Stochastic Geometry of Classical and Quantum Ising Models

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Stochastic Geometry

= Graphical Representation

= Path-Integral Approach to Spin Systems

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Main Examples:

- FK (Fortuin-Kesteleyn) Representation

- RC (Random Current) Representation

Plan of the Course

Lectures 1-2

- Quantum reformulation of Classical Ising model.
- General setup for path-integral representation in terms of Poisson processes of arrival (of matrices)
- Examples: FK and RC representations for Ising model in transverse field
- An Application: Exponential decay of truncated correlations at non-zero m.f.

Lecture 3 FK and RC representations

Lecture 4 Erdős-Rény random graphs.

Lecture 5 Curie-Weiss model in transverse field

Main Sources of Inspiration

[1] Aizenman, M., Nachtergaele, B.: Geometric aspects of quantum spin states. Comm. Math. Phys., **164**, 1, 17–63 (1994).

[2] Aizenman, M., Klein, A., Newman, C.M : Percolation methods for disordered quantum Ising models, Mathematics, Physics, Biology..., R. Kotecky, ed., 1–24, World Scientific, Singapore (1993)

[3] Campanino, M., Klein, A., Perez, J.F.: Localization in the ground state of the Ising model with a random transverse field. Comm. Math. Phys. **135**, 3, 499–515 (1991).

[4] Aizenman, M.: Geometric analysis of φ_4 fields and Ising models. I, II. Comm. Math. Phys. **86** , 1, 1–48 (1982)

Notation

- $(\Lambda, \mathcal{E})$ — Finite graph

- Classical spin configuration on Λ,

$$\sigma \in \{\pm 1\}^{\Lambda} \equiv \Omega_{\Lambda}$$

- $\mathbf{J} = \left\{ J_{ij} \geq 0 \right\}$ are coupling constants. By definition $J_{ij} > 0 \Leftrightarrow \{i, j\} \in \mathcal{E}$.

- $h \in \mathbb{R}$ is a magnetic field.

- Classical Hamiltonian

$$-\mathcal{H}^{I}_{\Lambda}(\sigma) = \sum_{(i,j) \in \mathcal{E}} J_{ij} \sigma_i \sigma_j + h \sum_{i \in \Lambda} \sigma_i .$$

Given $\beta \geq 0$ (inverse temperature) define the classical Ising-Gibbs probability distribution on Ω_Λ

$$\mu_\Lambda^{\beta,h}(\sigma) = \frac{1}{\mathcal{Z}_\Lambda^{\beta,h}} \mathrm{e}^{-\beta \mathcal{H}_\Lambda^I(\sigma)},$$

- Partition Function

$$\mathcal{Z}_\Lambda^{\beta,h} = \sum_{\sigma \in \Omega_\Lambda} \mathrm{e}^{-\beta \mathcal{H}_\Lambda^I(\sigma)}$$

- Mean value

$$\mu_\Lambda^{\beta,h}(\sigma_i) = \frac{1}{\mathcal{Z}_\Lambda^{\beta,h}} \sum_{\sigma \in \Omega_\Lambda} \sigma_i \mathrm{e}^{-\beta \mathcal{H}_\Lambda^I(\sigma)},$$

- Two point function

$$\mu_\Lambda^{\beta,h}(\sigma_i \sigma_j) = \frac{1}{\mathcal{Z}_\Lambda^{\beta,h}} \sum_{\sigma \in \Omega_\Lambda} \sigma_i \sigma_j \mathrm{e}^{-\beta \mathcal{H}_\Lambda^I(\sigma)}$$

Quantum Reformulation of the Classical Model

- ± 1 are understood as eigenvalues of Pauli matrix

$$\widehat{\mathsf{W}}^{\mathsf{z}} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The corresponding eigenfunctions are

$$|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |-1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this notation $\widehat{\mathsf{W}}^{\mathsf{z}}|\sigma\rangle = \sigma|\sigma\rangle$ for $\sigma = \pm 1$.

- Lifting of Classical configurations $\sigma \in \Omega_\Lambda$

$$\Omega_\Lambda \ni \sigma \to |\sigma\rangle \equiv \bigotimes_{i\in\Lambda} |\sigma_i\rangle \in \mathbb{X}_\Lambda \equiv \bigotimes_{i\in\Lambda} \mathbb{R}^2$$

- Action of $\widehat{\mathsf{W}}_i^{\mathsf{z}}$ on $|\sigma\rangle$

$$\widehat{\mathsf{W}}_i^{\mathsf{z}}|\sigma\rangle \equiv |\sigma_1\rangle \otimes \cdots \otimes \widehat{\mathsf{W}}^{\mathsf{z}}|\sigma_i\rangle \otimes \cdots = \sigma_i|\sigma\rangle$$

- For $i, j \in \Lambda$ the operators $\widehat{\mathsf{W}}^{\mathsf{z}}_i$ and $\widehat{\mathsf{W}}^{\mathsf{z}}_j$ (on $\mathbb{X}_\Lambda$) commute,

$$\widehat{\mathsf{W}}^{\mathsf{z}}_i \widehat{\mathsf{W}}^{\mathsf{z}}_j |\sigma\rangle = \sigma_i \sigma_j |\sigma\rangle$$

- Quantum Hamiltonian $\widehat{\mathsf{H}}_\Lambda$ is a linear self-adjoint operator on $\mathbb{X}_\Lambda$,

$$-\widehat{\mathsf{H}}_\Lambda \equiv \sum_{(i,j)\in\mathcal{E}} J_{ij} \widehat{\mathsf{W}}^{\mathsf{z}}_i \widehat{\mathsf{W}}^{\mathsf{z}}_j + h \sum_{i\in\Lambda} \widehat{\mathsf{W}}^{\mathsf{z}}_i.$$

- The action of $\widehat{\mathsf{H}}_\Lambda$ on $|\sigma\rangle$ is

$$\begin{aligned} -\widehat{\mathsf{H}}_\Lambda |\sigma\rangle &= \sum_{(i,j)\in\mathcal{E}} J_{ij} \sigma_i \sigma_j |\sigma\rangle + h \sum_{i\in\Lambda} \sigma_i |\sigma\rangle \\ &= \left(\sum_{(i,j)\in\mathcal{E}} J_{ij} \sigma_i \sigma_j + h \sum_{i\in\Lambda} \sigma_i \right) |\sigma\rangle \\ &= -\mathcal{H}^I_\Lambda(\sigma) |\sigma\rangle \end{aligned}$$

  where $\mathcal{H}^I_\Lambda(\sigma)$ is the classical Hamiltonian.

Classical case: Operators $\widehat{\mathsf{W}}^z_i, \widehat{\mathsf{W}}^z_j$ commute. Moreover, the operator

$$-\widehat{\mathsf{H}}_\Lambda \equiv \sum_{(i,j)\in\mathcal{E}} J_{ij}\widehat{\mathsf{W}}^z_i\widehat{\mathsf{W}}^z_j + h\sum_{i\in\Lambda}\widehat{\mathsf{W}}^z_i$$

is diagonal in the basis $\{|\sigma\rangle\}_{\sigma\in\Omega_\Lambda}$ of $\mathbb{X}_\Lambda$. In particular,

$$\mathrm{e}^{-\beta\widehat{\mathsf{H}}_\Lambda}|\sigma\rangle = \mathrm{e}^{-\beta\mathcal{H}^I_\Lambda(\sigma)}|\sigma\rangle \quad \forall\,\sigma\in\Omega_\Lambda$$

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Define the scalar product $\langle\bullet|\bullet\rangle$ on $\mathbb{X}_\Lambda$

$$\langle\sigma|\sigma'\rangle = \prod_{i\in\Lambda}\mathbb{1}_{\{\sigma_i=\sigma'_i\}} = \mathbb{1}_{\{\sigma=\sigma'\}}$$

Then $\{|\sigma\rangle\}_{\sigma\in\Omega_\Lambda}$ is an orthonormal basis of $\mathbb{X}_\Lambda$. Also,

$$\langle\sigma|\mathrm{e}^{-\beta\widehat{\mathsf{H}}_\Lambda}|\sigma\rangle = \mathrm{e}^{-\beta\mathcal{H}^I_\Lambda(\sigma)}$$

Furthermore,

$$\langle \sigma | \widehat{\mathsf{W}}_i^{\mathsf{z}} \mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} | \sigma \rangle = \sigma_i \mathsf{e}^{-\beta \mathcal{H}_\Lambda^I(\sigma)}$$

As well as,

$$\langle \sigma | \widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}} \mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} | \sigma \rangle = \sigma_i \sigma_j \mathsf{e}^{-\beta \mathcal{H}_\Lambda^I(\sigma)}$$

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Representation for the classical model

- Partition Function

$$\mathcal{Z}_\Lambda^{\beta,h} = \sum_{\sigma \in \Omega_\Lambda} \langle \sigma | \mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} | \sigma \rangle = \mathsf{Tr} \left(\mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} \right)$$

- Mean value

$$\mu_\Lambda^{\beta,h}(\sigma_i) = \frac{\mathsf{Tr}\left(\widehat{\mathsf{W}}_i^{\mathsf{z}} \mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} \right)}{\mathsf{Tr}\left(\mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} \right)}$$

- Two point function

$$\mu_\Lambda^{\beta,h}(\sigma_i \sigma_j) = \frac{\mathsf{Tr}\left(\widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}} \mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} \right)}{\mathsf{Tr}\left(\mathsf{e}^{-\beta \widehat{\mathsf{H}}_\Lambda} \right)}$$

General Case

• The space is as before $\mathbb{X}_\Lambda = \otimes_{i\in\Lambda}\mathbb{R}^2$ with the scalar product $\langle \bullet | \bullet \rangle$.

• $K_1, \ldots, K_m$ are self-adjoint operators (matrices) on $\mathbb{X}_\Lambda$, in general non-commuting.

• $\lambda_1, \ldots, \lambda_m$ are positive numbers.

• Hamiltonian: $-\widehat{\mathsf{H}}_\Lambda = \sum_1^m \lambda_\ell K_\ell$.

