Welcome to "Beyond the compasses. The geometry of curves", an exposition that has been shown in many cities in Italy and abroad, with more than 350.000 visitors. During the visit, we shall meet a number of curves –from the simplest ones, the straight line and the circle, to the most complex and surprising ones- and we shall discover their most important mathematical properties, in particular those regulating their applications to science, technics, and sometimes to everydays life.

Before starting, please relax: you shall not be obliged to learn any formula, nor to solve exercices. We shall discover instead that it is possible to be introduced to the ideas and methods of mathematics in a painless and –we hope- amusing way.

1. Straight lines and circles

What can be done with a piece of string?

The visit begins with a marker and a simple piece of string, with which you are invited to draw a straight line and a circumference. Walking on the street, you can sometimes see road workers about to dig a trench, that first draw up its outline by pulling a string between two stakes. We can also draw a straight line (note the verb *draw*) by pulling the string with two fingers and trying to follow it with the marker. If, instead, we want to draw a circle, we will roll the string around the marker, and we will make it go round by pinning the string's other end with a finger.

Drawing a straight line and a circle with a piece of string

The result in the two cases is very different. While usually the arcs of circle which we can draw (arcs - to draw a full circle one needs more precaution) are quite accurate, the segments of a straight line are usually disappointing.

The reason for this difference in behaviour is in the different function of the string. In the case of the circle, it is a tool. In the case of the straight line, it is a profile. With a profile one draws what is already there. We can draw a straight line because a string pulled by two fingers creates a straight line. The accuracy of the result in this case depends on the precision of the profile, and on the possibility of following it with the marker, and that is precisely what is difficult in our case.

Conversely, when drawing a circle, the string does not take up a circular form to follow the marker. We take advantage of a mathematical property of circumference, that is that all its points have the same distance from its centre. The string, pulled taut between the marker and with the end pinned to the table, ensures this very equidistance.

The same happens if, instead of using a string, we use more appropriate objects, like the ruler and the compass. Precision improves greatly with both, but there is no substantial change: the ruler is a profile, the compass an instrument. And although straight lines made with a ruler are always better than those made with a string (but circles are also rounder), the precision of an instrument will always be, at the same degree of complexity, better than the one of a profile.

How to draw a straight line, and why.

But what is the purpose of drawing a straight line? Is it not easier to draw with a computer, without using either a ruler or a compass? The question makes sense, and it would be useless if the issue was just that of drawing. But this is not the only problem. Let's look at any machine, a bicycle for example, or a blender. There are mobile parts in it, like the pedals or the wheels of the bicycle, or the blender's blades, that must move along predetermined trajectories, such as around a point. In this case there is no difficulty: it's enough to hinge the piece on the centre of rotation, so that it can only revolve around it. Every point of the piece thus traces a circumference around the hinge, which behaves like the finger holding the string to the table.

If, instead, the trajectory of the moving part is not circular, things become more difficult, such as in the rod we see coming out of the table, or in the piston's axis we see in the picture on the wall? We could certainly fix a ruler and make the rod slide on it, or, even better, go through two rings attached to the wall (and in this case the rod itself would act as a profile), but the movement would generate so much friction to make the functioning impossible.
In this situation, maybe even more than in the drawing, the difference between an instrument and a profile is fundamental. If we want the mechanism to work, we cannot use profiles, which always have sliding parts, but we must generate rectilinear motion with an instrument. The one we are seeing has been proposed by James Watt. It is a simple articulated quadrilateral, such that the midpoint of the smaller side - and thus the rod which is attached to it - moves up and down along a rectilinear trajectory.

The linkage of Watt

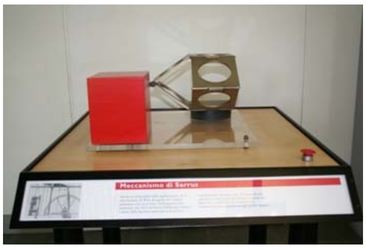
A table version of the same shows that the straight line is only approximate

But is it really a straight line? A table version of the same linkage shows the opposite.The midpoint draws an 8-shaped curve, with two *nearly* rectilinear parts, or, anyway, straight enough for the task. Watt's linkage uses these parts of the curve to maintain the rod constantly in a vertical position.

