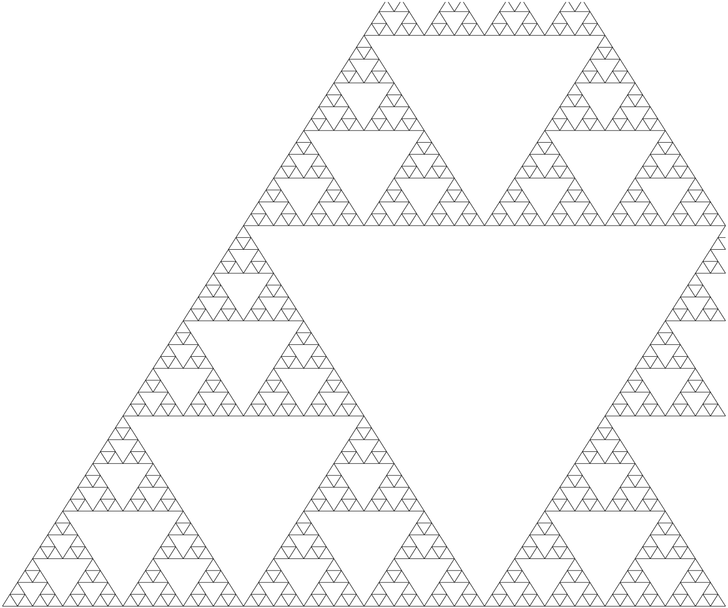
Random walk on random fractals: the Alexander-Orbach conjecture

Gady Kozma (speaker) and Asaf Nachmias

IMU 2008

The graphical Sierpinski gasket



Settings

- Let G be any infinite connected graph. Let $R(n)$ be a random walk on G i.e. $R(n+1)$ is chosen with equal probability among all the neighbors of $R(n)$ in G.

Settings

- Let G be any infinite connected graph. Let $R(n)$ be a random walk on G i.e. $R(n+1)$ is chosen with equal probability among all the neighbors of $R(n)$ in G.
- For any two vertices x and y let $d(x,y)$ be the *graph distance* between them, namely the length of the shortest path between x and y in G.

  Examine the dispacement $D_n = \mathbb{E}d(R(0),R(n))$

Settings

- Let G be any infinite connected graph. Let $R(n)$ be a random walk on G i.e. $R(n+1)$ is chosen with equal probability among all the neighbors of $R(n)$ in G.
- For any two vertices x and y let $d(x,y)$ be the *graph distance* between them, namely the length of the shortest path between x and y in G.

  Examine the dispacement $D_n = \mathbb{E}d(R(0), R(n))$
- If $G = \mathbb{Z}^d$ then $D_n \approx \sqrt{n}$. This behavior is called *diffusive*.

Settings

- Let G be any infinite connected graph. Let $R(n)$ be a random walk on G i.e. $R(n+1)$ is chosen with equal probability among all the neighbors of $R(n)$ in G.
- For any two vertices x and y let $d(x,y)$ be the *graph distance* between them, namely the length of the shortest path between x and y in G.

  Examine the dispacement $D_n = \mathbb{E}d(R(0), R(n))$
- If $G = \mathbb{Z}^d$ then $D_n \approx \sqrt{n}$. This behavior is called *diffusive*.
- If G is a binary tree, then $D_n \approx n$. This behavior is called *ballistic*. In our setting ("reversible"), this is possible only if the volume of balls grows exponentially.

Settings

- Let G be any infinite connected graph. Let $R(n)$ be a random walk on G i.e. $R(n+1)$ is chosen with equal probability among all the neighbors of $R(n)$ in G.
- For any two vertices x and y let $d(x,y)$ be the *graph distance* between them, namely the length of the shortest path between x and y in G.

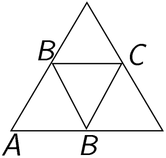
  Examine the dispacement $D_n = \mathbb{E}d(R(0), R(n))$
- If $G = \mathbb{Z}^d$ then $D_n \approx \sqrt{n}$. This behavior is called *diffusive*.
- If G is a binary tree, then $D_n \approx n$. This behavior is called *ballistic*. In our setting ("reversible"), this is possible only if the volume of balls grows exponentially.
- When $D_n \approx n^{1/\beta}$ for some $\beta > 2$, the process is called *subdiffusive*. Another name is *anomalous diffusion*.