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Task: Find a graphical representation for

- Partition Function

$$\mathcal{Z}_\Lambda^\beta = \mathrm{Tr}\left(\mathrm{e}^{-\beta \widehat{\mathsf{H}}_\Lambda}\right) = \mathrm{Tr}\left(\mathrm{e}^{\beta \sum_1^m \lambda_\ell K_\ell}\right)$$

- Mean value Given a (self-adjoint) matrix A,

$$\mathcal{Z}_\Lambda^\beta[A] = \mathrm{Tr}\left(A\mathrm{e}^{-\beta \widehat{\mathsf{H}}_\Lambda}\right)$$

Two Main Tools

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Lie-Trotter Formula

Let $A_1, \dots, A_n$ be self-adjoint matrices on $\mathbb{X}_\Lambda$. Then,

$$\mathrm{e}^{A_1+\dots+A_n} = \lim_{N\to\infty} \left(\prod_1^n \mathrm{e}^{A_\ell/N} \right)^N .$$

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Matrix Product Expansion Formula

Let $B_1, \dots, B_N$ be self-adjoint matrices. Let $\mathcal{B}$ be an orthonormal basis of $\mathbb{X}_\Lambda$, e.g. $\mathcal{B} = \{|\sigma\rangle\}_{\sigma\in\Omega_\Lambda}$. Then for any two $|\nu\rangle, |\nu'\rangle \in \mathcal{B}$,

$$\langle\nu|B_1 \dots B_N|\nu'\rangle$$

$$= \sum_{|\nu^1\rangle,\dots,|\nu^{N-1}\rangle\in\mathcal{B}} \langle\nu|B_1|\nu^1\rangle\langle\nu^1|B_2|\nu^2\rangle \dots \langle\nu^{N-1}|B_N|\nu'\rangle$$

Path Integral Representation

<u>Step1</u> Use of Lie-Trotter Formula.

$$e^{\beta \sum_1^m \lambda_\ell K_\ell} = \lim_{\Delta \to 0} \left(\prod_1^m e^{\Delta \lambda_\ell K_\ell} \right)^{\beta/\Delta}$$

$$= e^{\beta \sum \lambda_\ell} \lim_{\Delta \to 0} \left(\prod_{l=1}^m \{(1 - \Delta\lambda_\ell)\mathrm{I} + \Delta\lambda_\ell K_\ell\} \right)^{\beta/\Delta}$$

<u>Step2</u> Interpretation in Terms of Bernoulli Trials. Set $N = \beta/\Delta$.

For $\ell = 1, \ldots, m$ let $\xi_\ell = (\xi_\ell(1), \ldots \xi_\ell(N))$ be independent sequences of i.i.d. trials with

$$p_\ell = \lambda_\ell \Delta$$

Let $\mathbb{P}^\lambda_{\beta,\Delta}$ be the corresponding product probability measure on

$$\underbrace{\{0,1\}^N \times \cdots \times \{0,1\}^N}_{m \text{ times}} \equiv \Xi_N$$

Then, by the Product Expansion Formula

$$\left(\prod_{l=1}^{m}\{(1-\Delta\lambda_\ell)\mathrm{I}+\Delta\lambda_\ell K_\ell\}\right)^N$$

$$=\sum_{a\in\Xi_N}\mathbb{P}^{\lambda}_{\beta,\Delta}(\xi=a)\cdot\mathcal{K}_a.$$

where matrices $\mathcal{K}_a$ are defined via

$$\mathcal{K}_a \equiv \prod_{j=1}^{N}\left\{\prod_{\ell=1}^{m}((1-a_\ell(j))\mathrm{I}+a_l(j)K_\ell)\right\}.$$

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Remark:

$$\mathbb{P}^{\lambda}_{\beta,\Delta}\left(\exists j\ :\ \sum_{1}^{m}\xi_\ell(j)>1\right)=O(\Delta)$$

Sample Path Interpretation of $\mathcal{K}_a$

- Consider only a-s with $\sum_\ell a_\ell(j) = 0, 1$ for $j = 1, \ldots, N$.

- Arrival Times (belong to $[0, \beta]$)

$$a^{\Delta} \equiv \left\{ j\Delta : \sum_\ell a_\ell(j) = 1 \right\}$$
$$= \{ j\Delta : \ \exists \ell : a_\ell(j) = 1 \}$$

- Arrival Types For $t \in a^{\Delta}$,

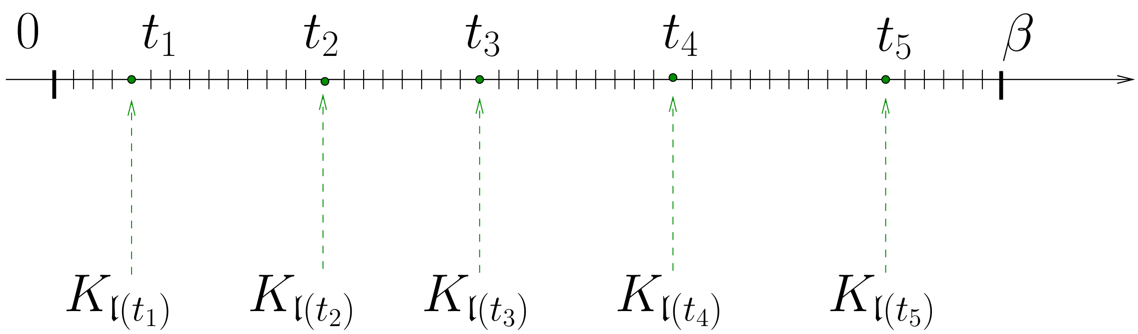
$$\mathfrak{l}(t) = \ell \quad \text{if} \quad a_\ell(t/\Delta) = 1$$

Then,

$$\boxed{\mathcal{K}_a = \prod_{t \in a^{\Delta}} K_{\mathfrak{l}(t)}}$$

$$\mathcal{K}_a = \prod_{t \in a^{\triangle}} K_{\mathfrak{l}(t)}$$

Example: $a^{\triangle} = \{t_1, t_2, t_3, t_4, t_5\}$



STEP3 Use of Product Expansion Formula.

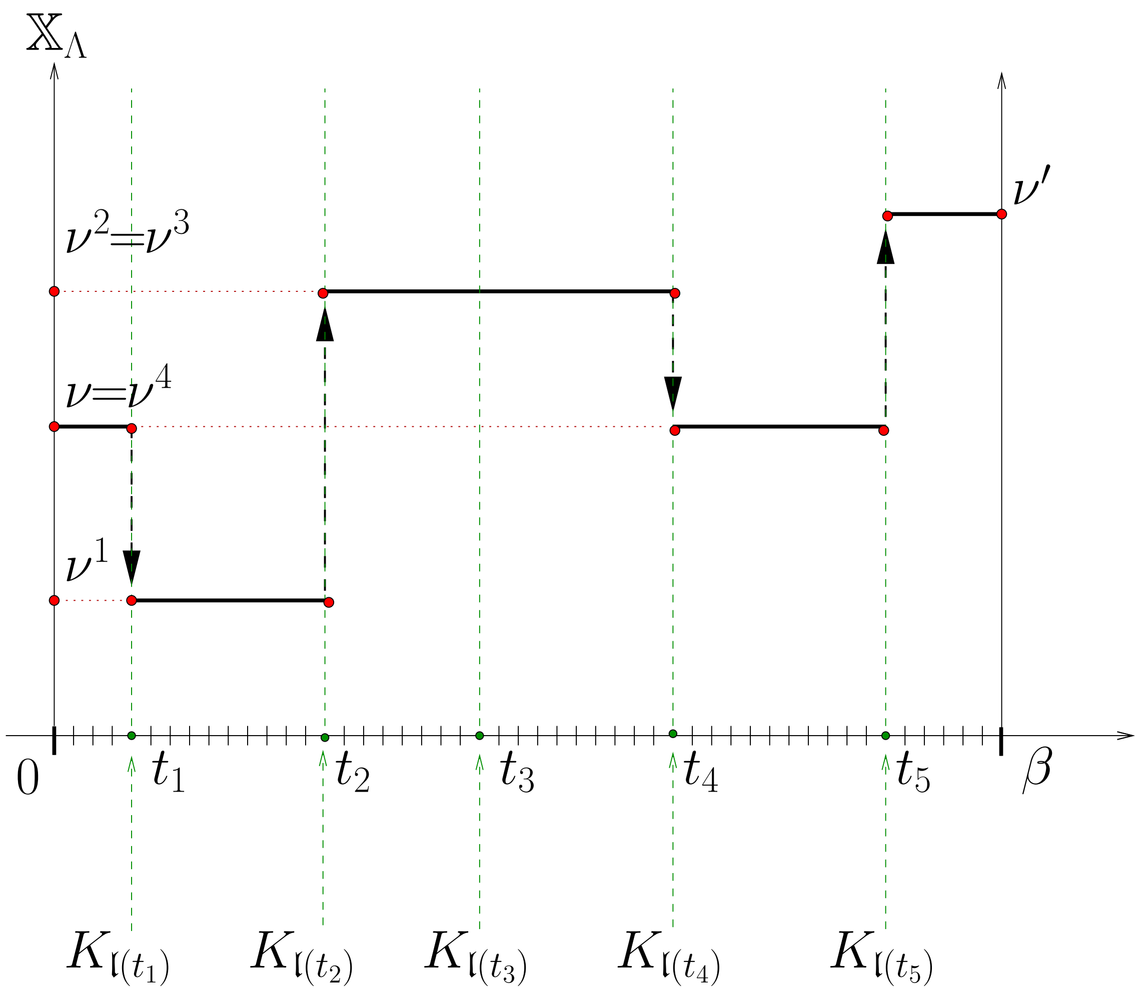
Representation of $\langle \nu | \mathcal{K}_a | \nu' \rangle$

$$\langle \nu | \mathcal{K}_a | \nu' \rangle = \sum_{|\nu^1\rangle, \ldots |\nu^4\rangle} \langle \nu | K_{\mathfrak{l}(t_1)} | \nu^1 \rangle \ldots \langle \nu^4 | K_{\mathfrak{l}(t_5)} | \nu' \rangle$$

$$= \sum_{|\nu(\cdot)\rangle \sim a^{\triangle}} \prod_{t \in a^{\triangle}} \langle \nu(t-) | K_{\mathfrak{l}(t)} | \nu(t) \rangle$$

Path of $|\nu(\cdot)\rangle \sim a^{\triangle} = \{t_1, \ldots, t_5\}$

- $|\nu(0)\rangle = |\nu\rangle$ and $|\nu(\beta)\rangle = |\nu'\rangle$
- $|\nu(\cdot\rangle$ can jump only at arrival times $t \in a^{\triangle}$



Poisson limit as $\Delta \downarrow 0$

• ξ_ℓ; $\ell = 1, \ldots, m$ are independent Poisson Processes of Arrivals (of operators K_ℓ) on $[0, \beta]$ with intensities λ_ℓ

Notation: $\mathcal{P}_\beta^\lambda(\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_m)$
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A piece-wise constant trajectory

$$|\nu(\cdot)\rangle \sim \xi \equiv \bigcup_1^m \xi_\ell$$

• If it has jumps only at the arrival times t of ξ.

