Random walk delayed on percolation clusters

Francis Comets and François Simenhaus

Université Paris 7 – Denis Diderot

3^{rd} La Pietra Week in Probability, Firenze

24^{th} of June 2008

Introduction

We consider a random walk in random environment where the environment is given by a subcritical percolation. The *quenched* process is a continuous random walk in $\mathbb{Z}^d$ subject to a drift and attraction to large clusters of the environment.

Environment

We consider a subcritical i.i.d. site percolation ω on $\mathbb{Z}^d$:
Fix $p < p_c(d)$ and a variable $\omega \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ with law

$$\mathbb{P} = Ber(p)^{\otimes \mathbb{Z}^d}.$$

Notation : C_x will denote the size of the cluster of x ($x \in \mathbb{Z}^d$)

Exponential decay of the cluster size

There exists $\xi(p,d) > 0$ such that

$$-\frac{1}{n} \ln \mathbb{P}(C_0 > n) \to \xi > 0 \qquad (n \to \infty)$$

The quenched law

Fix an environment ω and a direction $\ell \in S_{d-1}$. We need two parameters :

- $\lambda \geq 0$ for the strength of the drift
- $\beta \geq 0$ for the strength of the attraction by the clusters

We now define the continuous time Markov Chain $(Y_t)_{t\geq 0}$ with law P_ω by assigning its skeleton and jump rates.

Skeleton

The skeleton $(X_n)_{n\geq 0}$ is the drifted random walk on $\mathbb{Z}^d$ with transitions

$$\tilde{P}(X_{n+1} = x + e | X_n = x) = \frac{e^{\lambda \ell \cdot e}}{\sum_{|e'|=1} e^{\lambda \ell \cdot e'}}, \quad |e| = 1.$$

The quenched law

- Notice that the skeleton is independent from the environment ω.
- $d(\lambda)$ will denote the drift of $(X_n)_{n\geq 0}$.

Jump rates

The jump rate at site $x \in \mathbb{Z}^d$ is $e^{-\beta C_x}$.

The waiting time at site x are i.i.d. exponential variables with mean $e^{\beta C_x}$.

Clock Process

Consider a family of i.i.d. exponential variables $(\epsilon_i)_{i\in\mathbb{N}}$ with mean 1 (note Q the law of this family) and define the clock process as :

$$S_n = \sum_{i=0}^{n-1} \epsilon_i e^{\beta C_{X_i}}.$$

We now consider the process

$$Y_t = X_{S^{-1}(t)}, \quad t \geq 0,$$

and define P_ω as the law of $(Y_t)_{t\geq 0}$ under $\tilde{P} \otimes Q$.

Properties P_ω

P_ω is reversible and admits the reversible measure,

$$\mu_\omega(x) = e^{2\lambda \ell \cdot x + \beta C_x}, x \in \mathbb{Z}^d$$

We will also use the annealed law P :

$$P = \mathbb{P} \times P_\omega.$$

The Bouchaud Trap Model

This model can be seen as a BTM on $\mathbb{Z}^d$ (Ben Arous-Cerny)

- The skeleton is still a simple random walk, but now possibly drifted.
- The $(\tau_x)_{x\in\mathbb{Z}^d}$ are not i.i.d. as $\tau_x = e^{\beta C_x}$.

Popov and Vachkovskaia Model

S.Popov et M.Vachkovskaia studied the discrete time walk attracted by clusters with transitions,

$$P(X_{n+1} = x + e | X_n = x) = \frac{e^{\beta C_{x+e}}}{\sum_{|e'|=1} e^{\beta C_{x+e'}}}.$$

$\mu_\omega(x) = e^{\beta C_x}, x \in \mathbb{Z}^d$, is also a reversible measure for this dynamic ...but not for the same reasons.

Law of large numbers

Theorem : Law of large Numbers

For every $\lambda \geq 0$ and $\beta \geq 0$,

$$\lim_{t \to \infty} \frac{Y_t}{t} = \frac{d(\lambda)}{\mathbb{E} e^{\beta C_0}}$$

Main tool of the proof : the environment seen from the particle is an ergodic Markov chain under the annealed law.

Sub-diffusive case

Theorem : Subdiffusive case : $\beta \geq \xi$

- For every $d \geq 1$ and $\lambda > 0$,

$$\lim_{t\to\infty} \frac{\ln |Y_t|}{\ln t} = \frac{\xi}{\beta}, \quad P - p.s.$$

- If $\lambda = 0$ and $d \geq 2$,

$$\limsup_{t\to\infty} \frac{\ln |Y_t|}{\ln t} = \frac{\xi}{2\beta}, \quad P - p.s.$$

- If $\lambda = 0$ and $d = 1$,

$$\limsup_{t\to\infty} \frac{\ln |Y_t|}{\ln t} = \frac{1}{2}\left(\frac{\beta}{2\xi} + \frac{1}{2}\right)^{-1}, \quad P - p.s.$$

Quenched Invariance principle

Theorem : Quenched Invariance principle

If $\lambda = 0$ and $\mathbb{E}(e^{\beta C}) < \infty$, then $\mathbb{P} - p.s.$,

$$(\epsilon^{1/2} Y_{\epsilon^{-1}t})_{t \geq 0} \Rightarrow (B_t)_{t \geq 0}$$

Proof :

- $(Y_t)_{t \geq 0}$ is a martingale under the quenched law.
- Ergodic theorem gives the existence of a limit for the bracket of $(\epsilon^{1/2} Y_{\epsilon^{-1}t})_{t \geq 0}$.

Conclusions

- Convergence in law as for the BTM.
- Study of the model of Popov and Vachkovkaïa with a drift.
- Study of other Markovian dynamics admitting μ_ω as reversible measure.