Random walk on the Sierpinski gasket

- Denote by T_n the expected time a random walk on the Sierpinski gasket exits a triangle of order n. Our goal is to calculate T_n inductively.

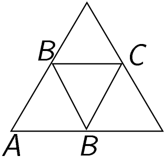
Random walk on the Sierpinski gasket

- Denote by T_n the expected time a random walk on the Sierpinski gasket exits a triangle of order n. Our goal is to calculate T_n inductively.
- Denote by $A = T_{n+1}$, B and C the expected exit times from the three ramification points.



Random walk on the Sierpinski gasket

- Denote by T_n the expected time a random walk on the Sierpinski gasket exits a triangle of order n. Our goal is to calculate T_n inductively.
- Denote by $A = T_{n+1}$, B and C the expected exit times from the three ramification points.

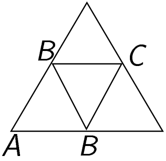


- Using the symmetries one gets

$$A = T_n + B$$
$$B = T_n + \tfrac{1}{4}B + \tfrac{1}{4}A + \tfrac{1}{4}C$$
$$C = T_n + \tfrac{1}{2}B$$

Random walk on the Sierpinski gasket

- Denote by $A = T_{n+1}$, B and C the expected exit times from the three ramification points.



- Using the symmetries one gets

$$A = T_n + B$$
$$B = T_n + \tfrac{1}{4}B + \tfrac{1}{4}A + \tfrac{1}{4}C$$
$$C = T_n + \tfrac{1}{2}B$$

- Solving one gets $T_{n+1} = A = 5T_n$, $B = 4T_n$ and $C = 3T_n$. Hence $T_n = 5^n$.

Sierpinski gasket II

- A simple induction shows that the diameter of an order-n triangle is 2^n. Therefore we get

$$D_{5^n} \approx 2^n$$

So

$$\beta = \frac{\log 5}{\log 2}.$$

Sierpinski gasket II

- A simple induction shows that the diameter of an order-n triangle is 2^n. Therefore we get

$$D_{5^n} \approx 2^n$$

So

$$\beta = \frac{\log 5}{\log 2}.$$

- Any *finitely ramified* fractal can be handled that way.

Definitions

- In any infinite graph G one can define a ball $B(x, r)$ with respect to the graph metric. Denote by $|B(x, r)|$ its volume, or simply number of vertices in it. If for some d_f,

$$|B(x, r)| = r^{d_f + o(1)}$$

(if it holds for one x then it holds for all x) then we say that G has *volume growth dimension* d_f. d_f is the graphical analog of the Hausdorff dimension.

Definitions

- In any infinite graph G one can define a ball $B(x,r)$ with respect to the graph metric. Denote by $|B(x,r)|$ its volume, or simply number of vertices in it. If for some d_f,

$$|B(x,r)| = r^{d_f+o(1)}$$

(if it holds for one x then it holds for all x) then we say that G has *volume growth dimension* d_f. d_f is the graphical analog of the Hausdorff dimension.

- Let $p_t(x,y)$ be the probability that random walk starting from x will be exactly at y at time t. If $G = \mathbb{Z}^d$ then $p_t(x,x) \approx t^{-d/2}$. To define a dimension using this relation, we say that if for some d_s

$$p_t(x,x) = t^{-d_s/2+o(1)}$$

(if it holds for one x then it holds for all x) then we call d_s the *spectral dimension*.

Examples

- For $\mathbb{Z}^d$, $d_s = d_f = d$.

Examples

- For $\mathbb{Z}^d$, $d_s = d_f = d$.
- For any transitive graph, $d_s = d_f$ and the common value is integer (Gromov 81, Trofimov 85, Delmotte 99).

Examples

- For $\mathbb{Z}^d$, $d_s = d_f = d$.
- For any transitive graph, $d_s = d_f$ and the common value is integer (Gromov 81, Trofimov 85, Delmotte 99).
- Let $\alpha > 0$ and examine the subset of $\mathbb{Z}^2$ given by

$$\{|y| \leq |x|^\alpha\}.$$

Then $d_f = d_s = 1 + \alpha$.

