium)

Phase transition at $\beta_c = \sqrt{2\log 2}$

High temperature: $\beta < \beta_c$

μ_N close to "uniform measure" on S_N (as for $\beta=0$)

(at least when projected on $k \ll N$ dimensions)

Low temperature: $\beta > \beta_c$

μ_N concentrate on the σ's where H_N is extreme.

$$Z_N = \sum_\sigma e^{-\beta\sqrt{N} H_N(\sigma)}$$

. Contribution of extreme values dominant

. $Z_N \sim$ stable.

. ~~μ_N~~ Order statistics: $H_N(\sigma^{(1)}) \leq H_N(\sigma^{(2)}) \leq \cdots$

$$\mu_N(\sigma^{(1)}) \geq \mu_N(\sigma^{(2)}) \geq \cdots \geq \mu_N(\sigma^{(2^N)})$$

This sequence converges to a Poisson-Dirichlet process. with $\alpha = \alpha(\beta) = \beta/\beta_c$

Rk: The tails of $e^{\beta\sqrt{N}H}$ are not heavy!

But:

$$P\left(e^{\beta\sqrt{N}H} \geq u\, e^{\alpha\beta^2 N} \;\middle|\; e^{\beta\sqrt{N}H} \geq e^{\alpha\beta^2 N} \right)$$

$$= P\left(H \geq \frac{1}{\beta\sqrt{N}}(\ln u + \alpha\beta^2 N) \;\middle|\; H \geq \frac{1}{\beta\sqrt{N}}(\alpha\beta^2 N) \right)$$

$$\sim \quad e^{-\frac{1}{2}\left(\frac{(\ln u)^2}{\beta^2 N} + 2\alpha\beta \ln u \right)} \longrightarrow \frac{1}{u^\alpha}$$

So, when observed in a window near $e^{\alpha\beta^2 N}$ the tails of $e^{\beta\sqrt{N}H}$ are close to an α-power law.

Rk: This is a general phenomenon (BA- Bogatchev - Molchanov).