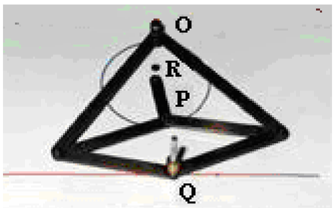
Other instruments to draw straight lines.

Watt's linkage, which because of its extreme simplicity is still in use today, solves in practice the problem of drawing a straight line, or at least a curve so near to a straight line to be practically indistinguishable in applications. After Watt, other instruments were found, certainly more complicated, that draw approximated lines; some of these can be seen and operated.

The theoretical problem still stands, however - is it possible to build an instrument that draws a true straight line, not just an approximation? A first positive answer is given by Sarrus' mechanism, in which the points on the upper plate all move along vertical lines. It is, however, a machine which is neither practical (Watt's linkage is much simpler and more reliable) nor satisfying from the theoretical point of view, since it operates in three-dimensional space and not on the plane. Technically, this also means it takes up a lot of space.



The exact solution to the problem is given by a mechanism invented in 1864 by A. Peaucellier, based on the properties of a specific mathematical transformation: the inversion compared to a circumference. The mechanism is made of a number of rods hinged so that, whatever way they are moved, the product of the distances of the P and Q point from O is always the same. To use a more technical language, points P and Q correspond through inversion compared to a circle whose centre is O.



One of the properties of inversion is that when point P traces a circumference, its corresponding point Q also traces a circumference. Only one case provides an exception, which is what we need - when the circumference traced by P goes through the centre O, the corresponding point Q traces not a circumference, but a straight line. One then understands the role of the PR rod, whose extremity R is fixed to the table. It has nothing to do with inversion, but it ensures that point P, which now can only rotate around R, draws a circumference. If PR is equal to RO, this circumference will go through the centre O, and then the corresponding point Q will draw a rect, or, more precisely, a segment.

The linkage of Čebičev

Two Čebičev's linkages are used to guide a saw

The mechanism of Sarrus

The linkage of Peaucellier

Articulated quadrilaterals.

Among the many connecting rod mechanisms that solve problems of a practical interest, the simplest is the articulated quadrilateral, which, because of its very simplicity and versatility, is the basis for many simple instruments that we see every day, some of which can be seen in the panel, from scales to Venetian blinds, to windscreen wipers, cranes, but also in some more sophisticated mechanisms like amputees' prostheses.

Four sides are the minimum to have a mobile mechanism. The triangle is a rigid, non-deformable figure, which because of this stability is used to build stable structures, such as towers, bridges and roofs. On the contrary, a square maintains a certain freedom of movement even when one fixes one if its sides, a freedom which makes it a very effective instrument to draw curves, or if you prefer, to have a piece move along a predetermined path.

Normally one of the sides of the quadrilateral is fixed, e.g. to the table, and remains immobile: thus it is possible to avoid putting it in, and just like in Watt's linkage, the quadrilateral is reduced to three interconnected rods, of which the first and last are fixed to the table by one extremity, around which they can only rotate.

Notwithstanding the extreme simplicity of the mechanism, articulated quadrilaterals are very versatile, and they have numerous applications. In particular, they are very useful to convert oscillatory movements in circular ones and viceversa, as it happens e.g. in a bicycle, where the alternated movement of the cyclist's legs generates the pedals' circular motion, or in the sewing machine, where the pedal's oscillating movement makes the machine's wheel spin.

If then you add two more rods to the quadrilateral, creating a triangle with the middle one, it is possible to draw a great many curves, even rather irregular ones, by properly adjusting the length of the added rods. In the machine shown, the additional rods are substituted with a sheet of Plexiglass, whose holes correspond to the vertices of the added triangle. Depending on the position of the hole, the mechanism describes curves, even very different ones.

The articulated quadrilateral in the pedal of a sewing machine

An application of the articulated quadrilateral

2. Conic sections

A light in the dark.

If we light up a wall with an electric torch keeping it perpendicular to the wall, the lit portion is more or less circular. Let's now begin tilting the torch upwards: the circle deforms and assumes an oblong shape, like a serving tray or a stadium: it's an ellipse.

