Entropic Repulsion for a Gaussian Interface Model (in the critical and supercritical dimensions)

Noemi Kurt, Universität Zürich

La Pietra, June 2008

The model

Membrane model

- $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, Borel σ-algebra
- Configuration $\varphi = \{\varphi_x\}_{x \in \mathbb{Z}^d} \in \Omega$
- Domain $\Lambda \subset \mathbb{Z}^d$ finite, $\varphi_x = 0 \ \forall x \in \Lambda^c$ (0-boundary conditions)
- Hamiltonian of the membrane model:

$$H_\Lambda(\varphi) = \sum_{x \in \mathbb{Z}^d} (\Delta \varphi_x)^2,$$

where $\Delta \varphi_x := (\Delta \varphi)_x = \frac{1}{2d} \sum_{y:|x-y|=1} \varphi_y - \varphi_x$ is the discrete (lattice) Laplacian.

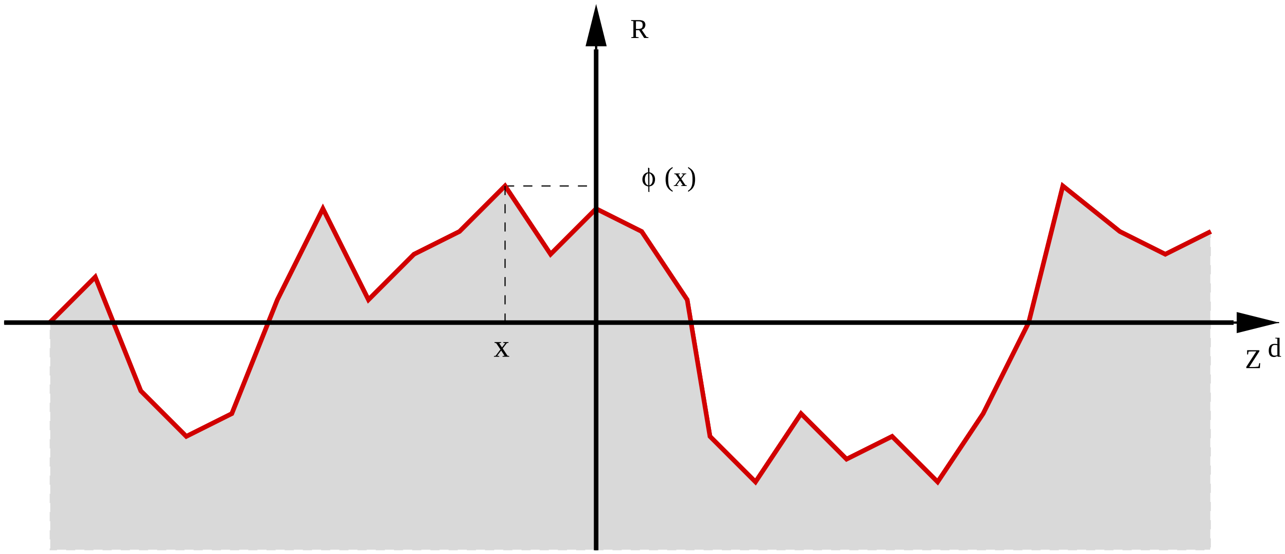
- Probability measure:

$$P_\Lambda(d\varphi) := \frac{1}{Z_\Lambda} \exp\left(-H_\Lambda(\varphi)\right) \prod_{x \in \Lambda} d\varphi_x \prod_{x \in \Lambda^c} \delta_0(d\varphi_x).$$

(Z_Λ normalising constant, $d\varphi_x$ Lebesgue measure, δ_0 Dirac mass).

The model

This models a random interface in statistical physics:
$\varphi_x := \varphi(x)$ height of the interface at site $x \in \mathbb{Z}^d$



Related model: "Lattice free field": $\tilde{H}_\Lambda(\varphi) = \sum_x (\nabla \varphi_x)^2$

Energetically favourable configurations for the membrane model have $\approx$ constant curvature, for the lattice free field the favourable configurations are $\approx$ flat.

Model for semiflexible membranes or semiflexible polymers.

Membrane model: Basic Properties

- P_Λ is the centered Gaussian field on Λ with covariance matrix

$$G_\Lambda = (\Delta^2_\Lambda)^{-1}$$

where $\Delta^2_\Lambda = (\Delta^2(x,y))_{x,y\in\Lambda}$.

