Near-critical percolation and the geometry of diffusion fronts

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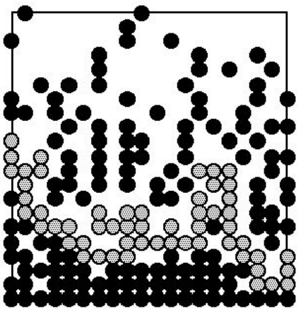
Motivations: inhomogeneity and universality

Inhomogeneity and universality

We would like to understand physical systems where inhomogeneity plays a central role: diffusion front models for instance (Gouyet, Rosso, Sapoval - 1985).



(Fig. J.F. Gouyet)

Inhomogeneity and universality

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Inhomogeneity and universality

- Numerical evidence that these interfaces are fractal, with dimension $D_f = 1.76 \pm 0.02$.
- The interface remains localized where the probability $p(z)$ of presence of a particle at site z is close to p_c, the percolation threshold.
- One observes various critical exponents (size of fluctuations, length...) that seem related to those of standard percolation: for instance $D_f \simeq 7/4$ is known to be the dimension of critical percolation interfaces.

Inhomogeneity and universality

- For the simulations, approximation that the status of the different sites (occupied / vacant) are independent of each other (inhomogeneous percolation process with parameter $p(z)$): *gradient percolation* model.

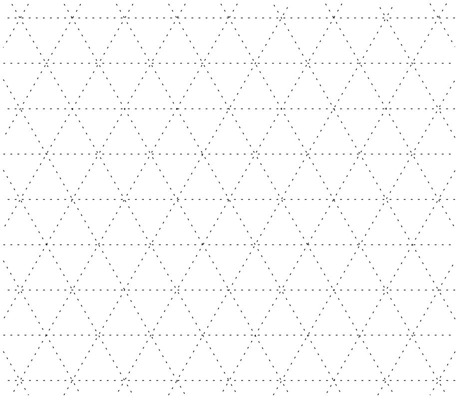
Inhomogeneity and universality

- For the simulations, approximation that the status of the different sites (occupied / vacant) are independent of each other (inhomogeneous percolation process with parameter $p(z)$): *gradient percolation* model.
- Here:
  - study of gradient percolation.
  - $\Rightarrow$ properties of diffusion fronts.

Standard percolation background

Site percolation

We work in the plane, and we consider the *triangular lattice*:



Site percolation

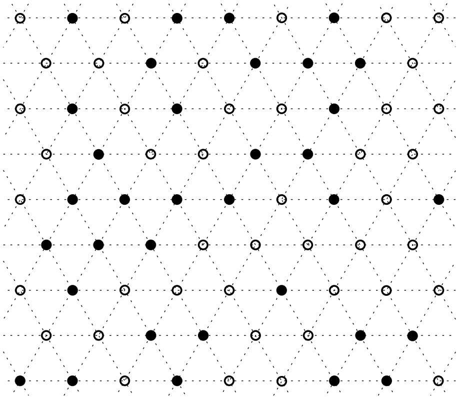
We fix a parameter $p \in [0,1]$, and we assume that:

- Each site is *occupied* (open / black) with probability p, *vacant* (closed / white) with probability $1-p$.
- The sites are *independent*.

The associated probability measure is denoted by $\mathbb{P}_p$.

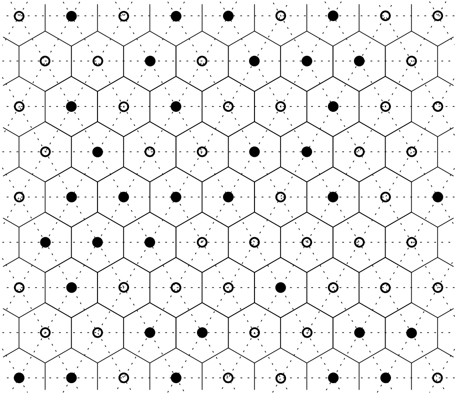
Site percolation

Hence,



Site percolation

We represent it usually with hexagons:

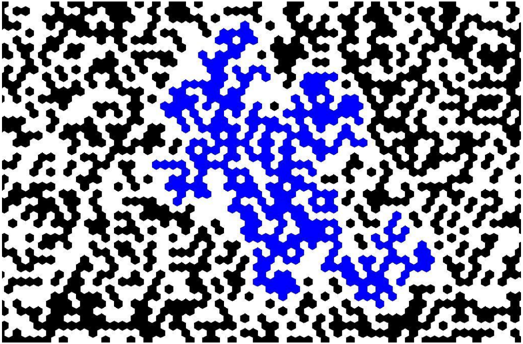


Site percolation

We obtain this kind of pictures:

Cluster of a site

The sites can be regrouped into connected components, called “clusters”:



Existence of a phase transition at $p_c = 1/2$

Percolation features a *phase transition*, at $p_c = 1/2$ on the triangular lattice:

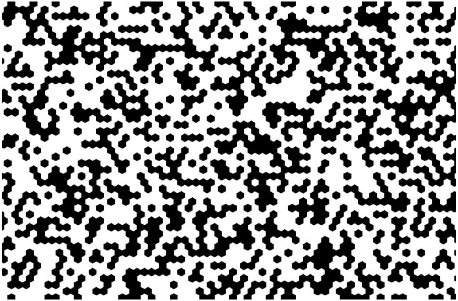
- If $p < 1/2$: a.s. no infinite cluster (*sub-critical* regime).
- If $p > 1/2$: a.s. a *unique* infinite cluster (*super-critical* regime).

If $p = 1/2$: *critical* regime, a.s. no infinite cluster.

Exponential decay

In *sub-critical* regime ($p < 1/2$), there exists a constant $C(p)$ such that

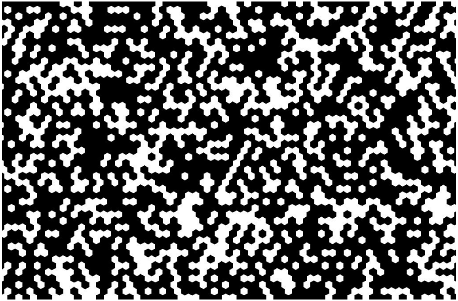
$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n) \leq e^{-C(p)n}.$$



Exponential decay

In *super-critical* regime ($p > 1/2$), we have similarly

$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n | 0 \not\rightsquigarrow \infty) \leq e^{-C(p)n}.$$



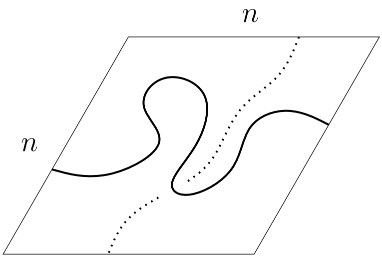
Critical regime

At the critical point $p = 1/2$, "there is no characteristic length": when we take some distance (scaling), we still observe the same behavior.

Critical regime

For symmetry reasons, we have for example (we denote by $\mathcal{C}_H$ the existence of a left-right crossing):

$$\mathbb{P}_{1/2}(\mathcal{C}_H([0,n] \times [0,n])) = 1/2.$$



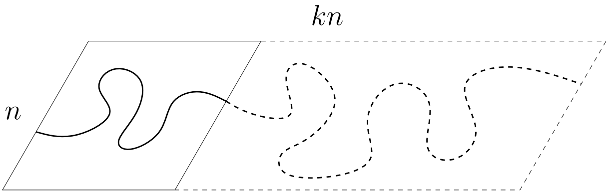
Critical regime

This implies the Russo-Seymour-Welsh theorem (key tool):

Theorem (Russo-Seymour-Welsh)

For each $k \geq 1$, there exists $\delta_k > 0$ such that

$$\mathbb{P}_{1/2}(\mathcal{C}_H([0, kn] \times [0, n])) \geq \delta_k.$$



Near-critical percolation

Two main ingredients:

(1) Study of critical percolation

(2) Scaling techniques

$\Rightarrow$ Description of percolation near the critical point.

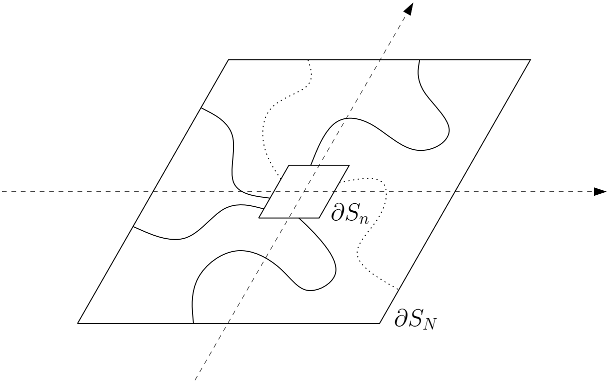
1st ingredient: study of critical percolation

A precise description of critical percolation was made possible by the introduction of SLE processes in 1999 by Schramm, and its subsequent study by Lawler, Schramm and Werner.