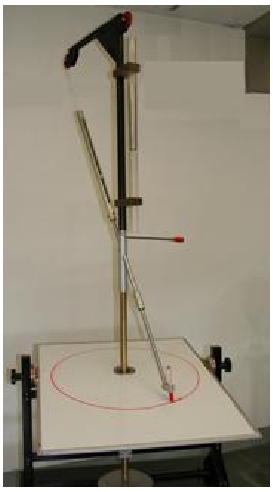
If we keep tilting the torch, the ellipse gets longer and longer. While one of its ends remains near us, the other moves further and further: if the wall were infinite, the lit area would become bigger and bigger, until for a certain inclination of the torch, it would become infinite. The figure thus obtained is a parabola.

If we tilt the torch even further, the lit area becomes even bigger, and it assumes the shape of a hyperbola.

The three figures which are obtained one after the other, or rather the curves that delimit them, are collectively called conic sections, since they are obtained sectioning a cone (in our case the cone of light projected by the torch) with a plane (the wall).

Conic sections: a circle, an ellipse, a parabola, a hyperbola

Conic sections are often found in the most common situations: a table lamp draws two hyperbolas on the wall, the shadow of a ball is an ellipse, a stone thrown by a sling takes a parabolic path. In the past, the theory of conic sections was essential to build sundials. In its apparent motion, the sun draws a circumference: the rays that pass by the tip of a sundial's stylus then form a cone, that cut by the wall creates a conic section, which at our latitudes is a hyperbola, on which the shadow of the stylus's tip moves.



The perfect compasses

One can draw an ellipse using the great three-dimensional compasses which the Arab geometers had called the *perfect compasses*. The inclined rod that rotates around the vertical axis describes a cone, which is intersected by the drawing plane. According to the latter's inclination, one can obtain a circumference (if the plane is horizontal) or an ellipse, longer the more inclined the plane. If one could increase indefinitely the plane's inclination, one would obtain first a parabola and then a hyperbola.

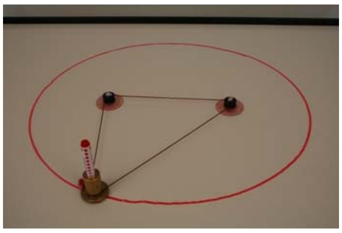
Other elliptical compasses, some of which are exhibited, can be built using the various properties of this curve: one can also find them on sale. A parabola or a hyperbola are more difficult to draw.

Echoes and reflections.

The simplest way to draw an ellipse is with a piece of string, a bit like with the circumference we drew at the start.

A circumference has all the points at the same distance from the centre, so we can draw it with a string, keeping one end fixed and rotating the other one with a marker. When the circumference gets longer and becomes an ellipse, the centre, so to say, divides up into two points: the foci. These have a characteristic property: if

you take any point on the ellipse and you unite them with the two foci, the sum of the lengths of the two segments is always the same.
This property can be used to draw an ellipse on the ground: fix two stakes on the foci and attach to them the two ends of a string. If now we bring a pencil around so that the string is always kept taut, the curve we have drawn is an ellipse, called the *gardener's ellipse*, because this method is often used to draw elliptical flower beds.



The gardener's ellipse

The same property can be used to build elliptical gears. If you take to identical ellipses, dispose them so that each of them can rotate around one of their foci, and if the distance between the stakes is equal to the length of the string that describes the ellipse, the two ellipses always remain tangent, and the rotation of one drags along the other. Moreover, if the first one rotates uniformly, the second has a variable velocity, higher the nearer the tangent point is to the fixed focus.

Elliptic gears

Another important property of the ellipse is that the line perpendicular to the ellipse in any of its points divides the angle formed by the string (that is, by the lines that link the point to the foci) in two halves. This property is relevant to light reflection. When a ray of light reflects on a mirror, be it flat or curved, the perpendicular to the mirror makes equal angles with both the incident ray and with the reflected ray, that is, with the incoming and the outgoing rays. But then, a ray of light which starts from a focus behaves like the string in the gardener's ellipse: after having reflected on the ellipse, it will strike the other focus.

An elliptic kitchen pan

The same is true for any type of ray: light, sound, heat. In every case, all the rays that originate from a focus, after a reflection on the ellipse, will concentrate in the other. This is the reason for the name *foci* ("fires"): if one places a source of heat on one of the foci, the heat concentrates in the other and it can light up a piece of paper or an inflammable material.