Examples

- For $\mathbb{Z}^d$, $d_s = d_f = d$.
- For any transitive graph, $d_s = d_f$ and the common value is integer (Gromov 81, Trofimov 85, Delmotte 99).
- Let $\alpha > 0$ and examine the subset of $\mathbb{Z}^2$ given by

$$\{|y| \le |x|^\alpha\}.$$

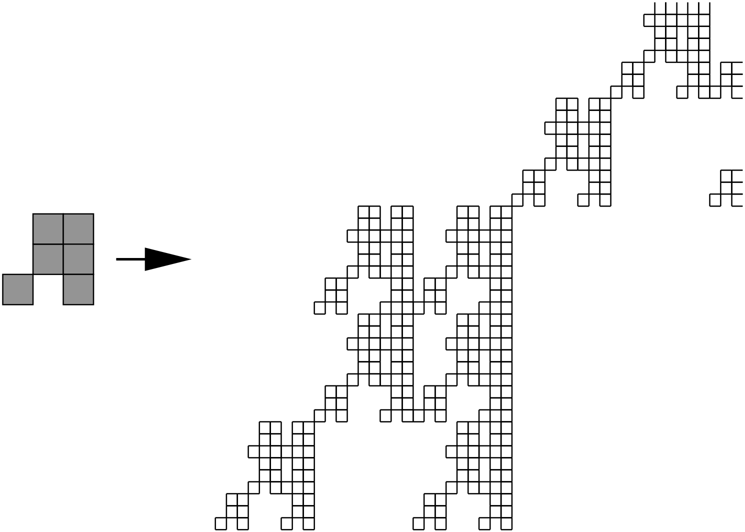
  Then $d_f = d_s = 1 + \alpha$.
- For the Sierpinski gasket,

$$d_s = 2\frac{\log 3}{\log 5} \qquad d_f = \frac{\log 3}{\log 2}$$

  and in particular $d_s < d_f$. (seeing the value of d_f is easy — a ball of radius 2^n around the "root" is simply a level n triangle which has volume 3^n).

Regularity results

- For any n and d and any connected subset of $\{1, \ldots, n\}^d$ one can construct an infinite connected graph, the *generalized Sierpinski carpet*.



Regularity results

- For any n and d and any connected subset of $\{1, \ldots, n\}^d$ one can construct an infinite connected graph, the *generalized Sierpinski carpet*.
- The usual Sierpinski carpet is created with $d = 2$, $n = 3$ and the subset being $\{1, 2, 3\}^2 \setminus \{(2, 2)\}$.

Regularity results

- For any n and d and any connected subset of $\{1,\ldots,n\}^d$ one can construct an infinite connected graph, the *generalized Sierpinski carpet*.
- The usual Sierpinski carpet is created with $d = 2$, $n = 3$ and the subset being $\{1,2,3\}^2 \setminus \{(2,2)\}$.

Theorem (Barlow & Bass, 1999)

For any generalized Sierpinski carpet, d_s, d_f and β are well defined, and one has $d_s = 2d_f/\beta$.

Regularity results

- For any n and d and any connected subset of $\{1, \ldots, n\}^d$ one can construct an infinite connected graph, the *generalized Sierpinski carpet*.
- The usual Sierpinski carpet is created with $d = 2$, $n = 3$ and the subset being $\{1, 2, 3\}^2 \setminus \{(2, 2)\}$.

Theorem (Barlow & Bass, 1999)

For any generalized Sierpinski carpet, d_s, d_f and β are well defined, and one has $d_s = 2d_f/\beta$.
Further, one has the following estimate

$$p_t(x, y) \approx Ct^{-d_s/2} \exp\left(-c\left(\frac{|x-y|^\beta}{t}\right)^{1/(\beta-1)}\right)$$

Theorem (Barlow, 2004)

Any value of β between 2 and $d_f + 1$ is possible. Similarly, any value of d_s between $2d_f/(d_f + 1)$ and d_f is possible.

Definition of p_c

- Let G be any infinite graph. Let $0 \leq p \leq 1$. Consider the random graph G_p that one gets by keeping every edge of G with probability p, independently for each edge.

Definition of p_c

- Let G be any infinite graph. Let $0 \le p \le 1$. Consider the random graph G_p that one gets by keeping every edge of G with probability p, independently for each edge.
- Let $\psi(p)$ be the probability that G_p has an infinite component. $\psi(p)$ is obviously an increasing function of p.

Definition of p_c

- Let G be any infinite graph. Let $0 \leq p \leq 1$. Consider the random graph G_p that one gets by keeping every edge of G with probability p, independently for each edge.
- Let $\psi(p)$ be the probability that G_p has an infinite component. $\psi(p)$ is obviously an increasing function of p.
- Changing any finite set of edges cannot destroy or create an infinite cluster. Therefore $\psi(p)$ is either 0 or 1.