(Proof: Summation by parts: $H_\Lambda(\varphi) = \sum_{x,y} \varphi_x \Delta^2_\Lambda(x,y)\varphi_y$.)

- Infinite volume limit. Let $\Lambda_N \nearrow \mathbb{Z}^d$. The limit

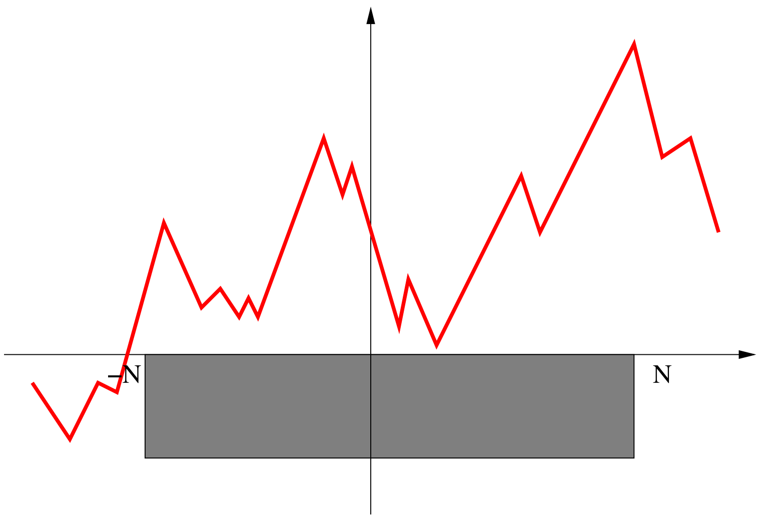
$$P := \lim_{N\to\infty} P_{\Lambda_N}$$

exists if and only if $d \geq 5$. In this case, it is the centered Gaussian field on $\mathbb{Z}^d$ with covariance matrix Δ^{-2}.

- Lattice free field: centered Gaussian field with covariances given by Δ_Λ^{-1} : can use (discrete) harmonic analysis and random walk representation.

Adding a hard wall

Let $\Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$, let $\Omega_N^+ := \{\varphi : \varphi_x \geq 0 \; \forall \, x \in \Lambda_N\}$.



Question: Behaviour of the interface conditioned on Ω_N^+?

Main Results $d \geq 5$

Theorem 1 (Sakagawa 2003/K. 2007) Let $d \geq 5$. Let $G := \text{var}(\varphi_0)$.

$$\lim_{N\to\infty} \frac{1}{N^{d-4}\log N} \log P(\Omega_N^+) = -4G\mathcal{C}$$

with $\mathcal{C} = \inf\left\{\frac{1}{(2d)^2}\int_{\mathbb{R}^d} |\Delta h|^2 dx;\, h \in H^2(\Lambda),\, h \geq 1 \text{ on } [-1,1]^d\right\}$.

Theorem 2 (Sakagawa 2003/K. 2007) Let $d \geq 5$, $\varepsilon > 0$ and $\eta > 0$. Then

$$\lim_{N\to\infty} \sup_{\substack{z\in\Lambda_N, \\ \Lambda_{N,\varepsilon}(z)\subset\Lambda_N}} P\left(\left|\frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} - \sqrt{8G}\right| \geq \eta \,\middle|\, \Omega_N^+\right) = 0\,,$$

where $\overline{\varphi}_{N,\varepsilon}(z) = \frac{1}{|\Lambda_{N,\varepsilon}(z)|}\sum_{x\in\Lambda_{N,\varepsilon}(z)} \varphi_x$
and $\Lambda_{N,\varepsilon}(z) = \{x \in \Lambda_N : \max_{1\leq i\leq d} |x_i - z_i| \leq \varepsilon N\}$.

Main results $d = 4$

Finite volume, domain Λ_N, wall $D_N \subset \Lambda_N$ with distance δN from the boundary.