A simple kitchen pan (of an approximately elliptical shape) with its bottom covered in water can be used to illustrate the phenomenon. If you touch the water with a finger on one of the foci (marked by a dot on the bottom), you create concentric waves that, after having reflected on the side of the pan, concentrate on the other focus.

Burning mirrors.

As the generating plane becomes more and more inclined, the ellipsis becomes more and more oblong, and the second focus moves away from the first. When it is transformed into a parabola, the second focus disappears (sometimes one says it has gone to infinite) and there is only one left. And while in an elliptical mirror the rays originating from a focus ended up in the other, in a parabolic mirror the rays that start from the only remaining focus are reflected parallel to the axis, and viceversa the rays parallel to the axis concentrate in the focus.

This latter property of the parabola can be used to build a burning mirror, that is a mirror that concentrates solar rays (which we may consider parallel because of the great distance of the Sun) in the focus, where they can light up inflammable materials. We have built an indoors burning mirror, substituting solar rays with those from a halogen lamp. We have put the lamp in the focus of a second parabolic mirror, from which the light rays come out parallel after being reflected; a second reflection on the first mirror concentrate them on the focus, where in a short time they light up a match.

The same principle is at the base of the parabolic microphone: the sound waves, which can be considered parallel if they come from afar, are reflected on the parabola and concentrated on the focus, where a microphone is places. This mechanism can hear very far away and faint noises. The great radiotelescopes and the parabolic dishes of satellite TV are also built using the focal properties of the parabola.

Parabolas are often found as solutions of technical and scientific problems. A stone thrown obliquely describes a parabola, just as the cable of a suspension bridge also assumes the shape of a parabola.

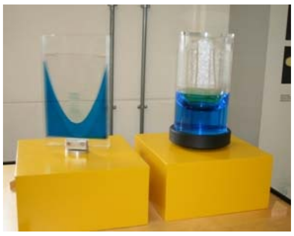
A mirage.

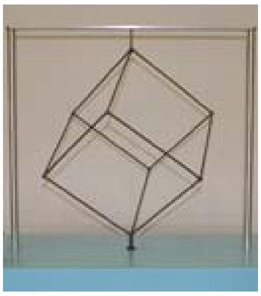
Using two identical parabolas we can create an interesting phenomenon. What you see is a sort of black box in the shape of a disc, with a flat hole on which a die is placed. You can light this die with a torch, and at first sight it has all the properties of a real object. But when you try to take or touch it, you see that there is no die: it's an optical illusion.

This mirage has a simple explanation. The box is actually made of two identical, superimposed parabolic mirrors, each of which has its focus more or less in the other's vertex. On the bottom of the lower mirror is the die, while the upper one has a hole through which it is possible to see inside. When one looks through this hole, or lights it up with a torch, the light ray makes two reflections before hitting the die, thus forming a real image of it, absolutely identical to the original.

Burning mirrors

A mirage





A parabola and a paraboloid

An elliptical chamber

Two large parabolas concentrate the voices

Hyperboloids of rotation

A rotating cube

Rotating.

In fact, burning mirrors and the mirage are made not with parabolas, but with surfaces obtained rotating parabolas around their axis. These surfaces are called rotation paraboloids. A paraboloid can be obtained rotating a liquid fast enough inside a cylindrical container. If instead the liquid is contained between two close planes, we have a parabola.

Similarly, when we rotate an ellipse or a hyperbola, we obtain a rotation ellipsoid or hyperboloid.

These surfaces also have reflecting properties similar to the ones of the paraboloid. We have built an elliptical chamber, obtained by rotating a half-ellipse along the axis. A phenomenon similar to the burning mirror happens in it: if we place ourselves in one of the foci and we speak towards the elliptical wall, even in a very low voice, those who are in the other focus receives the voice very clearly, while whoever is in between hears hardly anything.

A similar phenomenon happens if we use two parabolas: if those of the burning mirrors and of the mirage did concentrate the light, the great parabolas at the extremities of the corridor concentrate the sound waves. Speaking very softly in the focus of one of them one is heard distinctly in the other. Both these objects are in the nearby room.

The same focal properties of the parabola are used in the construction of the large radiotelescopes and of the parabolic antennas of the satellite TV.