Definition of p_c

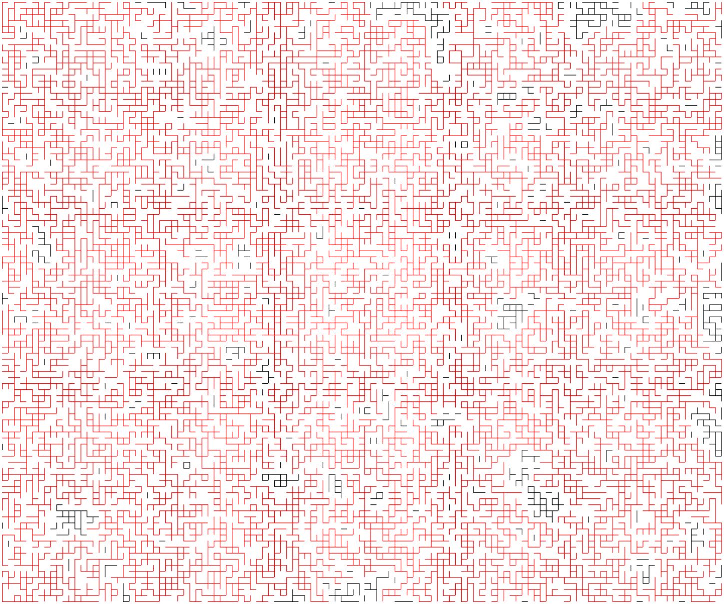
- Let G be any infinite graph. Let $0 \le p \le 1$. Consider the random graph G_p that one gets by keeping every edge of G with probability p, independently for each edge.
- Let $\psi(p)$ be the probability that G_p has an infinite component. $\psi(p)$ is obviously an increasing function of p.
- Changing any finite set of edges cannot destroy or create an infinite cluster. Therefore $\psi(p)$ is either 0 or 1.
- Therefore there exists some p_c, depending on G, such that $\psi(p) = 0$ for $p < p_c$ and $\psi(p) = 1$ for $p > p_c$.

Percolation on $\mathbb{Z}^2$, $p = 0.45$

Percolation on $\mathbb{Z}^2$, $p = 0.45$

Percolation on $\mathbb{Z}^2$, $p = 0.55$

Percolation on $\mathbb{Z}^2$, $p = 0.55$



Simple examples

- for $G = \mathbb{Z}$, $p_c = 1$ and $\psi(p_c) = 1$ (exercise).

Simple examples

- for $G = \mathbb{Z}$, $p_c = 1$ and $\psi(p_c) = 1$ (exercise).
- for G a d-regular tree, $p_c = \frac{1}{d-1}$ and $\psi(p_c) = 0$. This is equivalent to a Galton-Watson branching process.

Simple examples

- for $G = \mathbb{Z}$, $p_c = 1$ and $\psi(p_c) = 1$ (exercise).
- for G a d-regular tree, $p_c = \frac{1}{d-1}$ and $\psi(p_c) = 0$. This is equivalent to a Galton-Watson branching process.
- The complete graph on n vertices exhibits similar behvior (even though it is finite) with "$p_c = \frac{1}{n}$" and "$\psi(p_c) = 0$", Erdős & Rényi (1959).

$p \neq p_c$

- In the subcritical case, component sizes decay exponentially in the volume, i.e. for every $p < p_c$ there exist some $\lambda > 0$ such that

$$\mathbb{P}(|\mathcal{C}| > n) \leq e^{-\lambda n}$$

where $\mathcal{C}$ is the cluster containing the origin. Menshikov (1986), Aizenman & Barsky (1987).

$p \neq p_c$

- In the subcritical case, component sizes decay exponentially in the volume, i.e. for every $p < p_c$ there exist some $\lambda > 0$ such that
$$\mathbb{P}(|\mathcal{C}| > n) \leq e^{-\lambda n}$$
where $\mathcal{C}$ is the cluster containing the origin. Menshikov (1986), Aizenman & Barsky (1987).
- In the supercritical case there exists one infinite cluster (Burton & Keane, 1989). The sizes of the finite clusters decay exponentially in the surface area, i.e. for every $p > p_c$ there exists some λ such that
$$\mathbb{P}(n < |\mathcal{C}| < \infty) \leq e^{-\lambda n^{(d-1)/d}}$$
Grimmett & Marstrand (1990), Kesten & Zhang (1990).

