Quasi-stationary ROSt and the Continuous Cascades

Jason Miller

Department of Mathematics, Stanford University

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Random Overlap Structures (ROSt)

A ROSt is a measure on pairs (ξ, Q) where

- $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq 0)$ is such that $\sum_{n=1}^{\infty} \xi_n = 1$
- $Q = (q_{ij})$ is a positive semi-definite matrix of overlaps with $q_{ii} = 1$ for all i
- The overlap q_{ij} serves to quantify the degree to which particles ξ_i and ξ_j are related

ROSt Dynamics

For a fixed function $\psi \in C^2(\mathbb{R})$, consider the evolution

$$(\xi, Q) \mapsto \Phi(\xi, Q) := (\tilde{\xi}, \tilde{Q})$$

where

$$\tilde{\xi} = \left(\frac{\xi_n e^{\psi(\kappa_n)}}{\sum_j \xi_j e^{\psi(\kappa_j)}} \right)_{\downarrow},$$
$$\tilde{Q} = (q_{\pi(i)\pi(j)}),$$

and

- conditional on $Q = (q_{ij})$, (κ_n) is a Gaussian sequence with covariance Q,
- π is a permutation that restores the descending order.

Problem and Motivation

(ξ, Q) is quasi-stationary if

$$\Phi(\xi, Q) \stackrel{d}{=} (\xi, Q).$$

Goal: Characterize the quasi-stationary laws

- The evolution is related to the cavity dynamics of the SK model and the ROSt functional for its free energy
- Showing that the quasi-stationary laws necessarily have ultrametric overlaps, i.e. $q_{ij} \geq \min(q_{ik}, q_{kj})$ for all i, j, k, may help explain the validity of the Parisi ansatz

AS^2 Conjecture

Only known examples of quasi-stationary ROSt:

- Ruelle Probability Cascades (RPC)
  - constructed from a tree of Poisson processes with intensities of the form $xs^{-x-1}ds$
  - overlaps are ultrametric

Conjecture (Aizenman-Sims-Starr)

The only ROSt that are quasi-stationary in a "robust sense" are the RPCs. In particular, all such laws have ultrametric overlaps.

- Notion of robustness is not made precise

Previous Work

Theorem (Ruzmaikina-Aizenman 2005, Arguin 2007)

If (ξ, Q) *is*

- *quasi-stationary for the evolution, and*
- Q *is the identity matrix (increments iid),*

then ξ *is given by a mixture of Poisson-Dirichlet variables.*

Previous Work

Let $S_Q = \{q_{ij} : i \neq j\}$ be the set of values taken on by the entries of Q

Theorem (Arguin-Aizenman, 2007)

If (ξ, Q) *is*

- *robustly quasi-stationary, and*
- $|S_Q| < \infty$

then (ξ, Q) *is given by a mixture of finite level RPCs. In particular, all such laws have ultrametric overlaps.*

Robust Quasi-stationarity

Recall the evolution,

$$(\xi, Q) \mapsto \left(\left(\frac{\xi_n e^{\lambda\psi(\kappa_n)}}{\sum_j \xi_j e^{\lambda\psi(\kappa_j)}} \right)_{\downarrow}, Q^{\pi} \right), \quad Q^{\pi} = (q_{\pi(i)\pi(j)}) \qquad (\bigstar)$$

The following notion was proposed by Arguin and Aizenman:

Definition (Robust quasi-stationarity)

For each $r \in \mathbb{N}$ and $\lambda > 0$, the law of (ξ, Q) is stable under $(\bigstar)$ where (κ_n) is Gaussian conditional on Q with covariance $Q^{*r} = (q_{ij}^r)$

How Robustness is Used

Suppose (ξ, Q) is robustly quasi-stationary with $|S_Q| = n$

- $Q^{*r} \to I$ as $r \to \infty$ implies (ξ, Q) is quasi-stationary under the evolution by iid increments
  - ξ follows a Poisson-Dirichlet distribution and is independent of Q (Ruzmaikina-Aizenman, Arguin)
  - Q is weakly exchangeable so there exists a *random* measure ν on a Hilbert space $\mathcal{H}$ so that conditional on ν,

    $$q_{ij} \stackrel{d}{=} (\phi_i, \phi_j) + (1 - \|\phi_i\|^2)\delta_{ij}$$

    where (ϕ_i) is iid-ν (Dovbysh-Sudakov, 84).
- Robust quasi-stationarity heavily restricts the structure of ν
- It is possible to identify ν with a ROSt (η, P) that is itself robustly quasi-stationary and the number of values taken on by the entries of P is $n - 1$
- The law (ξ, Q) can be reconstructed from (η, P) in the same way that an n level RPC can be constructed from an $n - 1$ level RPC

The Infinite Case

- Arguin-Aizenman proof breaks down when $|S_Q| = \infty$
  - Analysis of the structure of ν requires $|S_Q| < \infty$
  - Induction fails in the presence of multiple cluster points in S_Q

The Infinite Case

- Say that S_Q does not have limit points from below if $p \in \overline{S_Q}$ implies
$$\sup\{q \in S_Q : q < p\} < p,$$

Theorem (JM)

If (ξ, Q) is robustly quasi-stationary and S_Q does not have limit points from below, then (ξ, Q) is given by a mixture of continuous RPCs. In particular, all such laws are ultrametric.

- Idea is to approximate (ξ, Q) by (η, P) where
  - (η, P) is robustly quasi-stationary, and
  - $|S_P| < \infty$

Future Work

- Give a full characterization of robustly quasi-stationary ROSt
  - Eliminate the assumption of "no limit points from below"
- Weaken or perhaps eliminate the assumption of robustness