The hyperboloid of rotation has the notable characteristic of being a ruled surface, that is of being constituted only of straight lines, as we can see from the hyperboloid obtained with strings.

This produces an unexpected phenomenon: rotating a straight line opportunely inclined, one can make it go through a hyperbola-shaped slit. The rod, rotating, describes a hyperboloid, which, intersected with a plane, leaves us with the two slits through which the rod passes with no difficulty.

The same surface is obtained by rotating a cube. While the upper and lower edges, that meet the rotation axis, form a cone, the intermediate ones, which do not meet the axis, generate a hyperboloid, which is visible thanks to the persistence of images on the retina.

Two properties of the ellipse and of the parabola.

We have dedicated one table to two interesting properties of the ellipse and of the parabola.

Assume that we have drawn one of these curves (say, the ellipse) on a plane and let us look at it from a point of the plane. We shall see the ellipse under a given angle, that varies according to the position of the eye. We ask now: from which points we see the ellipse under a right angle? The exhibit on the table shows that these points form a circumference. In a slightly more technical language: the locus of the points from which the ellipse is seen under a right angle is a circumference.

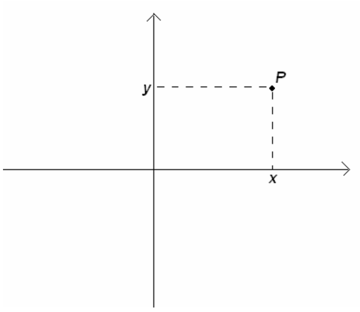
The same problem for a parabola has as solution a straight line.

3. Other curves

Analytic geometry.

Not even conic sections can satisfy all the needs of science and technology. It is therefore necessary to proceed further, and to consider even more complex curves.

A very effective way to represent these curves on the plane consists in using their equation.



If you trace two perpendicular lines on the plane, it is possible to individuate every point through its Cartesian coordinates, that more or less represent the distance of the point from the two axes. They are called thus in honour of Réné Descartes (Cartesius) who first used them systematically to describe curves.

The two coordinates are usually indicated as x and y. To each value of the couple (x,y) corresponds a point on the plane. When x and y vary in every possible way, the corresponding point describes the whole plane. If, instead, the coordinates are subject to an equation, the point that they represent is forced to move along a curve, of which the equation constitutes the analytical representation. For example, if one fixes x through the equation x=1, the corresponding curve is a vertical straight line; while the equation y=3 represents a horizontal line. More generally, a first-degree equation (that is, an equation in which the variables x and y appear at the power of one) represents a straight line, while a second-degree equation creates one of the conical sections (including the circumference).

A property of the ellipse and of the parabola

The cartesian coordinates

One can study curves with a third-degree equation, or fourth-degree, or of a higher and higher degree. For these, we find in the

Encyclopédie of Diderot and D'Alembert an universal machine that allows, adding layer upon layer, to trace curves of higher and higher degrees. The one that has been realised has three layers, and therefore traces third-degree curves. It is a rather heavy machine, whose complexity is mainly due to the need to reduce to the minimum the friction that would otherwise prevent it from functioning.

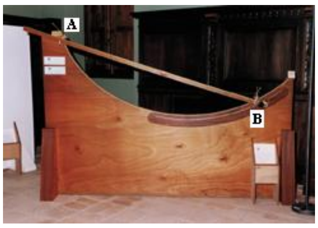
The cycloid.

Other curves have an equation that has no degree (or rather, they cannot be expressed through a polynomial). Some of these are distinguished by some special properties, that make them particularly useful and interesting. One of these is the cycloid.

The cycloid can be seen by attaching a light bulb to the wheel of a bicycle, possibly in the dark, or just rolling a circle on which we have marked a point, such as that hanging on the wall.

Although simple to describe, the cycloid is a relatively modern curve. Among the first, if not *the* first, to examine it was Galileo, who observed how elegantly it described the arch of a bridge. During the whole seventeenth century, the cycloid was an object of study by the best mathematicians, who determined its length, its enclosed area (which Galileo had first guessed as being three times the generating circle, as many geometers later demonstrated independently, among them Torricelli), its center of mass and other connected quantities.