Another important step: conformal invariance of critical percolation in the scaling limit (Smirnov - 2001), that allows to go from discrete to continuum.

Arm events

We use in particular the “arm-events”:



Arm events

Their probabilities decay like power laws, described by the "j-arm exponents":

Theorem (Lawler, Schramm, Werner, Smirnov)

We have

$$\mathbb{P}_{1/2}(0 \rightsquigarrow \partial S_n) \approx n^{-5/48}$$

and for each $j \geq 2$, for every fixed (non constant) sequence of "colors" for the j arms,

$$\mathbb{P}_{1/2}(j \text{ arms } \partial S_j \rightsquigarrow \partial S_n) \approx n^{-(j^2-1)/12}.$$

Arm events

It comes from *SLE* computations and the following connection:

Theorem (Smirnov, Camia-Newman)

When the mesh of the lattice $\delta \to 0$, the "radial exploration process" converges to a radial SLE_6.

$\Longrightarrow$ Link between arm events and some events for radial SLE_6 (ex: reaching the inner boundary of the annulus without closing any loop).

2nd ingredient: scaling techniques (H. Kesten)

We introduce the scale up to which the crossing probabilities remain non-trivial.

$\Rightarrow$ For any $\epsilon \in (0, 1/2)$, we define the **characteristic length**:

Definition

$$L_\epsilon(p) = \begin{cases} \min\{n \;\; s.t. \;\; \mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n])) \leq \epsilon\} & \text{if } p < 1/2 \\ \min\{n \;\; s.t. \;\; \mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n])) \geq 1-\epsilon\} & \text{if } p > 1/2 \end{cases}$$

"It still looks close to critical percolation"

For instance, the RSW theorem remains true for parallelograms of size $\leq L_\epsilon(p)$, and the probability to observe a path $0 \rightsquigarrow \partial S_n$ remains of the same order of magnitude.

We will need:

Lemma

For any $j \geq 1$ and any fixed colors,

$$\mathbb{P}_p(j \text{ arms } \partial S_j \rightsquigarrow \partial S_n) \asymp \mathbb{P}_{1/2}(j \text{ arms } \partial S_j \rightsquigarrow \partial S_n)$$

uniformly in p, $n \leq L_\epsilon(p)$.

“We quickly become sub-critical”

We have the following lemma, showing exponential decay with respect to $L_\epsilon(p)$ (control of speed for variable p):

Lemma

There exist constants $C_1, C_2 > 0$ such that for each n, each $p < 1/2$,

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq C_1 e^{-C_2 n / L_\epsilon(p)}.$$

This lemma implies in particular if $p > 1/2$:

$$\mathbb{P}_p(0 \rightsquigarrow \infty) \asymp \mathbb{P}_p(0 \rightsquigarrow \partial S_{L_\epsilon(p)}).$$

Scaling techniques

To sum up, $L_\epsilon(p)$ is at the same time:

- a scale on which everything looks like critical percolation.
- a scale at which connectivity properties start to change drastically.

We can also prove that

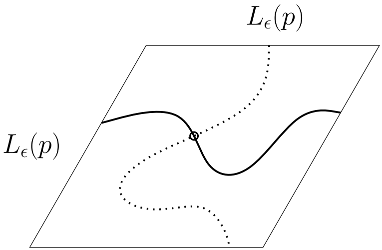
$$L_\epsilon(p) \asymp L_{\epsilon'}(p)$$

for any $\epsilon, \epsilon' \in (0, 1/2)$.

Consequences for the characteristic functions

These ingredients allow to obtain the critical exponents of standard percolation, associated to the characteristic functions (ξ, χ, θ ...). By counting "pivotal" sites,

$$|p - 1/2|(L_\epsilon(p))^2 \mathbb{P}_{1/2}(0 \rightsquigarrow^4 \partial S_{L_\epsilon(p)}) \asymp 1.$$



Consequences for the characteristic functions

Hence,

$$L_\epsilon(p) \approx |p - 1/2|^{-4/3} \qquad (p \to 1/2).$$

The density $\theta(p)$ of the infinite cluster satisfies:

$$\theta(p) = \mathbb{P}_p(0 \rightsquigarrow \infty) \asymp \mathbb{P}_p(0 \rightsquigarrow \partial S_{L_\epsilon(p)}),$$

