The Arab expansion. 1

In the middle of the seventh century, a still marginal people made a dramatic entrance to the world scene. Because of the weakness of both the Eastern Roman empire and the Sasanid Kingdom, brought about by long, demanding wars, the Arabs quickly conquered a huge territory and created an empire of unprecedented proportions. A century after the death of Muhammad, the Arab empire extended from Spain to India, unifying very distant regions and profoundly different cultures under the law of Islam.

Chronology of the Arab expansion.

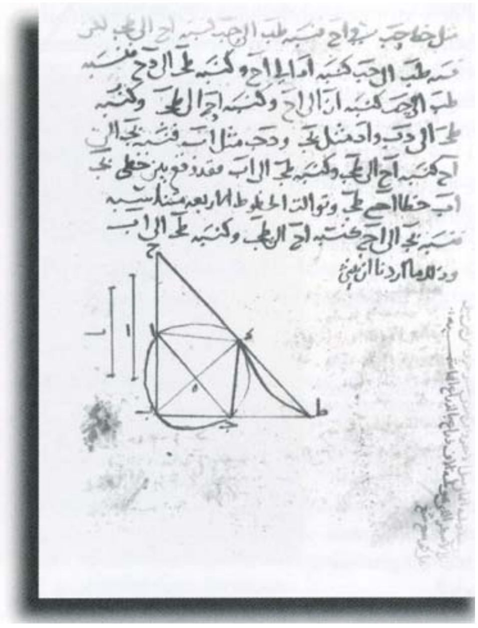
632	Death of Muhammad.
635	Conquest of Damascus.
636	Taking of Jerusalem.
637	Occupation of Syria and Palestine. Invasion of Persia. Conquest of Ctesifon.
639-41	Invasion of Egypt
640-44	Occupation of Iraq and Persia.
647	Beginning of the penetration into Mediterranean Africa.
673	Siege of Costantinople.
680	Conquest of Algeria.
681-82	Conquest of Morocco. Arab armies arrive at Atlantic Ocean.
698	Taking of Chartage.
711	Conquest of Spain. Occupation of Afghanistan and part of Pakistan. Taking of Bukhara and Samarkanda.
717-18	Second siege of Costantinople.
724	Taking of Tashkent and occupation of the Transoxania.
732	Battle of Poitiers and end of Arab expansion in the West.

The Arab-Islamic culture. 2

Despite the speed of the expansion and the inevitable destruction that occurred during the war of conquest, the new State immediately displayed great vitality. With the magnificence of the courts and the living standards of the subjects, it soon rivalled empires with very ancient traditions. Because of being in contact with different peoples and civilisations, and thanks to tolerant policies and to an unprecedented degree of intellectual curiosity, the Arabs were quickly able to assimilate different cultures and meld them into an original, vital synthesis. In this way, they created a culture which for many centuries was to be a model for less advanced societies, building a bridge between classical civilisation and the modern world.

When western Europe joined the worn threads of culture and art after the Dark Ages, it was to find in its contacts with the Arab world a heritage in almost every field of the knowledge, from astronomy to medicine, from philosophy to mathematics.

The transmission of the scientific knowledge. 3



The most progressive Khalif promoted and supported learned men, doctors and scientists, in translating scientific and philosophical texts and creating an Arab-Islamic culture. With the foundation of the Bayt al-Ḥikma (the “House of Knowledge”) by the Abbasid khalif al-Ma'mūn in Baghdad, the activity of translation acquired vast proportions, leading quickly to the assimilation of most Greek science. The most important works of classical mathematics were translated into Arabic, including most of the works of Euclid, Archimedes and Apollonius. In several cases, the translation into Arabic is all that bears witness to a lost Greek original.

Having been in contact with Indian mathematics, Arab scientists rapidly acquired the main outcomes, particularly the use of Indian numerals, positional notation and the related techniques of computation. The Arab mathematicians also collected the last echoes of Egyptian and Babylonian mathematics, thus operating the contamination and fusion of different concepts into a new and, in many ways, original science: algebra.