$p \neq p_c$

- In the subcritical case, component sizes decay exponentially in the volume, i.e. for every $p < p_c$ there exist some $\lambda > 0$ such that
$$\mathbb{P}(|\mathcal{C}| > n) \leq e^{-\lambda n}$$
where $\mathcal{C}$ is the cluster containing the origin. Menshikov (1986), Aizenman & Barsky (1987).
- In the supercritical case there exists one infinite cluster (Burton & Keane, 1989). The sizes of the finite clusters decay exponentially in the surface area, i.e. for every $p > p_c$ there exists some λ such that
$$\mathbb{P}(n < |\mathcal{C}| < \infty) \leq e^{-\lambda n^{(d-1)/d}}$$
Grimmett & Marstrand (1990), Kesten & Zhang (1990).
- In most senses, the supercritical cluster “looks like a stretched-out grid”. In neither case is it reasonable to claim that clusters are “fractal”.

$p = p_c$

Some conjectures coming from the physics literature:

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially**, i.e.

$$\mathbb{P}(|\mathcal{C}| > n) \approx n^{-1/\delta}$$

for some δ.

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially**, i.e.

$$\mathbb{P}(|\mathcal{C}| > n) \approx n^{-1/\delta}$$

for some δ.

(c). Universality: δ depends only on the dimension, and not on the specific grid (unlike, say, p_c).

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially**, i.e.

$$\mathbb{P}(|\mathcal{C}| > n) \approx n^{-1/\delta}$$

for some δ.

(c). Universality: δ depends only on the dimension, and not on the specific grid (unlike, say, p_c).

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2$.

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially**, i.e.

$$\mathbb{P}(|\mathcal{C}| > n) \approx n^{-1/\delta}$$

for some δ.

(c). Universality: δ depends only on the dimension, and not on the specific grid (unlike, say, p_c).

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2$. *In $d = 6$ there are logarithmic corrections.

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially**, i.e.

$$\mathbb{P}(|\mathcal{C}| > n) \approx n^{-1/\delta}$$

for some δ.

(c). Universality: δ depends only on the dimension, and not on the specific grid (unlike, say, p_c).

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2$. *In $d = 6$ there are logarithmic corrections. The conjecture for the value $\frac{91}{5}$ is related to a conjecture that the distribution of large finite clusters is conformally invariant.

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially*

(c). Universality: δ depends only on the dimension.

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2.$

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially*

(c). Universality: δ depends only on the dimension.

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2.$

What has been proved?

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially*

(c). Universality: δ depends only on the dimension.

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2$.

What has been proved?

- $d = 2$: a, Kesten (1980), “b, d” Smirnov (2001); Lawler, Schramm & Werner (2001).

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially*

(c). Universality: δ depends only on the dimension.

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2.$

What has been proved?

- $d = 2$: a, Kesten (1980), "b, d" Smirnov (2001); Lawler, Schramm & Werner (2001).
- $d > 6$: "a, b, c, d" Hara & Slade (1990).

$p = p_c$

Some conjectures coming from the physics literature:

(a). For $d > 1$ there is no infinite cluster at the critical p.

(b). The size of the critical cluster decays *polynomially*

(c). Universality: δ depends only on the dimension.

(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2.$

What has been proved?

- $d = 2$: a, Kesten (1980), "b, d" Smirnov (2001); Lawler, Schramm & Werner (2001).
- $d > 6$: "a, b, c, d" Hara & Slade (1990).
- $d = 3, 4, 5, 6$: not even a.

Definitions

We want "critical percolation conditioned to have an infinite cluster".

Definitions

We want "critical percolation conditioned to have an infinite cluster". Suggestions:

(a). (Kesten, 1986). Take $p > p_c$. Condition on 0 being in the infinite cluster. Take $p \to p_c$.

Definitions

We want "critical percolation conditioned to have an infinite cluster". Suggestions:

(a). (Kesten, 1986). Take $p > p_c$. Condition on 0 being in the infinite cluster. Take $p \to p_c$. The limit here is in the space $\mathcal{M}(\{0,1\}^{E(\mathbb{Z}^d)})$ i.e. the space of measures on configurations.

Definitions

We want "critical percolation conditioned to have an infinite cluster". Suggestions:

(a). (Kesten, 1986). Take $p > p_c$. Condition on 0 being in the infinite cluster. Take $p \to p_c$. The limit here is in the space $\mathcal{M}(\{0,1\}^{E(\mathbb{Z}^d)})$ i.e. the space of measures on configurations.