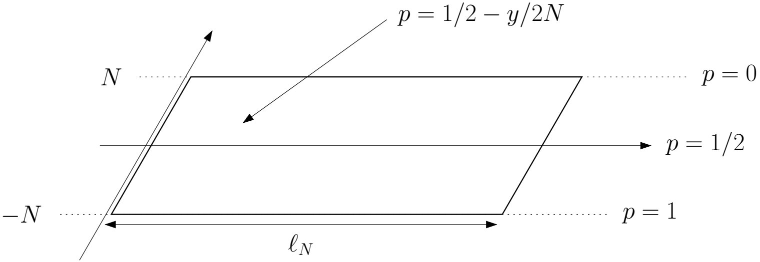
so that

$$\theta(p) \asymp \mathbb{P}_{1/2}(0 \rightsquigarrow \partial S_{L_\epsilon(p)}) \approx (L_\epsilon(p))^{-5/48} \approx (p - 1/2)^{5/36}.$$

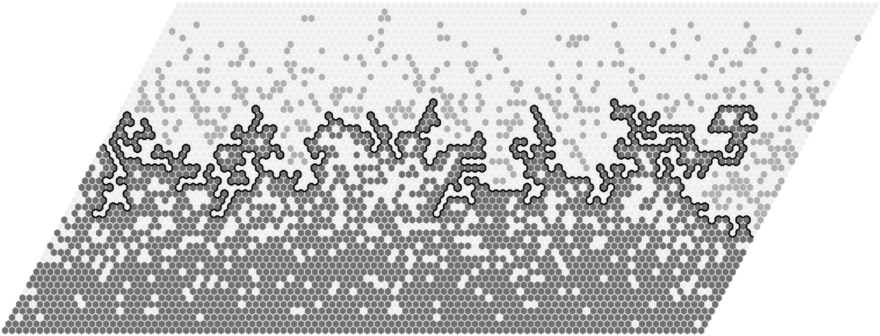
Gradient Percolation

Gradient percolation: setting

We consider the strip $\mathcal{S}_N = [0, \ell_N] \times [-N, N]$, in which the percolation parameter decreases linearly in y:



Gradient percolation: setting



Gradient percolation: setting

With this convention, all the sites on the lower boundary are occupied ($p = 1$), all the sites on the upper boundary are vacant ($p = 0$).

$\Rightarrow$ Two different regions appear:

- At the bottom of $\mathcal{S}_N$, the parameter is close to 1: "big" cluster of occupied sites.
- At the top of $\mathcal{S}_N$, the parameter is close to 0: "big" cluster of vacant sites.

Gradient percolation: setting

The characteristic phenomenon of this model is the appearance of a unique “front”, an interface touching simultaneously these two clusters.

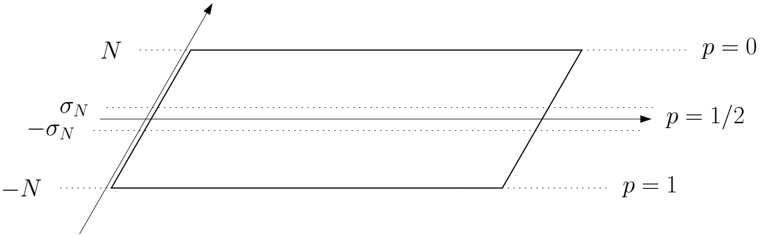
Hypothesis on ℓ_N: we will assume that for two constants $\epsilon, \gamma > 0$,

$$N^{4/7+\epsilon} \leq \ell_N \leq N^{\gamma}$$

Thus $\ell_N = N$ is OK.

Heuristics

The critical behavior of this model remains localized in a "critical strip" around $p = 1/2$, a strip in which we can consider percolation as almost critical.



Heuristics

The width σ_N of the critical strip satisfies

$$\sigma_N = L_\epsilon(1/2 \pm \sigma_N/2N).$$

The vertical fluctuations are of order σ_N.

The exponent for $L_\epsilon(p)$ implies that $\sigma_N \approx N^{4/7}$.

Heuristics

We show:

1. **uniqueness** of the front.
2. **decorrelation** of points at horizontal distance $\gg \sigma_N$.
3. **width** of the front of order σ_N.

Uniqueness

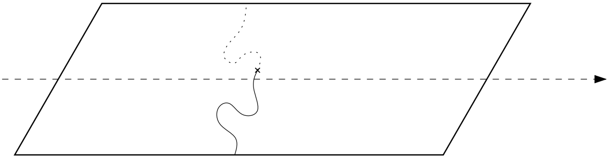
There exists with probability very close to 1 a unique front, that we denote by $\mathcal{F}_N$:

Lemma (N.)