• And if for every $t \in \xi$

$$\langle \nu(t-)|K_{\mathfrak{l}(t)}|\nu(t)\rangle \neq 0$$

where $\mathfrak{l}(t) \in \{1, \ldots, m\}$ is the arrival type at t.

Representation of Partition Function

$$\frac{\mathcal{Z}_{\Lambda}^{\beta}}{\mathrm{e}^{\beta\sum\lambda_\ell}} = \frac{\mathrm{Tr}\left(\mathrm{e}^{\beta\sum_\ell \lambda_\ell K_\ell}\right)}{\mathrm{e}^{\beta\sum\lambda_\ell}}$$

$$= \int \mathcal{P}_{\beta}^{\lambda}(\mathrm{d}\xi)\times\sum_{|\nu(\cdot)\rangle\sim\xi} \langle\nu(0)|\nu(\beta)\rangle \prod_{t\in\xi}\langle\nu(t-)|K_{\mathfrak{l}(t)}|\nu(t)\rangle$$

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Representation of Mean Values

$$\frac{\mathcal{Z}_{\Lambda}^{\beta}[A]}{\mathrm{e}^{\beta\sum\lambda_\ell}} = \frac{\mathrm{Tr}\left(A\mathrm{e}^{\beta\sum_\ell \lambda_\ell K_\ell}\right)}{\mathrm{e}^{\beta\sum\lambda_\ell}}$$

$$= \int \mathcal{P}_{\beta}^{\lambda}(\mathrm{d}\xi)\times\sum_{|\nu(\cdot)\rangle\sim\xi} \langle\nu(0)|A|\nu(\beta)\rangle \prod_{t\in\xi}\langle\nu(t-)|K_{\mathfrak{l}(t)}|\nu(t)\rangle$$

Classical FK Representation

Classical FK representation corresponds to the path-integral interpretation of the Hamiltonian

$$-\widehat{\mathsf{H}}_\Lambda = \sum_{(i,j)\in\mathcal{E}} J_{ij}\widehat{\mathsf{W}}^{\mathsf{z}}_i\widehat{\mathsf{W}}^{\mathsf{z}}_j + h\sum_{i\in\Lambda}\widehat{\mathsf{W}}^{\mathsf{z}}_i$$

$$= -\left(\sum_{(i,j)} J_{ij} + \sum_i h\right)\mathrm{I}$$

$$+ \sum_{(i,j)} 2J_{ij}\frac{\mathrm{I}+\widehat{\mathsf{W}}^{\mathsf{z}}_i\widehat{\mathsf{W}}^{\mathsf{z}}_j}{2} + \sum_i 2h\frac{\mathrm{I}+\widehat{\mathsf{W}}^{\mathsf{z}}_i}{2},$$

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Poisson Processes of Arrivals

- For $(i,j)\in\mathcal{E}$, Operators (matrices)

$$K_{ij} \equiv \frac{\mathrm{I}+\widehat{\mathsf{W}}^{\mathsf{z}}_i\widehat{\mathsf{W}}^{\mathsf{z}}_j}{2} \quad \text{with intensity } 2J_{ij}$$

- For $i\in\Lambda$, Operators

$$K_i \equiv \frac{\mathrm{I}+\widehat{\mathsf{W}}^{\mathsf{z}}_i}{2} \quad \text{with intensity } 2h$$

Actions of K_{ij} and K_i in the z-basis

$$\Omega_\Lambda \ni \sigma \longrightarrow |\sigma\rangle$$

- Recall: $\left(\mathrm{I} + \widehat{\mathrm{W}}_i^{\mathrm{z}}\widehat{\mathrm{W}}_j^{\mathrm{z}}\right)|\sigma\rangle = (1 + \sigma_i\sigma_j)|\sigma\rangle.$

$$\Rightarrow \quad \langle\sigma|K_{ij}|\sigma'\rangle = \mathbb{1}_{\sigma=\sigma'}\mathbb{1}_{\sigma_i=\sigma_j}$$

- Recall: $\left(\mathrm{I} + \widehat{\mathrm{W}}_i^{\mathrm{z}}\right)|\sigma\rangle = (1 + \sigma_i)|\sigma\rangle.$

$$\Rightarrow \quad \langle\sigma|K_i|\sigma'\rangle = \mathbb{1}_{\sigma=\sigma'}\mathbb{1}_{\sigma_i=+1}$$

Therefore

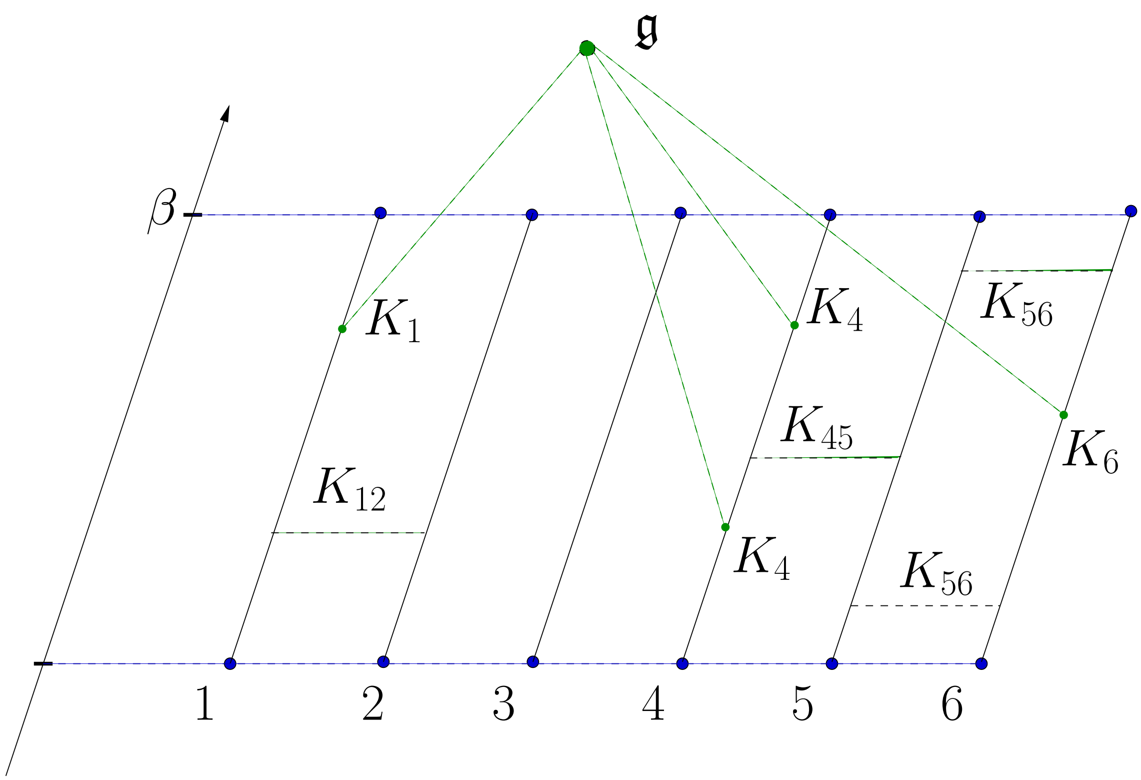
$$\prod_{t\in\xi}\langle\sigma(t-)|K_{\mathfrak{l}(t)}|\sigma(t)\rangle$$

$$= \prod_{t\in\xi}\mathbb{1}_{\sigma(t-)=\sigma(t)} \times \prod_{t\in\xi_{ij}}\mathbb{1}_{\sigma_i=\sigma_j} \times \prod_{t\in\xi_i}\mathbb{1}_{\sigma_i=+1}$$

Conclusions

• Only constant trajectories $\sigma \to |\sigma(\cdot)\rangle$ are compatible $|\sigma(\cdot)\rangle \sim \xi$
• Arrival of K_{ij} forces $\sigma_i = \sigma_j$
• Arrival of K_i forces $\sigma_i = +1$

Example: 2 compatible trajectories $|\sigma(\cdot)\rangle$:



• $\sigma_1(\cdot) = \sigma_2(\cdot) = +1$ • $\sigma_3(\cdot) \equiv \pm 1$
• $\sigma_4(\cdot) = \sigma_5(\cdot) = \sigma_6(\cdot) \equiv +1$

• There is a bond between (i,t) and (j,t) at arrival times t of $\xi_{i,j}$.
• There is a bond between (i,t) and $\mathfrak{g}$ arrival times t of ξ_i
• Any realization of ξ splits

$$\Lambda \cup \mathfrak{g} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_M$$

into disjoint union of maximal connected components.
• $\#_w(\xi) \equiv M$. Number of $|\sigma(\cdot)\rangle \sim \xi$ is $2^{\#_w(\xi)}$.
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Representation of partition Function

$$\frac{\mathcal{Z}_\Lambda^\beta}{e^{\beta\left(\sum_{ij} J_{ij} + \sum_i h\right)}} = \frac{\mathrm{Tr}\left(e^{-\beta \hat{H}_\Lambda}\right)}{e^{\beta\left(\sum_{ij} J_{ij} + \sum_i h\right)}}$$

$$= \mathcal{P}_\beta^{2J,2h}\left(2^{\#_w(\xi)}\right)$$

Define the (Random Cluster) Measure

$$\widetilde{\mathcal{P}}_{\beta}^{2J,2h}(\mathrm{d}\xi) = \frac{\mathcal{P}_{\beta}^{2J,2h}\left(2^{\#_w(\xi)}; \mathrm{d}\xi\right)}{\mathcal{P}_{\beta}^{2J,2h}\left(2^{\#_w(\xi)}\right)}$$

Representation of Mean Value

Since $\langle \sigma | \widehat{\mathsf{W}}_i^{\mathsf{z}} | \sigma \rangle = \sigma_i$

$$\mu_{\Lambda}^{\beta,h}(\sigma_i) = \frac{\mathrm{Tr}\left(\widehat{\mathsf{W}}_i^{\mathsf{z}} \mathrm{e}^{-\beta \widehat{\mathsf{H}}_\Lambda}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \widehat{\mathsf{H}}_\Lambda}\right)} = \widetilde{\mathcal{P}}_{\beta}^{2J,2h}(i \leftrightarrow \mathfrak{g})$$

Representation of Two Point Function

Since $\langle \sigma | \widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}} | \sigma \rangle = \sigma_i \sigma_j$

$$\mu_{\Lambda}^{\beta,h}(\sigma_i \sigma_j) = \frac{\mathrm{Tr}\left(\widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}} \mathrm{e}^{-\beta \widehat{\mathsf{H}}_\Lambda}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \widehat{\mathsf{H}}_\Lambda}\right)} = \widetilde{\mathcal{P}}_{\beta}^{2J,2h}(i \leftrightarrow j)$$

Classical RC Representation

Classical RC representation corresponds to the path-integral interpretation of the Hamiltonian

$$-\widehat{\mathsf{H}}_\Lambda = \sum_{(i,j)\in\mathcal{E}} J_{ij}\widehat{\mathsf{W}}^{\mathsf{z}}_i\widehat{\mathsf{W}}^{\mathsf{z}}_j + h\sum_{i\in\Lambda}\widehat{\mathsf{W}}^{\mathsf{z}}_i$$

in the x-basis of $\mathbb{X}_\Lambda$.