(b). (Kesten, 1986). Take critical percolation. Condition on $0 \leftrightarrow \partial B(0,r)$. Take $r \to \infty$.

Definitions

We want "critical percolation conditioned to have an infinite cluster". Suggestions:

(a). (Kesten, 1986). Take $p > p_c$. Condition on 0 being in the infinite cluster. Take $p \to p_c$. The limit here is in the space $\mathcal{M}(\{0,1\}^{E(\mathbb{Z}^d)})$ i.e. the space of measures on configurations.

(b). (Kesten, 1986). Take critical percolation. Condition on $0 \leftrightarrow \partial B(0,r)$. Take $r \to \infty$.

(c). (van der Hostad & Járai, 2004). Take critical percolation. Condition on $0 \leftrightarrow x$ for some $x \in \mathbb{Z}^d$. Take $x \to \infty$.

Definitions

We want "critical percolation conditioned to have an infinite cluster". Suggestions:

(a). (Kesten, 1986). Take $p > p_c$. Condition on 0 being in the infinite cluster. Take $p \to p_c$. The limit here is in the space $\mathcal{M}(\{0,1\}^{E(\mathbb{Z}^d)})$ i.e. the space of measures on configurations.

(b). (Kesten, 1986). Take critical percolation. Condition on $0 \leftrightarrow \partial B(0,r)$. Take $r \to \infty$.

(c). (van der Hostad & Járai, 2004). Take critical percolation. Condition on $0 \leftrightarrow x$ for some $x \in \mathbb{Z}^d$. Take $x \to \infty$.

(d). Condition on the size of the cluster being $> n$. Take $n \to \infty$.

Definitions

We want "critical percolation conditioned to have an infinite cluster". Suggestions:

(a). (Kesten, 1986). Take $p > p_c$. Condition on 0 being in the infinite cluster. Take $p \to p_c$. The limit here is in the space $\mathcal{M}(\{0,1\}^{E(\mathbb{Z}^d)})$ i.e. the space of measures on configurations.

(b). (Kesten, 1986). Take critical percolation. Condition on $0 \leftrightarrow \partial B(0,r)$. Take $r \to \infty$.

(c). (van der Hostad & Járai, 2004). Take critical percolation. Condition on $0 \leftrightarrow x$ for some $x \in \mathbb{Z}^d$. Take $x \to \infty$.

(d). Condition on the size of the cluster being $> n$. Take $n \to \infty$.

We will use (c). This object is called the *incipient infinite cluster*, or IIC for short.

- $d = 2$
  Kesten (1986) showed that random walk on the IIC of $\mathbb{Z}^2$ is subdiffusive in the sense that $\mathbb{E}|R(t)| \leq t^{1/2-eps}$.

- $d = 2$
  Kesten (1986) showed that random walk on the IIC of $\mathbb{Z}^2$ is subdiffusive in the sense that $\mathbb{E}|R(t)| \leq t^{1/2-eps}$. Note that this is not subdiffusive in the same sense as above because we measure distance by $|x|$ ("the extrinsic distance") rather than by $d(0,x)$ ("the intrinsic distance").

- $d = 2$
  Kesten (1986) showed that random walk on the IIC of $\mathbb{Z}^2$ is subdiffusive in the sense that $\mathbb{E}|R(t)| \leq t^{1/2-eps}$. Note that this is not subdiffusive in the same sense as above because we measure distance by $|x|$ ("the extrinsic distance") rather than by $d(0, x)$ ("the intrinsic distance").

  It is known that β_{ext} (if it exists) is between $2\frac{31}{48}$ and $3\frac{7}{48}$. This uses SLE theory. Even physicists do not have a good guess what is the correct value.

- $d = 2$
  Kesten (1986) showed that random walk on the IIC of $\mathbb{Z}^2$ is subdiffusive in the sense that $\mathbb{E}|R(t)| \leq t^{1/2-eps}$. Note that this is not subdiffusive in the same sense as above because we measure distance by $|x|$ ("the extrinsic distance") rather than by $d(0,x)$ ("the intrinsic distance").

  It is known that β_{ext} (if it exists) is between $2\frac{31}{48}$ and $3\frac{7}{48}$. This uses SLE theory. Even physicists do not have a good guess what is the correct value.
- Tree
  Kesten (1986) showed that random walk on the IIC of a regular tree has $\beta = 3$, $d_f = 2$ and $d_s = \frac{4}{3}$.