But surprises lay in wait until the end of the century, when the cycloid became the solution of two important mathematical problems.



One deals with the fall of weights, and it marks the birth of a totally new branch of mathematics: the calculus of variations. Let us suppose that we want to send a ball from a point A to a lower point B, but not along the vertical, We can imagine building a profile which unites the points A and B, and of making the ball roll along it. Obviously, there are infinite such profiles. We then ask ourselves: is there one that minimises the trajectory time?

At first sight, one might think that the solution is the straight line joining A and B. However, if we stop and think, we will see that the straight line is indeed the shortest line between two points, but not necessarily the fastest: it could be best to make the ball start more vertically, so that it would instantly gain enough speed to compensate for the longer path. This problem of the brachistochronous curve, or curve of the shortest time (from the Greek *brachistos*, shortest, and *chronos*, time), was posited by Johann Bernoulli as a challenge to the mathematicians of his time, and was solved among others by Newton and Leibniz - the fastest

A machine for the tracing of curves

The cycloid

The brachistocrone

curve is a cycloid. Bernoulli was so proud of his discovery, that he had on the title page of his Collected Works the drawing of a dog trying - in vain - to catch up with a cycloid, and the motto: *Supra invidiam*, above envy.

We have compared the cycloid with the straight line: letting two balls fall at the same time along those two curves, we can see how the time for the cycloid is considerably shorter than the one for the straight line.

The cycloidal pendulum.

The cycloid is also concerned in another problem, this time of a more technical kind. Galileo again had observed how the oscillations of a pendulum would take more or less the same time, independently from the amplitude of the oscillations. The first pendulum clocks originated from this observation by Galileo.

In reality, the pendulum's oscillations are not exactly isochronous: the time it takes to complete an oscillation does depend on the oscillation's amplitude, and it is longer for wider oscillations. Only for very small oscillations we can consider the time substantially constant, and it is those small oscillations that are used for pendulum clocks.

One can then wonder: how must a pendulum be so that the all its oscillations require exactly the same time, or in one word (also derived from Greek: *isos*, equal, and *chronos*, time) are strictly isochronous?

Let us see things from a slightly different point of view. In a normal pendulum, weight swings freely attached to a point, therefore along a circumference. Its oscillations are only approximately isochronous, and require more time the larger the arc of circumference. In this context, the question becomes: along what type of curve must a body oscillate so that the oscillations are perfectly isochronous?

The answer, once again, is: the cycloid. If we put two balls in two points of the cycloid, and we make them fall at the same time, they will hit each other exactly in the lower point of the curve, even if one started very near and the other very far from this point. In other words, the ball takes the same time to travel over the large arc and the small one, a sign that the oscillations are isochronous.

In a circumference this does not happen: the ball that starts nearer arrives earlier.

So, if we want to build a perfectly isochronous pendulum clock, we must have the weight oscillate along a cycloid. But how to make the weight move along this line without making it slide, that is without using a cycloidal profile? The situation is analogous to that of the first room, when we wanted to draw a straight line without using a profile. There we used a rather complex mechanism, Peaucellier's mechanism. Here we will use another trick: instead of allowing the weight to oscillate freely (in this case it would describe a circumference), we will condition its trajectory by making the string lie on two profiles.

In mathematical terms, the profiles will have to be shaped in such a way that their involute is a cycloid. And here we have a surprise: the involute of a cycloid is a second cycloid equal to the first! Consequently, the profiles must be arcs of a cycloid. In this case (and only in this case) the pendulum will oscillate along a cycloid, and will therefore be isochronous.

We have compared two pendulums: one free, moving along a circumference; and another cycloid, obtained with profiles at the top. Two photocellules measure their periods, that are visualised on the computers. As you may see, while the period of the ordinary pendulum decreases with the amplitude of its oscillations, the one of the cycloidal pendulum is strictly constant.

Spirals.

An ant starts from the beginning of a turntable's platter, and walks outward in a straight line. But if the platter starts turning at the moment when the ant starts walking, and if both the ant and the turntable maintain a constant speed, the ant will trace a spiral curve, called Archimedes' because it was firstly studied by the mathematician from Syracuse.

We can substitute the ant with a marker, which we move from the centre to the edge with the most constant speed possible. We will then see Archimedes' spiral form, with more spirals the slower we move the marker.