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x-basis of $\mathbb{R}^2$

$$|1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad |-1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Action of $\widehat{\mathsf{W}}^{\mathsf{z}}$ in x-basis of $\mathbb{R}^2$

$$\widehat{\mathsf{W}}^{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \widehat{\mathsf{W}}^{\mathsf{z}}|\pm 1\rangle = |\mp 1\rangle$$

Action of $\widehat{\mathsf{W}}^{\mathsf{z}}$ in x-basis of $\otimes_{i\in\Lambda}\mathbb{R}^2$

$$\widehat{\mathsf{W}}^{\mathsf{z}}_i|\nu\rangle \equiv |\nu_1\rangle \otimes \cdots \otimes |-\nu_i\rangle \otimes \cdots \equiv |\widehat{\nu}^{(i)}\rangle$$

Poisson Processes of Arrivals

- For $(i,j) \in \mathcal{E}$, Operators (matrices)

$$K_{ij} \equiv \widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}} \quad \text{with intensity } J_{ij}$$

Action of $K_{ij}|\nu\rangle = |\widehat{\nu}^{(ij)}\rangle$: Simultaneous flip of i-th and j-th component of ν.

- For $i \in \Lambda$, Operators

$$K_i \equiv \widehat{\mathsf{W}}_i^{\mathsf{z}} \quad \text{with intensity } h$$

Action of $K_i|\nu\rangle = |\widehat{\nu}^{(i)}\rangle$: Flip of i-th component of ν.

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Therefore

$$\prod_{t \in \xi} \langle \nu(t-)| K_{\mathfrak{l}(t)} |\nu(t)\rangle$$

$$= \prod_{t \in \xi_{ij}} \mathbb{1}_{\nu(t) = \widehat{\nu}^{(ij)}(t-)} \times \prod_{t \in \xi_i} \mathbb{1}_{\nu(t) = \widehat{\nu}^{(i)}(t-)}$$

Representation of Partition Function

$$\frac{\mathcal{Z}_\Lambda^\beta}{e^{\beta\left(\sum_{ij} J_{ij}+\sum_i h\right)}} = \frac{\mathrm{Tr}\left(e^{-\beta\hat{H}_\Lambda}\right)}{e^{\beta\left(\sum_{ij} J_{ij}+\sum_i h\right)}}$$

$$= \int \mathcal{P}_\beta^{J,h}(d\xi) \times \sum_{|\nu(\cdot)\rangle\sim\xi} \langle\nu(0)|\nu(\beta)\rangle \prod_{t\in\xi} \langle\nu(t-)|K_{\mathfrak{l}(t)}|\nu(t)\rangle$$

$$= \mathcal{P}_\beta^{J,h}(\partial\xi = \emptyset)\, 2^{|\Lambda|}$$

The event $\{\partial\xi = \emptyset\}$ means that there are Even number of flips of each coordinate $i \in \Lambda \cup \mathfrak{g}$.

Representation of Mean Value

$$\mu_\Lambda^{\beta,h}(\sigma_i) = \frac{\mathrm{Tr}\left(\hat{W}_i^z e^{-\beta\hat{H}_\Lambda}\right)}{\mathrm{Tr}\left(e^{-\beta\hat{H}_\Lambda}\right)} = \frac{\mathcal{P}_\beta^{J,h}(\partial\xi = \{i,\mathfrak{g}\})}{\mathcal{P}_\beta^{J,h}(\partial\xi = \emptyset)}$$

Representation of Two Point Function

$$\mu_\Lambda^{\beta,h}(\sigma_i\sigma_j) = \frac{\mathrm{Tr}\left(\hat{W}_i^z\hat{W}_j^z e^{-\beta\hat{H}_\Lambda}\right)}{\mathrm{Tr}\left(e^{-\beta\hat{H}_\Lambda}\right)} = \frac{\mathcal{P}_\beta^{J,h}(\partial\xi = \{i,j\})}{\mathcal{P}_\beta^{J,h}(\partial\xi = \emptyset)}$$

Switching Lemma

Let ξ and η be two independent random currents. Then,

$$\otimes\mathcal{P}_{\beta}^{J,h}\left(\partial\xi=\{i,j\};\partial\eta=A\right)$$
$$=\otimes\mathcal{P}_{\beta}^{J,h}\left(\partial\xi=\emptyset;\partial\eta=A\Delta\{i,j\};i\overset{\xi+\eta}{\longleftrightarrow}j\right)$$

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Consequence: Representation of truncated two-point functions

$$\mu_{\Lambda}^{\beta,h}\left(\sigma_i;\sigma_j\right)=\frac{\otimes\mathcal{P}_{\beta}^{J,h}\left(\partial\xi=\emptyset;\partial\eta=\{i,j\};i\overset{\xi+\eta}{\not\longleftrightarrow}\mathfrak{g}\right)}{\otimes\mathcal{P}_{\beta}^{J,h}\left(\partial\xi=\emptyset;\partial\eta=\emptyset\right)}$$

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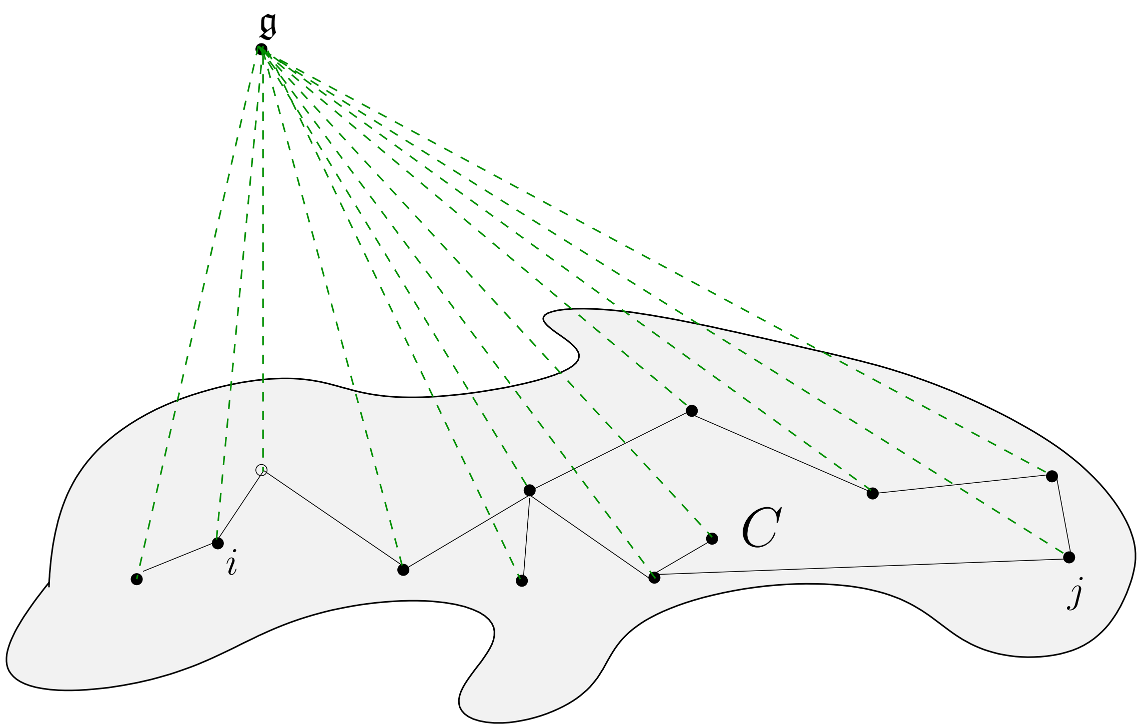
Application: $\Lambda\subset\mathbb{Z}^d$, J has finite range R and $h\neq0$. Then for any β,

$$\mu_{\Lambda}^{\beta,h}\left(\sigma_i;\sigma_j\right)\leq c_1\mathrm{e}^{-c_2|i-j|/R}$$

Proof: Let C be the connected cluster of i and j (in $\kappa \equiv \xi + \eta$).
One should pay a fixed price per site $k \in C$ for disconnecting k from $\mathfrak{g}$.



Ising Model in Transverse Field

Quantum Ising Hamiltonian in the transverse field is given by

$$-\hat{\mathsf{H}}_\Lambda = \sum_{(i,j)} J_{ij}\hat{\mathsf{W}}_i^{\mathsf{z}}\hat{\mathsf{W}}_j^{\mathsf{x}} + h\sum_i \hat{\mathsf{W}}_i^{\mathsf{z}} + \lambda \sum_i \hat{\mathsf{W}}_i^{\mathsf{x}},$$

where $\lambda \geq 0$, and (in the z-basis),

$$\hat{\mathsf{W}}^{\mathsf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \text{and} \quad \hat{\mathsf{W}}^{\mathsf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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Matrices $\hat{\mathsf{W}}^{\mathsf{z}}$ and $\hat{\mathsf{W}}^{\mathsf{x}}$ do not commute.