At last,

Conjecture (Alexander & Orbach, 1982)

For every $d > 1$, the IIC exhibits $d_s = \frac{4}{3}$.

At last,

Conjecture (Alexander & Orbach, 1982)

For every $d > 1$, the IIC exhibits $d_s = \frac{4}{3}$.

For $d < 6$ this was based on rough correspondance with numerical results, and better simulations "disproved" it.

At last,

Conjecture (Alexander & Orbach, 1982)

For every $d > 1$, the IIC exhibits $d_s = \frac{4}{3}$.

For $d < 6$ this was based on rough correspondance with numerical results, and better simulations "disproved" it.

For $d \geq 6$ this was based on various conjectures that were considered proven in the physics literature. Hence this was also considered to be a proven fact. In the mathematical community this became known as the Alexander-Orbach conjecture.

At last,

Conjecture (Alexander & Orbach, 1982)

For every $d > 1$, the IIC exhibits $d_s = \frac{4}{3}$.

For $d < 6$ this was based on rough correspondance with numerical results, and better simulations "disproved" it.

For $d \geq 6$ this was based on various conjectures that were considered proven in the physics literature. Hence this was also considered to be a proven fact. In the mathematical community this became known as the Alexander-Orbach conjecture.

We prove it under the same conditions of Hara & Slade i.e. d sufficiently large or $d > 6$ and a sufficiently spread-out lattice.

Electric resistance

- Let G be a finite connected graph, and let x and y be vertices of G. consider G as an electric network, where every vertex of G is a node, and every edge is a 1-ohm resistor. Denote the effective resistance between x and y by $R_{\text{eff}}(x, y)$.

Electric resistance

- Let G be a finite connected graph, and let x and y be vertices of G. consider G as an electric network, where every vertex of G is a node, and every edge is a 1-ohm resistor. Denote the effective resistance between x and y by $R_{\mathrm{eff}}(x,y)$.
- One way to connect random walk properties to the resistance is the *commute time identity*. It states that

$$\mathrm{Hit}(x,y)+\mathrm{Hit}(y,x)=2R_{\mathrm{eff}}(x,y)\cdot|E(G)|$$

  where $\mathrm{Hit}(x,y)$ is the expected time a random walker starting from x will first visit ("hit") y.
  We use this for G being a ball $B(0,r)$ in the IIC, with the entire boundary $\partial B(0,r)$ identified to one point, which we call ∂. If we knew that $|E(G)|\approx r^2$ and $R_{\mathrm{eff}}(0,\partial)\approx r$ we would get that

$$\mathrm{Hit}(0,\partial)+\mathrm{Hit}(\partial,0)\approx r^3$$

Electric resistance cont.

$$\mathrm{Hit}(0,\partial) + \mathrm{Hit}(\partial,0) \approx r^3$$

immediately gives the bound $\mathrm{Hit}(0,\partial B(0,r)) \le Cr^3$. The bound in the other direction uses that the graph is *strongly recurrent*. This also follows from the resistance estimate. Hence $\beta = 3$ would follow if we could prove that

$$|B(0,r)| \approx r^2 \qquad R_{\mathrm{eff}}(0,\partial B(0,r)) \approx r$$

(Barlow, Járai, Kumagai & Slade, 2008).

Electric resistance cont.

$$\mathrm{Hit}(0,\partial) + \mathrm{Hit}(\partial,0) \approx r^3$$

immediately gives the bound $\mathrm{Hit}(0,\partial B(0,r)) \le Cr^3$. The bound in the other direction uses that the graph is *strongly recurrent*. This also follows from the resistance estimate. Hence $\beta = 3$ would follow if we could prove that

$$|B(0,r)| \approx r^2 \qquad R_{\mathrm{eff}}(0,\partial B(0,r)) \approx r$$

(Barlow, Járai, Kumagai & Slade, 2008).