Theorem 3 (K. 2008) Let $d = 4$. Let $\gamma = \frac{8}{\pi^2}$. Let $D \subset [-1,1]^4$ and $D_N = ND \cap \mathbb{Z}^4$.Then

$$\lim_{N\to\infty} \frac{1}{(\log N)^2} \log P_{\Lambda_N}(\Omega^+_{D_N}) = -8\gamma \mathcal{C}(D),$$

where $\mathcal{C}(D) = \inf\{\frac{1}{2}\int_V |\Delta h|^2 dx : \ h \in H_0(\Lambda), h \geq 1 \text{ a.e. on } D\}$.

Theorem 4 (K. 2008) Let $d = 4$. Let $\eta > 0$ and $\varepsilon > 0$. Then

$$\sup_{\substack{x\in D_N, \\ V_{\varepsilon N}(x)\subset D_N}} \lim_{N\to\infty} P_{\Lambda_N}\Big(\overline{\varphi}_{\varepsilon N}(x) \leq (2\sqrt{2\gamma} - \eta)\log N \,\Big|\, \Omega^+_{D_N}\Big) = 0.$$

Steps in the proof

Strategy similar to the one for the gradient model (Bolthausen, Deuschel, Zeitouni 1995, Bolthausen, Deuschel, Giacomin 2001)

1. Decay of covariances

$d \geq 5$: $G(0,x) \sim |x|^{4-d}$, intersections of random walks
$d = 4$: logarithmic

2. Lower bound on $P(\Omega_N^+)$

Relative entropy argument

3. Upper bound on $P(\Omega_N^+)$

Conditioning argument, coarse graining, Gaussian tail estimates.
Critical dimension: Hierarchy of levels of conditioning

1. Step: Covariances
Infinite volume

Proposition (Sakagawa 2003) $d \geq 5$. Then

$$\lim_{|x|\to\infty} \frac{G(0,x)}{|x|^{4-d}} = \eta\,.$$

$\eta = \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{R}^d} \exp\left(i\zeta \cdot \theta - \frac{1}{(2d)^2}|\theta|^4 t\right) d\theta dt$, for any $\zeta \in \mathbb{S}^{d-1}$.
Proof: Via Fourier transform.

Proposition: Random walk representation. $d \geq 5$. Then

$$G(x,y) = \mathbb{E}^{x,y}\left(\sum_{n,m=0}^{\infty} 1_{\{X_n = Y_m\}}\right)$$

$(X_n)_{n\in\mathbb{N}}, (Y_m)_{m\in\mathbb{N}}$ two independent simple random walks on $\mathbb{Z}^d$, $\mathbb{E}^{x,y}$ their expectation conditional on start in x and y respectively.

Finite volume

Define $(x, y \in \Lambda_N)$

$$R_N(x,y) := \mathbb{E}^{x,y}\left(\sum_{n=0}^{\tau_N}\sum_{m=0}^{\tau_N} 1_{\{X_n=Y_m\}}\right)$$

X, Y independent simple random walks, $\tau_N = \inf\{n \geq 0 : X_n \notin \Lambda_N\}$.

Fact: $R_N(x,y) \neq G_{\Lambda_N}(x,y)$

But: $R_N(x,y) - G_{\Lambda_N}(x,y)$ satisfies a discrete boundary value problem $(x \in \Lambda_N$ fixed)

$$\begin{aligned} \Delta^2(R_N(x,y) - G_{\Lambda_N}(x,y)) &= 0, & y \in \Lambda_N \\ R_N(x,y) - G_{\Lambda_N}(x,y) &= R_N(x,y), & y \in \partial_2\Lambda_N. \end{aligned}$$

Finite volume

- $G_{\Lambda_N}(x,x) \leq R_N(x,x)\ \forall x \in \Lambda_N$

- $\sup_{y \in \Lambda_{\delta N}} |R_N(x,y) - G_{\Lambda_N}(x,y)| \leq c(d) N^{4-d}$
  if $x \in \Lambda_{\delta N}, 0 < \delta < 1$
  via regularity considerations for solutions of the discrete bounday value problems

- Estimates on $R_N(x,y)$ via local central limit theorem for simple random walks

Corollary

(a) $d \geq 5, x \in \Lambda_{\delta N}$: $\mathrm{var}_N(\varphi_x) = c(d) + O(N^{4-d})$

(b) $d = 4, x \in \Lambda_{\delta N}$: $\mathrm{var}_N(\varphi_x) = \gamma \log N + c(\delta)$ $(\gamma = \frac{8}{\pi^2})$