$\hat{\mathsf{H}}_\Lambda$ is not diagonal

FK Representation

- Decomposition of $-\widehat{\mathsf{H}}_{\Lambda}$

$$
-\left(\sum_{(i,j)} J_{ij} + \sum_i h + \sum_i \lambda\right)\mathrm{I}
+ \sum_{(i,j)} 2J_{ij}\frac{\mathrm{I}+\widehat{\mathsf{W}}_i^{\mathsf{z}}\widehat{\mathsf{W}}_j^{\mathsf{z}}}{2} + \sum_i 2h\frac{\mathrm{I}+\widehat{\mathsf{W}}_i^{\mathsf{z}}}{2}
+ \sum_i \lambda(\widehat{\mathsf{W}}_i^{\mathsf{x}} + \mathrm{I}).
$$

- Basis of $\mathbb{X}_{\Lambda}$: z-basis $\{|\sigma\rangle\}_{\sigma\in\Omega_{\Lambda}}$

======================

$$
\widehat{\mathsf{W}}^{\mathsf{x}} + \mathrm{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \langle\sigma|\widehat{\mathsf{W}}^{\mathsf{x}} + \mathrm{I}|\sigma'\rangle \equiv 1
$$

Poisson Processes of Arrivals

- Links between $(i,j) \in \mathcal{E}$, Operators (matrices)

$$K_{ij} \equiv \frac{\mathrm{I} + \widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}}}{2} \quad \text{with intensity } 2J_{ij}$$

Action: Arrival of K_{ij} forces $\sigma_i = \sigma_j$

- Links between $i \in \Lambda$ and $\mathfrak{g}$, Operators

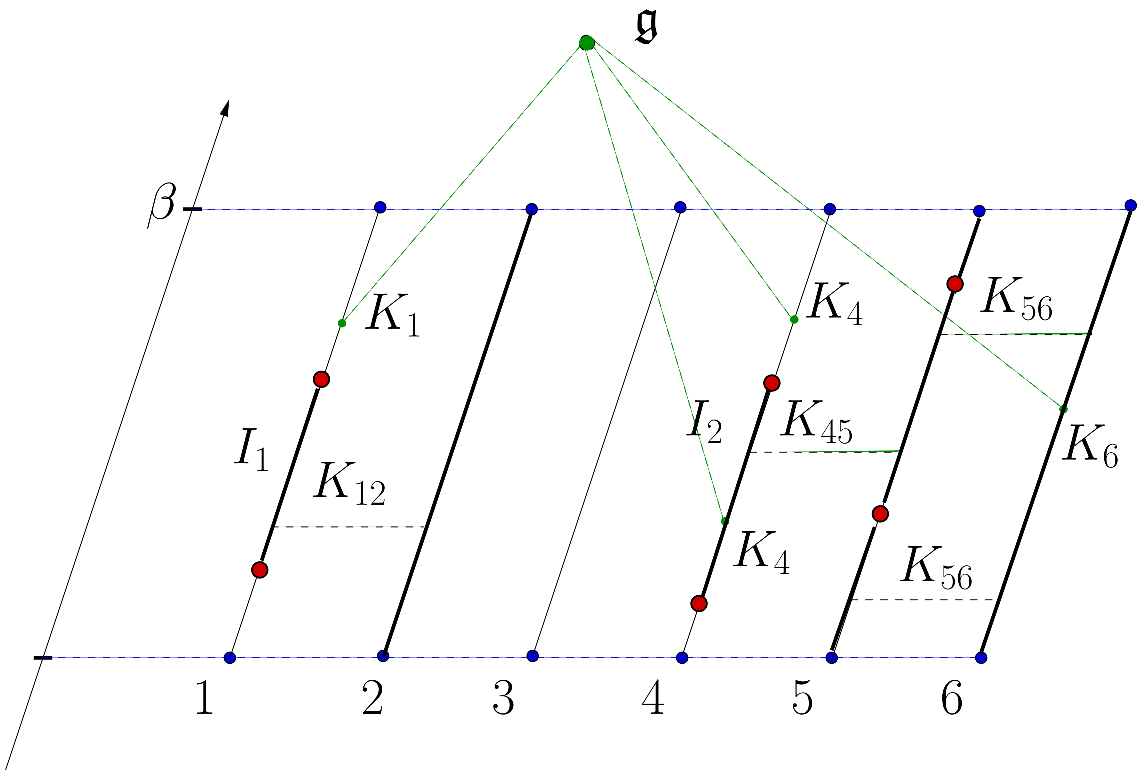
$$K_i^h \equiv \frac{\mathrm{I} + \widehat{\mathsf{W}}_i^{\mathsf{z}}}{2} \quad \text{with intensity } 2h$$

Action: Arrival of K_i^h forces $\sigma_i = +1$

- Holes in $[0, \beta]_i$, $i \in \Lambda$,

$$K_i^\lambda \equiv \left(\widehat{\mathsf{W}}_i^{\mathsf{x}} + \mathrm{I}\right) \quad \text{with intensity } 2\lambda$$

Action: Arrival of K_i^λ enables a flip of σ_i



- $I_1, I_2, \ldots$ maximal connected components
- Compatible trajectories $|\sigma(t)\rangle \sim \xi$ are constant on $I_1, I_2, \ldots$.
- If I_k is linked to $\mathfrak{g}$, then $\sigma = 1$ on I_k
- For each compatible $|\sigma(t)\rangle \sim \xi$

$$\langle \sigma(t-)|K_{\mathfrak{l}(t)}|\sigma(t)\rangle = 1$$

- Number of compatible trajectories is $2^{\#_w(\xi)}$

Representation of partition Function

$$\frac{\mathcal{Z}_{\Lambda}^{\beta}}{e^{\beta\left(\sum_{ij} J_{ij}+\sum_{i} h+\sum_{i} \lambda\right)}} = \frac{\mathrm{Tr}\left(e^{-\beta \widehat{H}_{\Lambda}}\right)}{e^{\beta\left(\sum_{ij} J_{ij}+\sum_{i} h+\sum_{i} \lambda\right)}}$$

$$= \mathcal{P}_{\beta}^{2J,2h,\lambda}\left(2^{\#_w(\xi)}\right)$$

Define the (Random Cluster) Measure

$$\widetilde{\mathcal{P}}_{\beta}^{2J,2h,\lambda}(d\xi) = \frac{\mathcal{P}_{\beta}^{2J,2h,\lambda}\left(2^{\#_w(\xi)}; d\xi\right)}{\mathcal{P}_{\beta}^{2J,2h,\lambda}\left(2^{\#_w(\xi)}\right)}$$

Representation of Mean Value

$$\frac{\mathrm{Tr}\left(\widehat{W}_i^z e^{-\beta \widehat{H}_{\Lambda}}\right)}{\mathrm{Tr}\left(e^{-\beta \widehat{H}_{\Lambda}}\right)} = \widetilde{\mathcal{P}}_{\beta}^{2J,2h,\lambda}\left((i,0) \leftrightarrow \mathfrak{g}\right)$$

Representation of Two Point Function

$$\frac{\mathrm{Tr}\left(\widehat{W}_i^z \widehat{W}_j^z e^{-\beta \widehat{H}_{\Lambda}}\right)}{\mathrm{Tr}\left(e^{-\beta \widehat{H}_{\Lambda}}\right)} = \widetilde{\mathcal{P}}_{\beta}^{2J,2h,\lambda}\left((i,0) \leftrightarrow (j,0)\right)$$

RC Representation

- Decomposition of $-\widehat{\mathsf{H}}_\Lambda$

$$-\left(\sum_i \lambda\right) \mathrm{I}$$

$$+\sum_{(i,j)} J_{ij} \widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}} + \sum_i h \widehat{\mathsf{W}}_i^{\mathsf{z}} + \sum_i 2\lambda \frac{\widehat{\mathsf{W}}_i^{\mathsf{x}} + \mathrm{I}}{2}$$

- Basis of $\mathbb{X}_\Lambda$: x-basis $\{|\nu\rangle\}_{\nu \in \{\pm 1\}^\Lambda}$

In the x basis

$$\widehat{\mathsf{W}}^{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \frac{\widehat{\mathsf{W}}^{\mathsf{x}} + \mathrm{I}}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Poisson Processes of Arrivals

- Flips: For $(i,j) \in \mathcal{E}$, Operators (matrices)

$$K_{ij} \equiv \widehat{\mathsf{W}}_i^{\mathsf{z}} \widehat{\mathsf{W}}_j^{\mathsf{z}} \quad \text{with intensity } J_{ij}$$

Action of $K_{ij}|\nu\rangle = |\widehat{\nu}^{(ij)}\rangle$: Simultaneous flip of i-th and j-th component of ν.

- Flips: For $i \in \Lambda$, Operators

$$K_i^h \equiv \widehat{\mathsf{W}}_i^{\mathsf{z}} \quad \text{with intensity } h$$

Action of $K_i|\nu\rangle = |\widehat{\nu}^{(i)}\rangle$: Flip of i-th component of ν.

- Marks: For $i \in \Lambda$, Operators

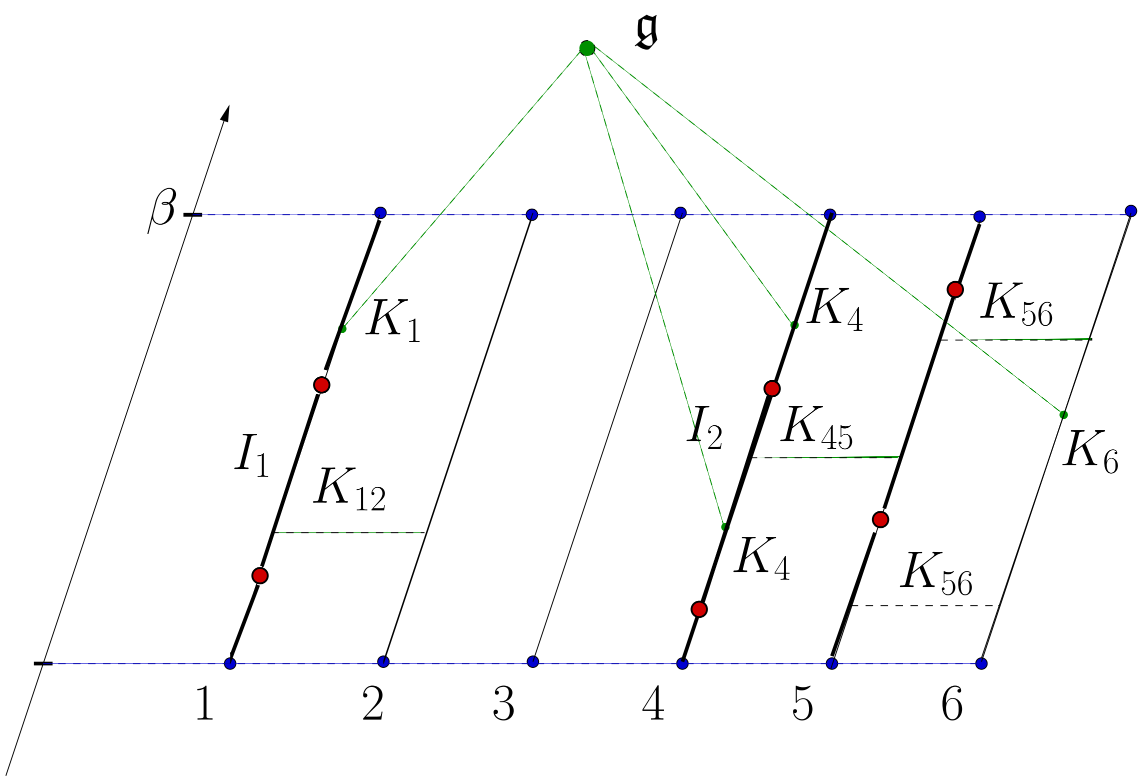
$$K_i^\lambda \equiv \frac{\widehat{\mathsf{W}}_i^{\mathsf{x}} + \mathrm{I}}{2} \quad \text{with intensity } 2\lambda$$