4

The flourishing of Arabic mathematics.

The first original mathematical works conceived by Arabian culture date back to the ninth century, a period by which the process of assimilation of the peoples of the Arabian empire had already occurred. Therefore, rather than speaking about Arabic mathematics in the literal sense, it is more accurate to speak about Islamic mathematics.

In fact, the first important mathematician, al-Khwārizmi (c. 780-850), had already been born in Central Asia, just like the astronomer al–Bīrūnī (973-c. 1040), while the mathematician and poet Omar al-Khayyām (1048-c. 1131) was Iranian.

In the tenth and eleventh centuries, mathematics were at their peak. Thanks to the widespread assimilation of the classical tradition and the contribution of numerous scholars from every part of the Islamic world, Arabic science underwent unprecendeted development during these years. It attained a level of knowledge so high that the model it sets contemporary civilisations is unattainable.

Some mathematicians who were particularly active during this period are more significant than others: Abū Kāmil (c. 850- c. 930), Abu'l Wafa (940-997), the Banu Musa and al-Haytam, known in the Western world as Alhazen (965-1039).

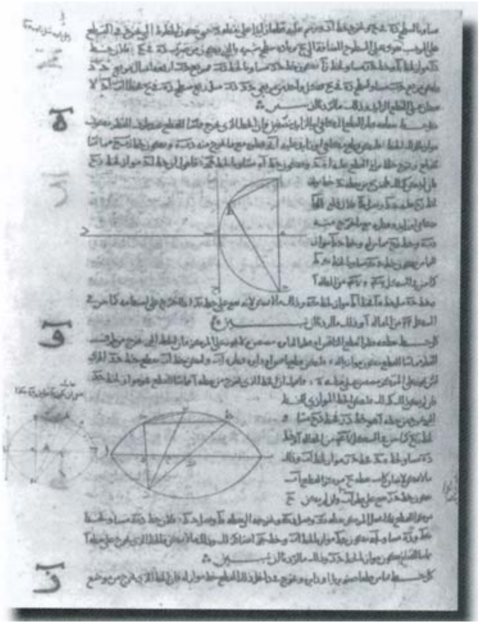
The sky pours candid petals from the clouds
You would say that a rain of flowers scatters across the garden
In the cup like a lily I pour the rosy wine
From the purple cloud a rain of jasmines comes down.

5

Arabic terms in western mathematics.

Together with Indo-Arabic figures and positional notation, European mathematics assimilated a terminology derived from Arabic. In many cases, it was a relatively reliable transliteration, while in others it was a translation of the corresponding Arabic terms, in turn translations from Greek or Sanskrit. The presence of Arabic terms was particularly considerable during the Middle Ages, even in the translations from Arabic of the Greek classics. When many originals were read in the sixteenth century, several of the Arabic geometrical terms were replaced by the corresponding Greek ones and inevitably disappeared. Only those terms that did not have any corresponding Greek terms remained.

The sources of the *Liber Abaci*: al-Khwārizmi and Abu Kāmil. 6



Abū Ja'far Muḥammad ibn Mūsā was called al-Khwārizmi because his family, and possibly himself, came from the central Asiatic town of Khwārizm. His name, Latinised as Algorismus, is the origin of the term *algorithm* that today indicates a computational procedure. Of his life, it is known only that he lived in the first half of the ninth century. He was an astronomer, geographer and historian, but owes his fame to his two mathematical works: *The indian calculus*, of which only the Latin versions of the XIIth and XIIIth centuries are known, and *Algebra* (*Al-Kitāb al-muktaṣar fi ḥīsāb al-jabr wa'l-muqābala*).

In the last mentioned work, al-Khwārizmi integrates notions deriving from Indian mathematics (the use of zero and the positional notation) and the *Elements* by Euclid. The second book especially is used to give a geometrical demonstration of the rules for the solution of equations of the second degree.