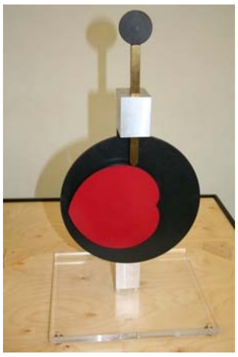
An interesting application of the spiral is found in sewing machines, in the part that spins the thread around the spool. The thread coming from the skein is kept taut, and it is rolled around the spool, which turns and moves up and down, in order to allow an uniform distribution of the thread. And this is where the spiral comes in. To make sure that the thread is spun uniformly on all the parts of the spool, the oscillation movement needs to keep a constant speed. If the oscillation movement were faster in the centre and slower towards the ends, when the spool must change directions (which is what would happen unless one takes specific measures), the thread would not be distributed uniformly, but would tend to pile up at the spool's extremities.

The cycloidal profile

Two pendulums in comparison

Archimede's spiral

A sewing machine



Therefore, one needs a mechanism that makes the spool oscillate with a constant speed. This is obtained by regulating the oscillation of the spool through a profile made of two coupled spiral arcs. Here we see this specific mechanism of the sewing machine, and a working enlarged reproduction of it. The vertical rod moves alternatively up and down, always with the same speed.

Caustics.

The description of particular curves could go on for quite some time. Those who are interested can see on the computer some simulations of mechanisms to draw various types of curves. We will proceed with the visit and examine both the general structure of curves and the specific properties of some of them.

We have seen how conical sections, and in particular parabolas, have the ability to concentrate light rays in one point. Other curves do not, in general, have this property. This however does not mean that the reflected rays scatter completely, illuminating space more or less uniformly. Very often they concentrate not in a point, but on a curve: the caustic. Like the *focus* of conical sections, the name of this curve derives from "burning"; in fact, the term *caustic* means *burning*. In reality, the name does not correspond to an actual ability to light up a fire, at least as long as the light source has a low power, like a light bulb.

Caustics are often seen in everyday life, such as when a kitchen bowl is lit obliquely: the rays reflecting on the vertical side draw a curve on the bottom, which is called a reflection caustic, because it is obtained through the reflection of light rays.

Caustics can also be obtained through refraction, when the rays coming from a point penetrate a medium with varying density.

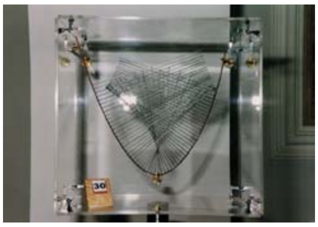
Envelopes.

Reflected or refracted rays are all tangent to the caustic they form. More in general, if one takes a family of straight lines (as in the caustics the reflected or refracted rays), they can be uniformly distributed on the plane, as it happens when for example they are parallel, but they can also concentrate onto a curve, to which they are all tangent. This curve is called the envelope of the lines. Thus a reflection caustic is the envelope of the rays reflected by a mirror.

The mechanism for rolling the thread around the spoon, and its larger model

A caustics of reflection

One can create envelopes of lines by pulling strings between various points on the plane or in space. The bicycle wheel above



your head, the perpendicular lines to the parabola, show curves obtained as envelopes of straight lines.

To build an envelope one need not start with straight lines - the same can be done with curves. Every curve of the family will be tangent to the envelope, that is it will have the same tangent line in the contact point. On the computer, it is possible to see various examples of envelopes of circles and also of more complex curves.

An interesting example comes from ballistics. Ignoring the resistance of air, the shell shot by an artillery piece will describe a parabola, the shape of which will depend on the power of the cannon and on the shooting angle. Let us suppose now that we have a cannon which always shoots with the same power, but that can vary the angle. Where must one stand to be sure one is outside the cannon's range?

The answer is simple: one traces all the shells' trajectories with varying angles, and one stands outside the region these cover. This region is delimited by the curve enveloping all the parabolas, which in this case is also a parabola.

Let us note that the caustics, and envelopes in general, are not curves that physically exist. What exist are the rays of light, or more in general the lines of the family. The curve that they envelop appears only because they concentrate on it. Thus the light rays concentrating on the caustic shed more light on the area of the plane corresponding to the curve, and they draw it. The same happens with the lines normal to the parabola, which can be seen in the entrance hall. They trace a curve that can be seen, although it does not exist - only the lines enveloping it exist.