Action: Arrival of K_i^λ at time t forces $\nu_i(t) = +1$

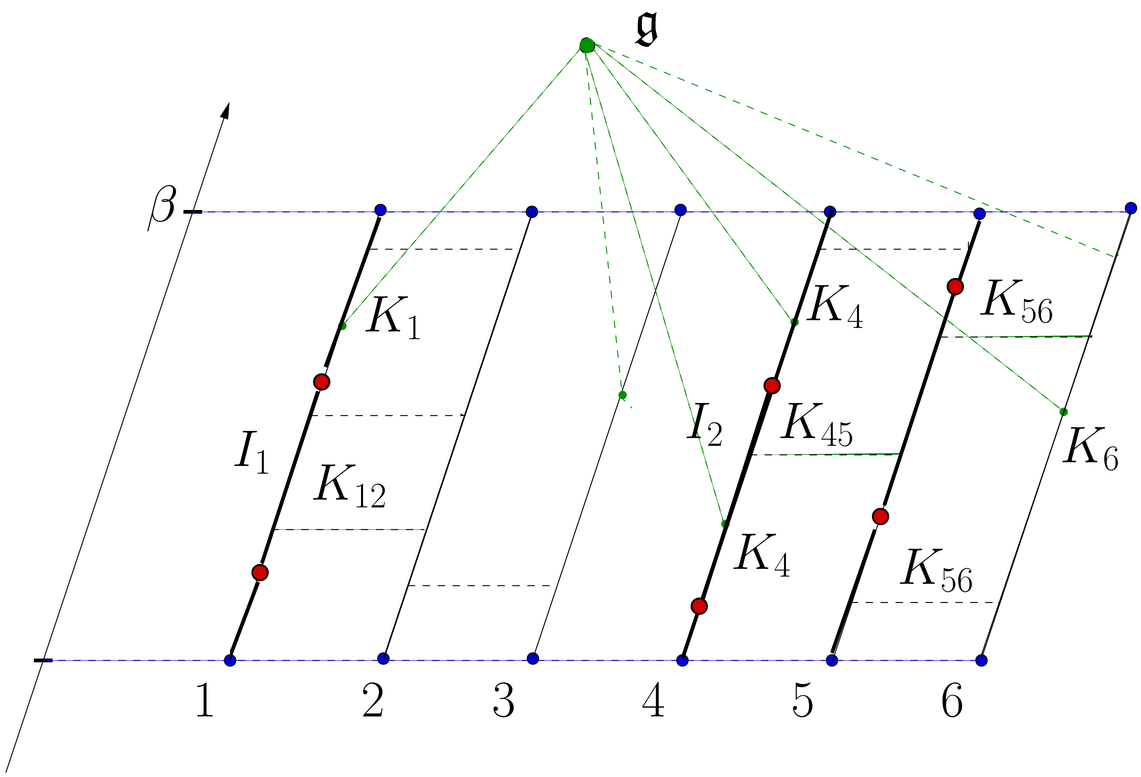
• $I_1, I_2, \ldots$ are marked intervals.

• Compatible trajectories $|\nu(t)\rangle \sim \xi$ have even number of flips on each marked interval and even number of flips on each $[0, \beta]_i$.

Example: Realization of ξ which has no compatible trajectories at all



Example: Realization of ξ which has 2^3 compatible trajectories



Notation:

- $\#_m(\xi)$: number of $i \in \Lambda$, such that $[0,\beta]_i$ contains no marks.
- $\partial\xi$: union of $\mathfrak{g}$ and all intervals, marked or $[0,\beta]_i$, whichever receives odd number of flips.

Representation of Partition Function

$$\frac{\mathcal{Z}_{\Lambda}^{\beta}}{e^{\beta(\sum_{ij} J_{ij}+\sum_i h)+\sum_i \lambda}} = \frac{\mathrm{Tr}\left(e^{-\beta \hat{H}_{\Lambda}}\right)}{e^{\beta(\sum_{ij} J_{ij}+\sum_i h+\lambda)}}$$

$$= \int \mathcal{P}_{\beta}^{J,h,2\lambda}(d\xi)\times\sum_{|\nu(\cdot)\rangle\sim\xi} \langle \nu(0)|\nu(\beta)\rangle \prod_{t\in\xi}\langle\nu(t-)|K_{\mathfrak{l}(t)}|\nu(t)\rangle$$

$$= \mathcal{P}_{\beta}^{J,h,2\lambda}\left(\partial\xi=\emptyset; 2^{\#_m(\xi)}\right)$$

Representation of Mean Value Define I_i marked interval containing $(i,0)$.

$$\frac{\mathrm{Tr}\left(\hat{W}_i^z e^{-\beta\hat{H}_{\Lambda}}\right)}{\mathrm{Tr}\left(e^{-\beta\hat{H}_{\Lambda}}\right)} = \frac{\mathcal{P}_{\beta}^{J,h,2\lambda}\left(\partial\xi=I_i\cup\mathfrak{g}; 2^{\#_m(\xi)}\right)}{\mathcal{P}_{\beta}^{J,h,2\lambda}\left(\partial\xi=\emptyset; 2^{\#_m(\xi)}\right)}$$

Representation of Two Point Function

$$\frac{\mathrm{Tr}\left(\hat{W}_i^z\hat{W}_j^z e^{-\beta\hat{H}_{\Lambda}}\right)}{\mathrm{Tr}\left(e^{-\beta\hat{H}_{\Lambda}}\right)} = \frac{\mathcal{P}_{\beta}^{J,h,2\lambda}\left(\partial\xi=I_i\cup I_j; 2^{\#_m(\xi)}\right)}{\mathcal{P}_{\beta}^{J,h,2\lambda}\left(\partial\xi=\emptyset; 2^{\#_m(\xi)}\right)}$$

Quantum Curie-Weiss Model

$$-\widehat{H}_N^{CW} = \frac{1}{2N} \sum_{ij} \widehat{W}_i^z \widehat{W}_j^z + \lambda \sum_{i=1}^{N} \widehat{W}_i^x$$

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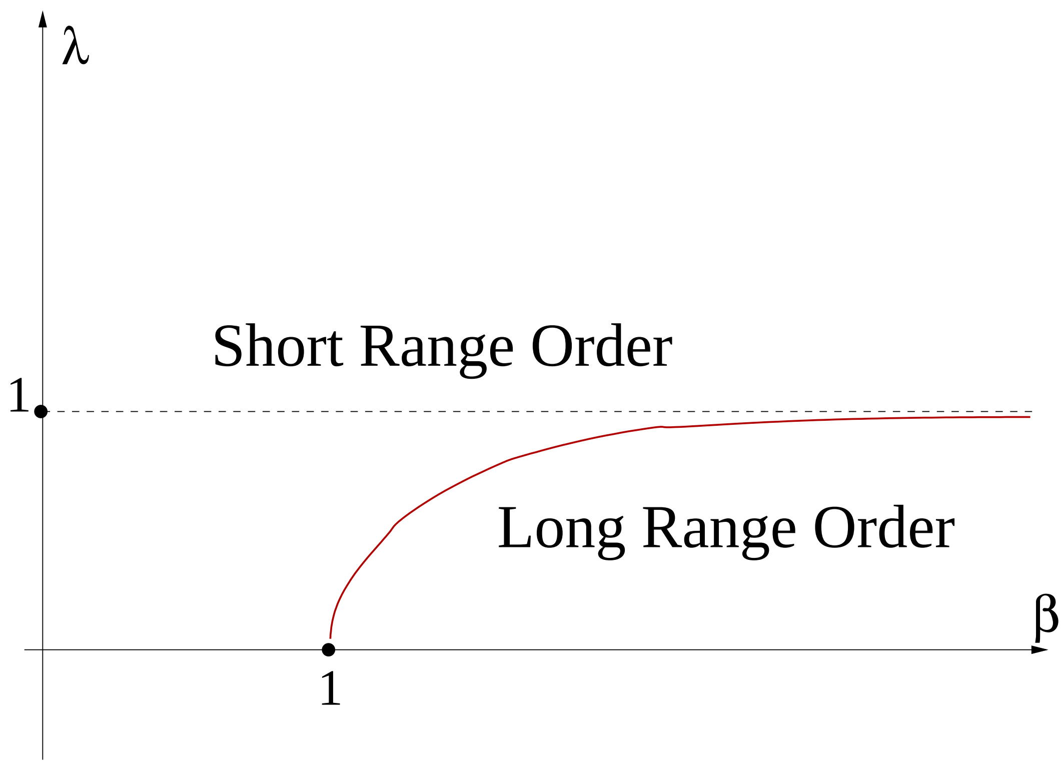
Critical Curve in the (β, λ) Plane

$$\mathfrak{f}(\beta, \lambda) \equiv \frac{1}{\lambda} \tanh(\beta\lambda) = 1.$$



Classical Curie-Weiss Model

$$-\mathcal{H}_N^{\mathsf{CW}}(\sigma) = \frac{1}{2N}\sum_{ij}\sigma_i\sigma_j \simeq \frac{N}{2}\left(\frac{1}{N}\sum_1^N \sigma_i\right)^2$$
$$\equiv \frac{1}{2N} m_N^2(\sigma)$$

Probability Distribution

$$\mu_N^\beta(\sigma) \simeq \mathrm{e}^{\frac{1}{2N} m_N^2(\sigma)} \mathbb{P}_N(\sigma)$$

where $\mathbb{P}_N$ - ± 1 Bernoulli.

Large Deviation Approach: The distribution of $m_N^2(\sigma)$ is sharply concentrated around

$$argmax\left\{\frac{1}{2}\beta m^2 - I(m)\right\} = argmax\left\{\frac{1}{\beta}\Lambda(\beta h) - \frac{1}{2}h^2\right\}$$

where

$$\Lambda(h) = \log\frac{\mathrm{e}^h + \mathrm{e}^{-h}}{2} \qquad I(m) = \sup_h\left\{hm - \Lambda(h)\right\}.$$

Critical Points

$$h = \frac{\mathrm{d}}{\beta \mathrm{d}h} \Lambda(\beta h) = \tanh(\beta h).$$

Conclusion: $\beta_c = 1$. $\beta > 1 \Leftrightarrow$ the spontaneous magnetization $m^*(\beta) > 0$.