There exists $\delta' > 0$ such that for each N sufficiently large,

$$\mathbb{P}(\textit{the boundaries of the two "big" clusters coincide}) \geq 1 - e^{-N^{\delta'}}.$$

Uniqueness



A site x is on the front *iff* there exist **two arms**, one occupied to the bottom of $\mathcal{S}_N$, and one vacant to the top.

Moreover, this is a **local** property (depending on a neighborhood of x of size $\approx \sigma_N$) $\Rightarrow$ **decorrelation** of points at horizontal distance $\gg \sigma_N$.

Width of the front

The scale $\sigma_N \approx N^{4/7}$ is actually the order of magnitude of the vertical fluctuations:

Theorem (N.)

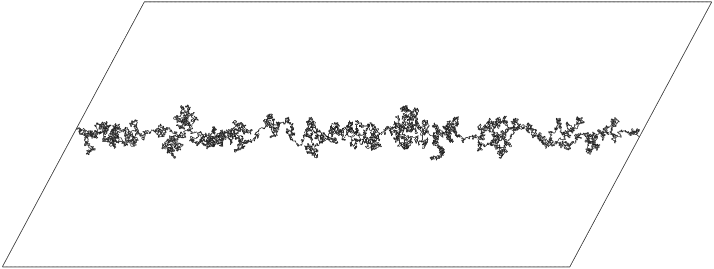
- *For each* $\delta > 0$*, there exists* $\delta' > 0$ *such that for* N *sufficiently large,*

$$\mathbb{P}(\mathcal{F}_N \subseteq [\pm N^{4/7-\delta}]) \leq e^{-N^{\delta'}}.$$

- *For each* $\delta > 0$*, there exists* $\delta' > 0$ *such that for* N *sufficiently large,*

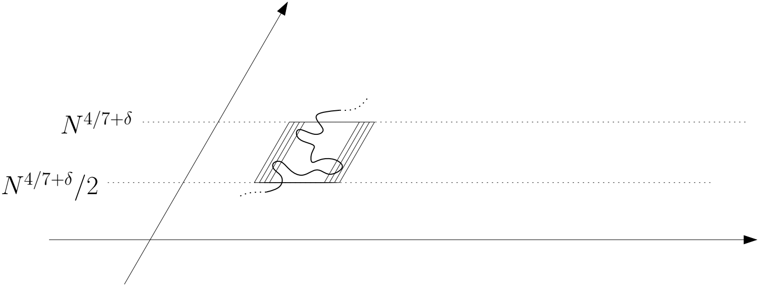
$$\mathbb{P}(\mathcal{F}_N \nsubseteq [\pm N^{4/7+\delta}]) \leq e^{-N^{\delta'}}.$$

Width of the front



Width of the front

$\mathcal{F}_N \subseteq [\pm N^{4/7+\delta}]$ with high probability:



Representative example: length of the front

To estimate a quantity related to the front:

(1) Only the edges in the critical strip must be counted.

(2) For these edges, being on the front is equivalent to the existence of two arms of length σ_N:

$$\mathbb{P}(e \in \mathcal{F}_n) \approx (\sigma_N)^{-1/4} \approx N^{-1/7}$$

(2 arm exponent: $1/4$).

Representative example: length of the front

Thus, for the length T_N of the front:

Proposition (N.)

For each $\delta > 0$, we have for N sufficiently large:

$$N^{3/7-\delta}\ell_N \leq \mathbb{E}[T_N] \leq N^{3/7+\delta}\ell_N.$$

For $\ell_N = N$, this gives $\mathbb{E}[T_N] \approx N^{10/7}$.

Noteworthy property: In a box of size σ_N, approximately N points are located on the front.

Variance of T_N

The decorrelation of points at horizontal distance $\gg N^{4/7}$ implies:

Theorem (N.)

