Competing Particle Systems Evolving by i.i.d. Increments

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Problem

- Consider a random locally finite upper bounded point configuration $x_1 \geq x_2 \geq \dots$ on $\mathbb{R}$ (later on $\mathbb{R}^d$)
- Add to each point an increment from an i.i.d. sequence $(\pi_n)_{n\geq 1}$
- Rearrange the points in descending order

In other words:

$$(x_n)_{n\geq 1} \mapsto (x_n + \pi_n)_{\downarrow} \tag{I.1}$$

What are the quasi-stationary measures of such evolutions (i.e. measures on configurations for which the distribution of the gaps is invariant)?

Interesting for example in the context of the SK model of spin glasses (x_n's are interpreted as energy levels of particles)

Previously known results

Ruzmaikina-Aizenman Theorem. If π_n have a density and a finite moment generating function, then quasi-stationary states are superpositions of Poisson point processes with densities of the type $ae^{-ax}dx$, $a > 0$.

Arguin-Aizenman Theorem. If $(\pi_n)_{n\geq 1}$ is centered Gaussian with covariance matrix Q, Q has finitely many different entries and $(x_n)_{n\geq 1}$ satisfies a growth condition, then indecomposable robustly quasi-stationary measures under a one parameter family of evolutions driven by $(\pi_n)_{n\geq 1}$ are superpositions of Ruelle Probability Cascades.

Here we are interested in the i.i.d. case, but with general increments ($\Rightarrow$ e.g. lattice type increments) and $d > 1$.

Key issues

1. How to define a total order on $\mathbb{R}^d$ which can be used to define quasi-stationarity and for which large deviations arguments work?

Want

- level sets of $\succ$ to be infinite rectangles up to boundary
- $x \succ y,\ y \succ x \ \Rightarrow\ x = y$ ($\Rightarrow$ unique leader w.r.t. $\succ$)

Possible solution:

Define a line $l \subset \mathbb{R}^d$ and $p : \mathbb{R}^d \to l$ the "projection" to the closest point which is not $\geq$ to the projected point, set $x \succ y$ if $p(x) \geq p(y)$, for x, y with $p(x) = p(y)$ use inductive construction (in addition l must also satisfy a small technical property in the lattice case)

2. How to treat the case of lattice type increments?

For the Poissonization Theorem (relating the quasi-stationary measure μ to Poisson point processes) need a unique leader of the process, so we **assume that μ is simple** (no two particles are at the same point). Otherwise it is not known if the quasi-stationary states are of Poisson type.

For our large deviations arguments we use Bahadur-Rao in $d = 1$ (holds for lattice and non-lattice r.v.!), for $d > 1$ we prove an extension of Bahadur-Rao.

Main result

Theorem.

Let the π_n have an everywhere finite Laplace transform and if $d > 1$ let the increments have a density or be of lattice type.

Then every simple measure μ which is quasi-stationary for the evolution with i.i.d. increments is a superposition of Poisson point processes.

Moreover, the intensity measures λ of the Poisson point processes are exactly the solutions of Choquet-Deny equations $\lambda * \pi_a = \lambda$ where π is the distribution of the increments and π_a is the measure resulting from π by a shift by a.

Sketch of the proof

1. Poissonization Theorem. Let μ be a simple quasi-stationary measure of the evolution above, $\{F_N\}_{N\geq 1}$ be the family of functions defined by

$$F_N(x) = \sum_{m\geq 1} \mathbb{P}_\pi(x_m + \pi_1 + \cdots + \pi_N \geq x)$$

where $(x_n)_{n\geq 1}$ is a fixed starting configuration of the particles. Then for any non-negative continuous function with compact support $f \in C_c^+(\mathbb{R})$ it holds

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int d\mu \; \widehat{G}_{F_N}(f). \tag{V.2}$$

Here, $\widetilde{G}_\mu$ denotes the modified probability generating functional of μ given by

$$\widetilde{G}_\mu(f) = \mathbb{E}_\mu\left[\exp\left(-\sum_n f(x_1 - x_n)\right)\right] \quad \text{(V.3)}$$

and $\widehat{G}_{F_N}$ denotes the modified probability generating functional of the Poisson point process on $\mathbb{R}$ with intensity measure λ_N uniquely determined by

$$\lambda_N([a, b)) = F_N(a) - F_N(b).$$

2. μ **is a superposition of Poisson point processes.** Need in some sense "tightness" of $(\lambda_N)_{N\geq 1}$. To prove this we prove and use a generalized version of the Bahadur-Rao Theorem:

Multidimensional Bahadur-Rao Theorem. Let $(Y_n)_{n\geq 1}$ be an i.i.d. sequence of centered $\mathbb{R}^d$-valued random variables having an everywhere finite moment generating function and set

$S_N \equiv \sum_{n=1}^{N} Y_n$. Then in case that $d = 1$ and the Y_n are non-lattice or $d > 1$ and the Y_n have a density we have for all $x \in \mathbb{R}^d$ and $\mathbb{R}^d \ni q \geq 0$:

$$\frac{\mathbb{P}(S_N \geq x + qN)}{\mathbb{P}(S_N \geq qN)} \sim \exp(-\eta(\nu(q)) \cdot x) \tag{V.4}$$

uniformly in q where $\sim$ means that the quotient of the two expressions tends to 1, $\eta = \eta(q)$ is the unique solution of

$$\gamma(q) = \eta \cdot q - \Lambda(\eta), \tag{V.5}$$

Λ is the logarithmic moment generating function, γ is the Fenchel-Legendre transform and $\nu(q)$ is the minimizer of γ over the set $\{y \in \mathbb{R}^d | y \geq q\}$. In case that the Y_n are lattice with values in $A\mathbb{Z}^d + b$, equation (V.4) holds for all $x \in A\mathbb{Z}^d$ and all lattice points $q \geq 0$ again uniformly in q where A is a real $d \times d$ matrix and $b \in \mathbb{R}^d$.

3. Computation of the Poisson intensities.

Prove that

$$\lambda \mapsto \lambda * \pi \tag{V.6}$$

makes the tail of λ "steeper" with "equal steepness" iff $\lambda * \pi_a = \lambda$ for an $a \in \mathbb{R}$. Solve the Choquet-Deny equation with the Choquet-Deny Theorem.

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