Stochastic Geometric Approach

Arrival of links between i and j: Poisson on $[0, \beta]$ with intensity $2/N$

- $p_N = 1 - \mathrm{e}^{2\beta/N} \sim 2\beta/N$.
- $\mathbb{P}_N^\gamma$ Erdős-Rény random graph with $p = \gamma/N$
- $\widetilde{\mathbb{P}}_N^\gamma \simeq 2^{\#(\xi)} \mathbb{P}_N^\gamma$ the FK random graph

$$\mu_N^\beta(\sigma_i \sigma_j) = \widetilde{\mathbb{P}}_N^{2\beta}(i \longleftrightarrow j)$$

$$\simeq \widetilde{\mathbb{P}}_N^{2\beta}(\text{GiantComponent})$$

Critical Value for Erdős-Rény Random Graphs

$$\mathbb{P}_N^{\gamma}(\text{GiantComponent}) \to 0 \Leftrightarrow \gamma \leq 1.$$

Edwards-Sokal Construction

• Sample edges from $\widetilde{\mathbb{P}}_N^{\gamma}$
• Paint all connected clusters into red and blue with proability 1/2 each
• Conditional on M sites and $N - M$ blue sites the distribution of $\widetilde{\mathbb{P}}_N^{\gamma}$ is $\mathbb{P}_M^{\gamma} \otimes \mathbb{P}_{N-M}^{\gamma}$

Conclusion 1 (Immediate): If $\beta > 1$, then

$$\widetilde{\mathbb{P}}_N^{2\beta}(\text{GiantComponent}) \to 1$$

Indeed: Either $M \geq N/2$ or $N - M \geq N/2$
Conclusion 2 (Slightly more involved): If $\beta \leq 1$, then

$$\widetilde{\mathbb{P}}_N^{2\beta}(\text{GiantComponent}) \to 0$$

Quantum Random Graphs $\mathbb{P}_N^{\beta,\lambda}$

• Links: For each couple (i, j) arrive with intensity $1/N$
• Holes: For each $i = 1, \ldots, N$ arrive with intensity λ.

Size of a connected component $I = \cup I_l$:

$$|\mathcal{C}| = \sum_l |I_l|$$

Critical Curve in the (β, λ) Plane

$$\mathfrak{h}(\beta,\lambda) \equiv \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) - \beta\mathrm{e}^{-\lambda\beta} = 1$$

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Theorem (I, Levit):

Long Range Order

1. If $\mathfrak{h}(\beta,\lambda) < 1$,

$$\mathbb{P}_N^{\beta,\lambda}\left((i,t) \longleftrightarrow (j,s)\right) = O\left(\frac{\log N}{N}\right)$$

uniformly in $t, s \in [0,\beta]$ and $i \neq j$.

2. If $\mathfrak{h}(\beta,\lambda) > 1$ and $\beta < \infty$ then there exists $\rho = \rho(\beta,\lambda) \in (0,1)$, such that

$$\mathbb{P}_N^{\beta\lambda}\left((i,t) \longleftrightarrow (j,s)\right) = \rho(\beta,\lambda)^2 \left(1 + o(1)\right),$$

uniformly in $t, s \in [0,\beta]$ and $i \neq j$.

• $\mathcal{M}$ - size of the maximal connected component

• $\mathcal{M}^{\text{next}}$ - size of the next to maximal connected component.

Giant Componnets

1. If $\mathfrak{h}(\beta,\lambda) < 1$,

$$\mathbb{P}_N^{\beta,\lambda}(\mathcal{M} > c \log N) = o(1)$$

2. If $\mathfrak{h}(\beta,\lambda) > 1$ and $\beta < \infty$

$$\mathbb{P}_N^{\beta,\lambda}\left(\left|\frac{\mathcal{M}}{\beta N} - \rho\right| < \epsilon; \mathcal{M}^{\text{next}} < c \log N\right) \to 1$$

3. If $\beta = \infty$ and $\lambda < 2$, then there is unique giant component intersecting $t = 0$ section.

Comparison with Branching Random Walks on $\mathbb{S}_\beta$

1. Generate a random interval $I \subseteq \mathbb{S}_\beta$ around 0 The end-points of I would imitate two successive holes. Since the holes arrive with intensity λ the length $|I|$ should be distributed as $\min\{\Gamma(2,\lambda),\beta\}$.

2. Given a realization of I generate descendants according to the unit rate Poisson process on I.

Mean Value of Descendants X

- $\mathbb{E}(X|I) = |I|$.
- Let $V \sim \Gamma(2, \lambda)$ Then,

$$\mathbb{E}\left(|I|\right) = \mathbb{E}\left(V; V < \beta\right) + \beta \mathbb{P}\left(V \geq \beta\right),$$

- Now,

$$\mathbb{P}\left(V \geq \beta\right) = \int_{\beta}^{\infty} \lambda^2 t \mathrm{e}^{-\lambda t} \mathrm{d}t = (\lambda\beta + 1)\mathrm{e}^{-\lambda\beta}.$$

- In the same fashion,

$$\mathbb{E}\left(V; V \leq \beta\right) = \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) - \left(\beta^2\lambda + 2\beta\right)\mathrm{e}^{-\lambda\beta}.$$

- Consequently,

$$\mathbb{E}(X) = \mathbb{E}\left(|I|\right) = \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) - \beta\mathrm{e}^{-\lambda\beta},$$

Critical Curve for Quantum FK Model via Percolation Arguments

?

LD Approach

- Partial Trotterization: Set $M = \beta/\Delta$.

$$\frac{\mathrm{e}^{-\beta \widehat{\mathrm{H}}_N^{\mathrm{CW}}}}{\mathrm{e}^{\lambda N}} =$$

$$\lim_{\Delta \to 0} \left(\prod_{(i,j)} \mathrm{e}^{\frac{\Delta}{2N} \widehat{\mathrm{W}}_i^{\mathrm{z}} \widehat{\mathrm{W}}_j^{\mathrm{z}}} \prod_i \left\{ (1 - \Delta\lambda)\mathrm{I} + \Delta\lambda(\widehat{\mathrm{W}}_i^{\mathrm{x}} + \mathrm{I}) \right\} \right)^M$$

- The matrices $\mathrm{e}^{\frac{\Delta}{N} \widehat{\mathrm{W}}_i^{\mathrm{z}} \widehat{\mathrm{W}}_j^{\mathrm{z}}}$ are diagonal in the z-basis,

$$\langle \sigma | \mathrm{e}^{\frac{\Delta}{2N} \widehat{\mathrm{W}}_i^{\mathrm{z}} \widehat{\mathrm{W}}_j^{\mathrm{z}}} | \sigma \rangle = \mathrm{e}^{\frac{\Delta}{2N} \sigma_i \sigma_j}$$

- $\xi = (\xi_1, \dots, \xi_N)$ Poisson processes of arrival of holes on $\mathbb{S}_\beta$. A classical trajectory $\sigma : \mathbb{S}_\beta \to \{\pm 1\}^N$ is compatible $\sigma(\cdot) \sim \xi$ if i-th components $\sigma_i(\cdot)$ jump only at arrival times of ξ_i.

- Number of compatible trajectories $2^{\sum_i \#(\xi_i)}$

Poisson Limit:

$$\frac{\mathrm{Tr}\left(e^{-\beta \widehat{H}_N^{CW}}\right)}{e^{\lambda N}} =$$

$$\int \mathcal{P}_N^{\beta,\lambda}(d\xi) \sum_{\sigma \sim \xi} \exp\left\{ \int_0^\beta \frac{1}{N} \sum_{(i,j)} \sigma_i(t)\sigma_j(t) dt \right\}.$$

- Define

$$\widetilde{\mathcal{P}}_N^{\beta,\lambda}(d\xi) = \frac{2^{\#(\xi)} \mathcal{P}_N^{\beta,\lambda}(d\xi)}{\mathcal{P}_N^{\beta\lambda}\left(2^{\#(\xi)}\right)} = \otimes \widetilde{\mathcal{P}}^{\beta,\lambda}$$

- Define $\mathcal{Q}^{\beta,\lambda}$ the measure on piecewise constant classical trajectories $\sigma : \mathbb{S}_\beta \to \{\pm 1\}$:

STEP 1 Sample holes ξ from $\widetilde{\mathcal{P}}^{\beta,\lambda}$

STEP 2 Paint each connected component of ξ into ± 1 with probability $1/2$ each

$$\mathcal{Q}_N^{\beta,\lambda} \equiv \bigotimes_1^N \mathcal{Q}^{\beta,\lambda}$$

Representation Formula

$$\mathsf{Tr}\left(\mathsf{e}^{-\beta \hat{\mathsf{H}}_N^{\mathsf{CW}}}\right) \simeq \mathcal{Q}_N^{\beta,\lambda}\left(\mathsf{e}^{\frac{N}{2}\int_0^\beta m_N^2(t)\mathsf{d}t}\right)$$

where

$$m_N(t) = \frac{1}{N}\sum_{i=1}^{N} \sigma_N(t)$$

Variational Problem (VP) (on $\mathbb{L}_2(\mathbb{S}_\beta)$)

$$\sup_m \left\{\frac{1}{2}\int_0^\beta m^2(t)\mathsf{d}t \; - \; I(m)\right\} \triangleq \sup_m \mathfrak{G}(m),$$

where

$$I(m) \; = \; \sup_h \left\{(h,m)_\beta - \Lambda(h)\right\}$$

and

$$\Lambda(h) \; = \; \log \mathcal{Q}^{\beta,\lambda}\left(\mathsf{e}^{(h,\sigma)_\beta}\right).$$

Theorem (L. Chayes, Crawford, I, Levit)

Set $\mathfrak{f}(\beta,\lambda) = \frac{1}{\lambda}\tanh(\beta\lambda)$

The variational problem (VP) has constant maximizers $\pm m^*(\lambda,\beta)\mathbf{1}$, where the spontaneous z-magnetization m^* satisfies:

- If $\mathfrak{f}(\lambda,\beta) \leq 1$, then $m^* = 0$.
- If $\mathfrak{f}(\lambda,\beta) > 1$, then $m^* > 0$, and, consequently there are two distinct solutions to (VP)

Furthermore, away from the critical curve the solutions $\pm m^*\mathbf{1}$ are stable in the following sense: There exists $c = c(\lambda,\beta) > 0$ and a strictly convex symmetric function U with a $U(r) \sim r\log r$ growth at infinity, such that

$$\mathfrak{G}(\pm m^* \cdot \mathbf{1}) - \mathfrak{G}(m) \geq c \max\left\{ \|m \pm m^* \cdot \mathbf{1}\|_\beta^2 , \int_0^\beta U(m'(t))\mathrm{d}t \right\},$$

where,

$$\|m \pm m^* \cdot \mathbf{1}\|_\beta^2 = \min\left\{ \|m - m^* \cdot \mathbf{1}\|_\beta^2 , \|m + m^* \cdot \mathbf{1}\|_\beta^2 \right\}.$$

As a result, the variational problem (VP) is also stable in the supremum norm $\|\cdot\|_{\text{sup}}$. Namely, there exists a constant $c_{\text{sup}} = c_{\text{sup}}(\lambda, \beta) > 0$, such that

$$\mathfrak{G}(\pm m^* \cdot \mathbf{1}) - \mathfrak{G}(m) \geq \exp\left\{ - \frac{c_{\text{sup}}}{\|m \pm m^* \cdot \mathbf{1}\|_{\text{sup}}} \right\}.$$

Finally we have the following expression for the decay of $m^* > 0$ in the super-critical region near the critical curve:

$$m^* = m^*(\lambda, \beta) = \sqrt{\frac{6\beta\left(\mathfrak{f}(\lambda, \beta) - 1\right)}{s_4(\lambda, \beta)}}\,(1 + o(1))$$

where the implicit constants depend on β and λ but are bounded below in compact regions of the parameter space.