If for some $\epsilon > 0$, $\ell_N \geq N^{4/7+\epsilon}$, then

$$\frac{T_N}{\mathbb{E}[T_N]} \longrightarrow 1 \quad \textit{in } L^2, \textit{ as } N \to \infty.$$

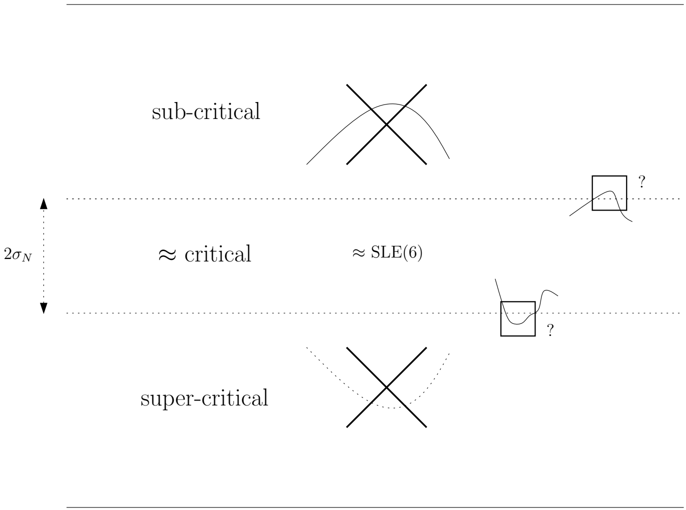
$\Rightarrow$ Concentration of T_N around $\mathbb{E}[T_N] \approx N^{3/7}\ell_N$.

Outer boundaries of the front

We can introduce the lower and upper boundaries of the front: we denote by U_N^+ and U_N^- their respective lengths. The proof of the results on the length T_N can easily be adapted, and the 3-arm exponent (equal to $2/3$) gives:

$$U_N^{\pm} \approx N^{4/21} \ell_N.$$

Summary


sub-critical
?
$2\sigma_N$
$\approx$ critical
$\approx$ SLE(6)
?
super-critical

Existence of scaling limits

Using standard arguments due to Aizenman and Burchard, one can show the existence of *scaling limits*. The right way to scale is by using the characteristic length

$$\sigma_N^{\epsilon} = \sup\{\sigma \quad \text{s.t.} \quad L_{\epsilon}(1/2 \pm \sigma/2N) \geq \sigma\}.$$

One can check that

$$\sigma_N^{\epsilon} \asymp \sigma_N^{\epsilon'},$$

and scaling by a quantity much smaller or much larger does not produce non-trivial limits.

Similarities with critical percolation interfaces

The potential scaling limits have the same dimension $7/4$, the same exponents as SLE_6.

$\Longrightarrow$ Can they be expressed in terms of SLE_6?

Discrete Asymmetry

But we also have:

Proposition

Consider a box of size σ_N centered on the line $y = -2\sigma_N$: it contains $\approx N$ sites of the front, but

$$\#\textit{black sites} - \#\textit{white sites} \approx N^{4/7} \gg \sqrt{N}.$$

And in fact, in the scaling limit, the law of the front will be singular with respect to SLE_6. This is related to *off-critical* percolation (super-critical percolation on a characteristic length).

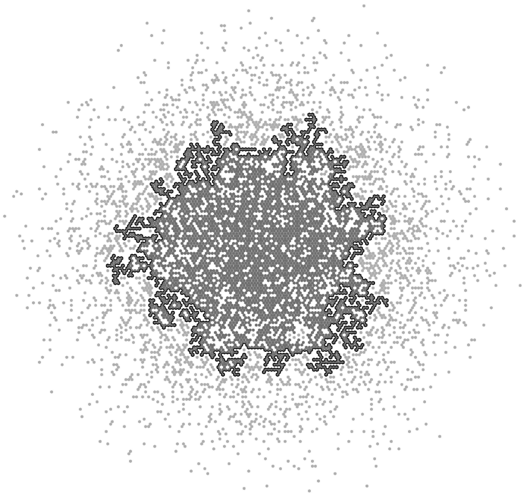
Application: geometry of diffusion fronts

Description of the model

We start at time $t = 0$ with a large number n of particles located at the origin, and we let them perform independent random walks.

At each time t, we then look at the sites containing at least one particle. These occupied sites can be regrouped into connected components, or "clusters", by connecting two occupied sites if they are adjacent on the lattice.

Description of the model



Different regimes

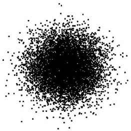
As time t increases, different regimes arise:

- At first, a very dense cluster around the origin forms. This cluster grows as long as t remains small compared to n.

Evolution for $n = 10000$ particles: $t = 10$

Evolution for $n = 10000$ particles: $t = 100$

Evolution for $n = 10000$ particles: $t = 500$

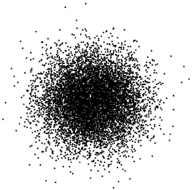


Different regimes

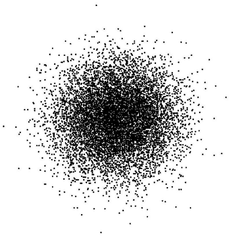
As time t increases, different regimes arise:

- At first, a very dense cluster around the origin forms. This cluster grows as long as t remains small compared to n.
- When t gets comparable to n, the cluster first continues to grow up to some time $t_{\mathrm{max}} = \lambda_{\mathrm{max}} n$.