In some cases it is important to avoid the formation of caustics, and to have a light which is as uniform as possible. For example, in photography the uniformity of illumination is essential. The Fortuny lamp, which is still in use, thanks to the shape of its reflecting surface, supplies a light density which is constant in every point.

Curvature and its environs.

Just as of all lines that pass through a point *P* of a curve, the tangent is the one that best approximates the curve, of all the circles that pass through *P* there is one which best adapts to the curve's shape near *P*. This circle, whose centre is on the line perpendicular to the curve (or to its tangent, which is the same thing), is called the osculatory circle.

We can thus measure the curvature of a curve. The tangent allows us to determine the direction of a curve *C*. If we imagine a

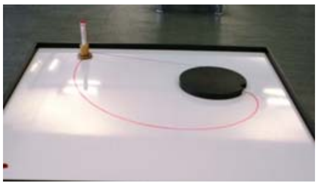
The threads in a wheel envelope a curve similar to the caustic

The envelope of the normals to a parabola

The Fortuny lamp

point moving along *C*, we can think that in every moment the point moves in the same direction than the tangent. Analogously, the curvature of *C* will be given by that of its osculatory circle, and since a circle is more curved the smaller its radius, we can measure *C*s curvature through the inverse of the osculatory circle's radius, also called curvature radius.

As the point *P* varies on the curve, the curvature centres (the centres of the osculatory circles) describe a second curve, which is called the evolute of the first. This curve is also the envelope of the lines perpendicular to the given curve.



Reciprocally, the first curve is the involute (or evolvent) of the second. The involute of a curve can be materially obtained by attaching a piece of string to the profile of the curve, then slowly detaching it, being careful to keep the detached part always taut. The free extremity of the string will then describe the involute. On the table, you can draw the involute of the circle.

New tendencies.

One of the characters of the course we have followed so far is the shift from simple to complex: as techniques were refined and knowledge increased, it became possible to study more and more complex curves, which were used to deal with otherwise insoluble problems. At the same time, one assisted to a similar passage from the particular to the general: instead of studying the genesis and properties of this or that curve, one dealt with whole classes of curves, and elaborated concepts that could work with any of them.
Modern geometry has continued along this road and made it even more obvious, in both directions: more generality, more complexity. Correspondingly, mathematic techniques have become more and more abstract, to the point of making it difficult, if not impossible, to describe them even approximately.

Since we don't want to completely renounce giving an idea of modern developments of curve geometry, we chose two examples that represent the two tendencies: geodetics and fractals.

On a flat surface, as for example in a city square, the shortest path between two points is a straight line. But if we move along a curved surface, like the surface of the Earth, it is not possible to go in a straight line, and in its place there is a curve, called geodetic, whose length is the minimum among all curves that touch both its ends. Just like as we pulled a string taut to get a straight line at the beginning of the exhibition, we could find the geodetic of a convex surface by pulling a string between two points. In the case of the Earth, which is more or less a sphere, the geodetics are the

The involute of the circumference

A polar route

the maximum circles, those that divide the sphere in two equal parts.

To go from one point to another it is best to move along the maximum circle that goes through the two points. This is the reason for polar routes in intercontinental air travel: on the maps they seem much longer, because they are drawn on a plane, but it is enough to pull an elastic band on a globe to see how actually the shortest route between Pisa and Los Angeles goes near the North Pole.

The second example regards the very concept of curve. Those we have seen so far correspond to the intuitive idea of a curve that we all have; they are a single-dimensioned object, which we can think can be obtained bending a piece of string. If they have some special points, such as the edge of a polygon, these are finite and isolated.

Around the end of the nineteenth century, *pathological* curves with surprising properties begin to pop up. One of which is Peano's curve, which fills a square and poses the problem of the meaning of dimensions: another is Koch's curve, which has no tangent in any point. Finally, in recent years, some new object have emerged, the fractals, which have a fractionary dimension and the surprising property that every part of them, no matter how small, is similar to the whole. These forms, whose generation is relatively simple, produce images of a remarkable beauty: images with which we conclude our visit in the world of curves.