Strong Coupling Limit Structure of $\mathcal{Q}^{\beta,\lambda}$

- Consider 1D Periodic lattice

$$\mathcal{L}_\Delta = \{0, \Delta, 2\Delta, \ldots, \beta - \Delta\}$$

- Consider Ising Model $\mathcal{Q}_\Delta^{\beta,\lambda}$ on $\{\pm 1\}^{\mathcal{L}_\Delta}$ with the coupling constant $e^{-2J} = \lambda\Delta$.

$$\mathcal{Q}_\Delta^{\beta,\lambda} \Longrightarrow \mathcal{Q}^{\beta,\lambda}$$

Properties of $\mathcal{Q}^{\beta,\lambda}$

- $\mathcal{Q}^{\beta,\lambda}$ possesses the FKG property

- $\mathcal{Q}^{\beta,\lambda}$ satisfies a qualitative version of the the GHS inequality: Given $h \in \mathbb{R}_+$ define

$$\mathcal{Q}_h^{\beta,\lambda}(\mathrm{d}\sigma) = \frac{\mathcal{Q}^{\beta,\lambda}\left(\mathrm{e}^{h(\sigma,\mathbb{1})_\beta}\,;\,\mathrm{d}\sigma\right)}{\mathcal{Q}^{\beta,\lambda}\left(\mathrm{e}^{h(\sigma,\mathbb{1})_\beta}\right)}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{V}\mathrm{ar}_h^{\beta,\lambda}\left((\sigma,\mathbb{1})_\beta\right) \le -ch\mathrm{e}^{-2\beta h}.$$

- $\mathcal{Q}^{\beta,\lambda}$ is reflection positive:
Let $0 < t_1 < \cdots < t_n < \beta/2$ and let $f : \{\pm 1\}^n \to \mathbb{R}$. Set $s_k = \beta - t_k$. Then,

$$\mathcal{Q}^{\beta,\lambda}\left(f(\sigma_{t_1},\ldots,\sigma_{t_n})f(\sigma_{s_1},\ldots,\sigma_{s_n})\right) \ge 0.$$

Implications of Reflection Positivity

Moment generating function

$$\Lambda(h) = \log \mathcal{Q}^{\beta,\lambda}\left(\mathrm{e}^{(h,\sigma)_\beta}\right).$$

satisfies

$$\Lambda(h) \le \frac{1}{\beta}\int_0^\beta \Lambda\left(h(t)\mathbb{1}\right)\mathrm{d}t$$

Consequently, for $h \in \mathbb{L}_2(\mathbb{S}_\beta)$,

$$\begin{aligned}
&\Lambda(h) - \frac{1}{2}\int_0^\beta h^2(t)\mathrm{d}t \\
&\quad \le \int_0^\beta \left\{\frac{1}{\beta}\Lambda\left(h(t)\mathbb{1}\right) - \frac{1}{2}h^2(t)\right\}\mathrm{d}t \\
&\quad \le \beta \sup_{h\in\mathbb{R}}\left\{\frac{1}{\beta}\Lambda\left(h\mathbb{1}\right) - \frac{1}{2}h^2\right\}
\end{aligned}$$

One Dimensional Variational Problem and Critical Curve

Recall: $\mathfrak{f}(\beta, \lambda) \equiv \frac{1}{\lambda}\tanh(\beta\lambda)$.

- Maximizers of

$$\max_{h \in \mathbb{R}} \left\{ \frac{1}{\beta} \Lambda \left(h \mathbb{1} \right) - \frac{1}{2} h^2 \right\},$$

are of the form $\pm h^*$, where $h^* > 0$ if and only if $\mathfrak{f}(\lambda, \beta) > 1$.

- Compute,

$$\frac{\mathrm{d}}{\mathrm{d}h} \left\{ \frac{1}{\beta} \Lambda(h\mathbb{1}) - \frac{1}{2} h^2 \right\} = \frac{1}{\beta} \mathcal{Q}_h^{\beta,\lambda} \Big((\sigma, \mathbb{1})_\beta \Big) - h.$$

Negative for h large enough $\Rightarrow$ maximum is attained.

First use of the GKS Inequality

$$\frac{\mathrm{d}}{\mathrm{d}h}\mathcal{Q}_h^{\beta,\lambda}\Big((\sigma, \mathbb{1})_\beta\Big) = \mathbb{V}\mathrm{ar}_h^{\beta,\lambda}\Big((\sigma, \mathbb{1})_\beta\Big).$$

Hence $h \mapsto \mathcal{Q}_h^{\beta,\lambda}\Big((\sigma, \mathbb{1})_\beta\Big)$ is concave on $\mathbb{R}_+$.

- Consequently non-trivial maximaizers iff

$$1 < \frac{1}{\beta}\mathbb{V}\mathrm{ar}_0^{\beta,\lambda}\Big((\sigma, \mathbb{1})_\beta\Big) = \mathfrak{f}(\beta, \lambda)$$

Second use of the GKS Inequality

- Stability of the 1D Variation Problem: For $\mathfrak{f}(\beta, \lambda) > 1$ there exists $d : [-1, 1] \mapsto \mathbb{R}_+$ such that

$$\mathfrak{G}(m\mathbb{1}) + d(m) \leq \mathfrak{G}(\pm m^* \mathbb{1})$$

and

$$d(m) \geq c\mathrm{e}^{-2\beta|h|} \min\left\{(m - m^*)^2, (m + m^*)^2\right\}$$

- $\mathbb{L}_2(\mathbb{S}_\beta)$-Stability of the (VP):

$$\mathfrak{G}(m(\cdot)) \leq \int_0^\beta \left\{\frac{1}{2}m^2(t) - \frac{1}{\beta} I(m(t)\mathbb{1})\right\} \mathrm{d}t$$
$$= \frac{1}{\beta}\int_0^\beta \mathfrak{G}(m(t)\mathbb{1})\mathrm{d}t$$

Use of the FKG Ineqiality

Fact: Exists $\eta > 0$ such that $\widetilde{\mathcal{P}}_h^{\beta,\lambda} \prec \mathcal{P}^{\beta,\eta}$

- Stability of (VP) in the sup-norm

$$m_N(t) = \frac{1}{N} \sum_i \sigma_i$$

Let $\mathcal{R} = \{0 < t_1 < t_2 < \cdots < t_n = t_0 < \beta\}$ be a partition of $\mathbb{S}_\beta$. Consider

$$z_N^{\mathcal{R}} \equiv (m_N(t_1) - m_N(t_0), \ldots, m_N(t_n) - m_N(t_{n-1}))$$

Random vectors $z_N^{\mathcal{R}}$ satisfy a LD principle with rate function

$$I^{\mathcal{R}} = \max_{g_1,\cdots,g_n} \left\{ \sum_i g_i z_i - \Lambda^{\mathcal{R}}(g_1, \ldots, g_n) \right\}$$

where

$$\Lambda^{\mathcal{R}} = \mathcal{Q}_h^{\beta,\lambda} \left(\mathrm{e}^{\sum_i g_i(\sigma(t_i)-\sigma(t_{i-1}))} \right)$$

In view of Edwards-Sokal Representation

$$\mathrm{e}^{\sum_i g_i(\sigma(t_i)-\sigma(t_{i-1}))}$$

$$\leq \prod_i \left(1 + \mathbb{1}_{\{\xi([t_{i-1},t_i])>0\}}(\mathrm{e}^{2g_i} + \mathrm{e}^{-2g_i})\right).$$

The RHS is monotone in ξ. Hence by FKG,

$$\mathcal{Q}_h^{\beta,\lambda}\left(\mathrm{e}^{\sum_i g_i(\sigma(t_i)-\sigma(t_{i-1}))}\right)$$

$$\leq \prod_i \left(1 + \left(1 - \mathrm{e}^{-\eta|t_i - t_{i-1}|}\right)(\mathrm{e}^{2g_i} + \mathrm{e}^{-2g_i})\right).$$

Consequently,

$$\Lambda^{\mathcal{R}}(g_1, \ldots, g_n) \leq \eta \sum_i |t_i - t_{i-1}| H(g_i),$$

where $H(g) = (\mathrm{e}^{2g} + \mathrm{e}^{-2g})$.

By duality,

$$I^{\mathcal{R}}(z_1,\ldots,z_n) \geq \sum_i |t_i - t_{i-1}| U_\eta \left(\frac{z_i}{|t_i - t_{i-1}|} \right),$$

with,

$$U_\eta(z) = \eta H^* \left(\frac{z}{\eta} \right),$$

and H^* is the Legendre transform of H.

• Conclusion: For every partition $\mathcal{R}$,

$$I(m) \geq \sum_i |t_i - t_{i-1}| U_\eta \left(\frac{m(t_i) - m(t_{i-1})}{|t_i - t_{i-1}|} \right)$$

• U_η is smooth, strictly convex, $U_\eta(m) \sim |m| \log |m|$ at infinity (but $U_\eta(0) = -2\eta$)

$$I(m) \geq \begin{cases} \infty, & \text{if } m \text{ is not a.c.} \\ \int_0^\beta U_\eta(m'(t))\mathrm{d}t, & \text{otherwise} \end{cases}$$