The estimate of the resistance would follow if we could show that there exists $\ge cr$ *pivotal edges*, i.e. edges whose removal would disconnect 0 from $\partial B(0,r)$. Which will follow, more-or-less, if we show that

$$\mathbb{P}(0 \leftrightarrow \partial B(0,r)) \approx \frac{1}{r}$$

(Nachmias & Peres, 2008)

Volume growth exponent

- On a tree,

$$\mathbb{E}(\partial B(0,2r)) = (\mathbb{E}(\partial B(0,r)))^2.$$

Volume growth exponent

- On a tree,

$$\mathbb{E}(\partial B(0,2r)) = (\mathbb{E}(\partial B(0,r)))^2.$$

- We would like to show something similar for $\mathbb{Z}^d$, but there are local interactions so the best we can hope for is

$$\mathbb{E}(\partial B(0,2r)) \geq c(\mathbb{E}(\partial B(0,r)))^2. \qquad (*)$$

Volume growth exponent

- On a tree,

$$\mathbb{E}(\partial B(0,2r)) = (\mathbb{E}(\partial B(0,r)))^2.$$

- We would like to show something similar for $\mathbb{Z}^d$, but there are local interactions so the best we can hope for is

$$\mathbb{E}(\partial B(0,2r)) \geq c(\mathbb{E}(\partial B(0,r)))^2. \qquad (*)$$

- If we could show that, we would be fine, since the minute $\mathbb{E}(\partial B(0,r))$ crosses some constant, it will start exploding.

Volume growth exponent

- On a tree,

$$\mathbb{E}(\partial B(0,2r)) = (\mathbb{E}(\partial B(0,r)))^2.$$

- We would like to show something similar for $\mathbb{Z}^d$, but there are local interactions so the best we can hope for is

$$\mathbb{E}(\partial B(0,2r)) \geq c(\mathbb{E}(\partial B(0,r)))^2. \qquad (*)$$

- If we could show that, we would be fine, since the minute $\mathbb{E}(\partial B(0,r))$ crosses some constant, it will start exploding.
- We could not show $(*)$ either, because the boundary of $B(0,r)$ is too fragile. We instead showed

$$\mathbb{E}(B(0,2r)) \geq \frac{c}{r}(\mathbb{E}(B(0,r)))^2.$$

which works just as well.

Resistance exponent

- A crucial point is the determination of δ by Barsky & Aizenman (1991),

$$\mathbb{P}(|\mathcal{C}_0| > n) \approx \frac{C}{\sqrt{n}}.$$

Resistance exponent

- A crucial point is the determination of δ by Barsky & Aizenman (1991),

$$\mathbb{P}(|\mathcal{C}_0| > n) \approx \frac{C}{\sqrt{n}}.$$

- This means we can restrict our attention to the case that $|B(0,r)| \leq Cr^2$, and hence for some $j \in [\frac{1}{3}r, \frac{2}{3}r]$, $|\partial B(0,j)| \leq Cr$.

Resistance exponent

- A crucial point is the determination of δ by Barsky & Aizenman (1991),

$$\mathbb{P}(|\mathcal{C}_0| > n) \approx \frac{C}{\sqrt{n}}.$$

- This means we can restrict our attention to the case that $|B(0,r)| \leq Cr^2$, and hence for some $j \in [\frac{1}{3}r, \frac{2}{3}r]$, $|\partial B(0,j)| \leq Cr$.
- Since, to get to distance r, we need to survive to $\frac{1}{3}r$, and then one of the $\leq Cr$ vertices in $\partial B(0,j)$ must survive another $\frac{1}{3}r$, we get (roughly)

$$\mathbb{P}(0 \leftrightarrow \partial B(0,3r)) \leq \frac{C}{r} + Cr \cdot (\mathbb{P}(0 \leftrightarrow \partial B(0,r)))^2$$

where the first term comes from Barsky & Aizenman.

Resistance exponent

- A crucial point is the determination of δ by Barsky & Aizenman (1991),

$$\mathbb{P}(|\mathcal{C}_0| > n) \approx \frac{C}{\sqrt{n}}.$$

- This means we can restrict our attention to the case that $|B(0,r)| \leq Cr^2$, and hence for some $j \in [\frac{1}{3}r, \frac{2}{3}r]$, $|\partial B(0,j)| \leq Cr$.
- Since, to get to distance r, we need to survive to $\frac{1}{3}r$, and then one of the $\leq Cr$ vertices in $\partial B(0,j)$ must survive another $\frac{1}{3}r$, we get (roughly)

$$\mathbb{P}(0 \leftrightarrow \partial B(0,3r)) \leq \frac{C}{r} + Cr \cdot (\mathbb{P}(0 \leftrightarrow \partial B(0,r)))^2$$

where the first term comes from Barsky & Aizenman.
- An induction then shows (roughly) the required estimate.

Thank you