Evolution for $n = 10000$ particles: $t = 1000$



Evolution for $n = 10000$ particles: $t = 1463 = \lambda_{\max} n$



Different regimes

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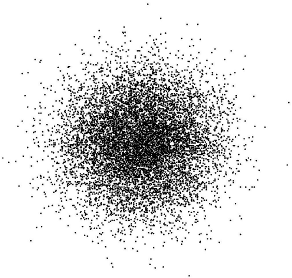
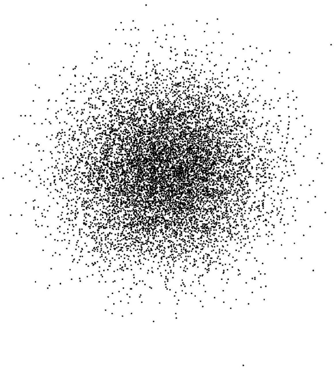
- At first, a very dense cluster around the origin forms. This cluster grows as long as t remains small compared to n.
- When t gets comparable to n, the cluster first continues to grow up to some time $t_{\text{max}} = \lambda_{\text{max}} n$.
- It then starts to decrease and it finally dislocates at some critical time $t_c = \lambda_c n$ – and never re-appears ($t_c / t_{\text{max}} = e$ is universal).

Evolution for $n = 10000$ particles: $t = 2500$

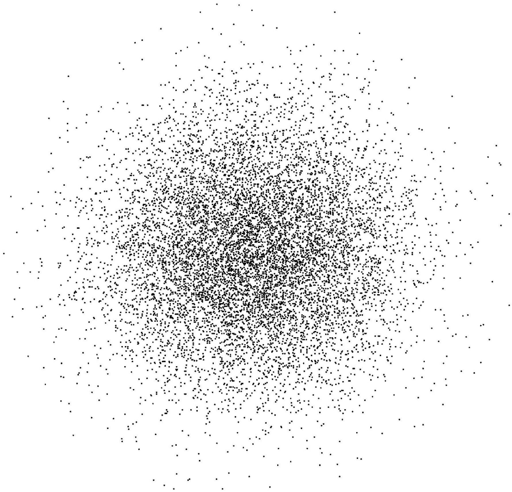


Evolution for $n = 10000$ particles: $t = 3977 = \lambda_c n$



Evolution for $n = 10000$ particles: $t = 5000$

Evolution for $n = 10000$ particles: $t = 10000$



Main ingredients

We first need a strong form of the Local Central Limit Theorem. The distribution of a simple random walk after t steps satisfies:

$$\pi_t(z) = \frac{\sqrt{3}}{2\pi t} e^{-\|z\|^2/t} + O\left(\frac{1}{t^2}\right).$$

($\sqrt{3}/2$ comes from the “density” of sites on the triangular lattice).

Main ingredients

The probability of occupation for a site z is

$$1 - (1 - \pi_t(z))^n \simeq 1 - e^{-n\pi_t(z)}.$$

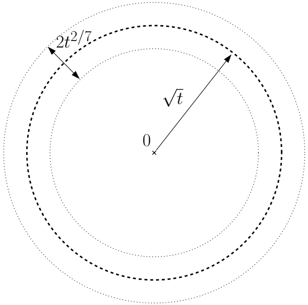
This is equal to 1/2 for

$$r^*_{n,t} = \sqrt{t \log \frac{\lambda_c}{t/n}}$$

if $t \leq \lambda_c n$, with $\lambda_c = \sqrt{3}/2\pi \log 2$ (and it remains $< 1/2$ otherwise).

Main ingredients

An approximation using Gradient percolation is valid. The boundary remains localized in an annulus of width $\approx (\sqrt{t})^{4/7} = t^{2/7}$ around $r = r^* \asymp \sqrt{t}$. They represent a negligible fraction of the sites, we can thus use a *Poissonian approximation*.



Results: case $\lambda < \lambda_c$

Theorem (N.)

Consider $t_n = \lambda n$, with $\lambda < \lambda_c$. Then with probability tending to 1,

- *There exists a unique macroscopic interface surrounding 0.*
- *It remains localized in the annulus of width $\approx t^{2/7}$ around $r = r^* \asymp \sqrt{t}$.*
- *Its length behaves like $t^{5/7}$ and its roughness can be described via the universal exponent 7/4 (it is locally of Hausdorff dimension 7/4).*

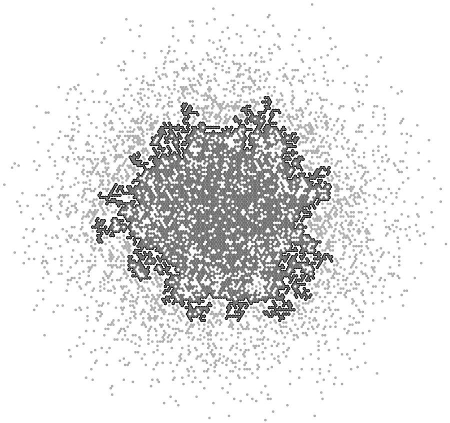
Remark: case $t \ll n$

Similar results are valid in the case $t = n^\alpha$, $\alpha < 1$.

Only the **transition window** (from parameter $1/2 + \epsilon$ to $1/2 - \epsilon$) is different. This window is of size $\asymp \sqrt{t}/\sqrt{\log t}$ around $r^* \asymp \sqrt{t \log t}$ (localized transition).

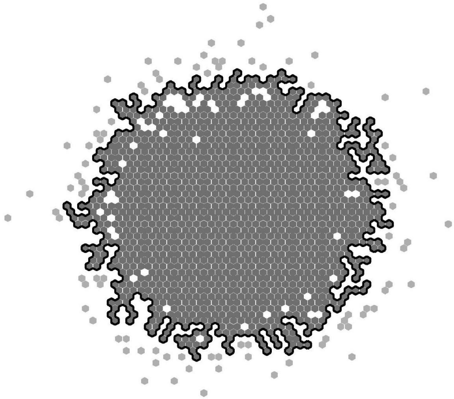
Remark: case $t \ll n$

When $t \asymp n$, gradual transition:



Remark: case $t \ll n$

When $t \ll n$, abrupt transition:



Results: case $\lambda > \lambda_c$

In the case $t_n = \lambda n$ with $\lambda > \lambda_c$, the whole picture can be "dominated" by a sub-critical percolation.

Hence for some constant $c = c(\lambda)$,

$$\mathbb{P}(\text{every cluster is of size} \leq c \log n) \to 1.$$

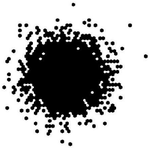
Model with a source

We now consider a model with a Poissonian source at the origin:

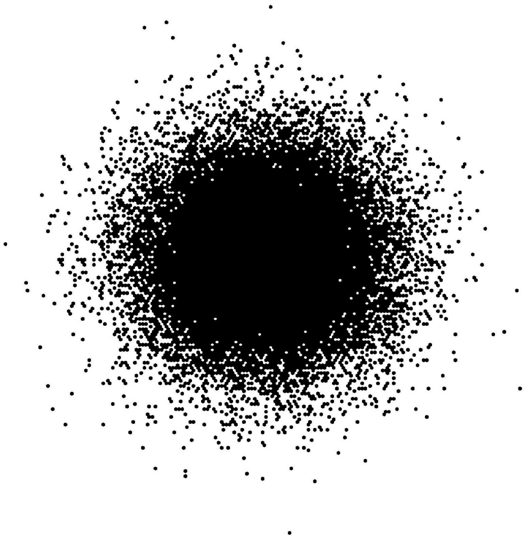
- Particles arrive at rate $\mu > 0$.
- Once arrived, they perform independent random walks.

Model with a source ($\mu = 50$): $t = 10$

Model with a source ($\mu = 50$): $t = 100$



Model with a source ($\mu = 50$): $t = 1000$



Model with a source

The occupation parameter is now

$$1 - e^{-\mu \rho_t(z)},$$

with

$$\rho_t(z) = \pi_0(z) + \ldots + \pi_t(z) \simeq \frac{\sqrt{3}}{2\pi} \int_{\|z\|^2/t}^{+\infty} \frac{e^{-u}}{u} du.$$

Model with a source

The behavior is then the same as previously for any $\mu > 0$ (no phase transition):

- There exists a unique macroscopic interface surrounding 0.
- It remains localized in the annulus of width $\approx t^{2/7}$ around $r = r^* \asymp \sqrt{t}$.
- Its length behaves like $t^{5/7}$ and it is locally of Hausdorff dimension $7/4$.

End

Thank you !