As in the case of al-Khwārizmi, almost no biographical information is known about Abū Kāmil. Since he is also known as al Ḥāsib al-Miṣrī, the counter from Egypt, he was probably born there. He almost certainly lived between 850 and 930.

Abū Kāmil was probably the first of the Arab mathematicians to study integer solutions of indeterminate problems, as the Greek mathematician Diophantus had done. In his algebra, he used powers of the unknown greater than two and studied equations with irrational coefficients. Many of the examples provided by al-Khwārizmi and Abū Kāmil can be found in Fibonacci's works.

7

Pisa and the Mediterranean Sea in the Thirteenth Century.

Pisa is a metropolis of "Rum", a well known and vast territory. Its markets and buildings prosper and flourish and it extends over a wide area; it has plenty of gardens and its fields for sowing stretch out as far as the eye can see. Pre-eminent is its location, amazing its exploits. Pisa is gifted with lofty forts, fertile lands, plentiful waters and wonderful monuments. The Pisans own ships and horses and are well trained in the maritime ventures against all the other countries.

This description by the Arab geographer al-Idrisi summarises the prosperous period of economic pre-eminence that Pisa enjoyed in the twelfth century, along with her considerable military power.

Taking advantage from the civil wars that occured in the Western Arabian world, Pisa and Genoa had acquired the control of the western Mediterranean Sea and were on the verge of beginning a war for supremacy that would finish with the Meloria battle in 1284.

The Almohades and the development of trades.

8

Initially, relations between Pisa and the Arab Maghreb were characterised by permanent conflict. Definitive periods of war of varying lengths were interspersed by relatively quiet periods during which sudden, brief destructive acts and sackings continued to occur. From 1150, reciprocal commercial interests and the establishment of a centralised power in the Maghrebian territories brought about a period of considerable peace.

After a period of great instability, the Muslim Western world saw the expansion of the Almoravides (al-Murābiṭūn). They were soon replaced by the Almohades (al-Muwah-ḥidūn), who lent political unity to the Maghreb and also won some victories in Spain that hampered the process of *Reconquista*.

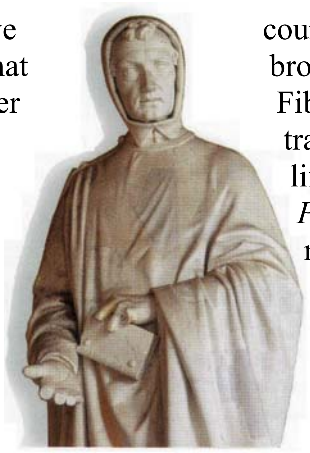
Since 1133, Pisa inaugurated a policy of cooperation with the new sovereign of Tunis, that led to the drafting of several peace treaties, periodically renewable and containing several clauses for the development and protection of commerce.

From the peace treaty of 1186 between Pisa and Tunis.

1. The Pisan merchants are allowed to trade in the Almohade's kingdom, within the territories of Ceuta, Orano, Bugia and Tunis and with the absolute prohibition of disembarking and staying in any other town of the empire except on account of circumstances beyond one's control. In any case, outside of the above-mentioned harbours, it is forbidden to buy or sell or even talk with the inhabitants of the territories. The Spanish town of Almeria is exempt from this prohibition, but the Pisan merchants are allowed to stop there only to get a new supply of food or to have their ships repaired. The violations of these rules shall be punished either by death or slavery depending on the free will of the sovereign.
2. The Pisans commit themselves to severely punish each action carried out against the Muslim citizens of the Khalif.
3. The Pisans are forbidden to transport citizens of the Khalif on their ships. In this case, punishments shall be very severe.
4. The duty on sold goods will be the tenth of their price, "according to customs".
5. Freedom of commerce, freedom of navigation and the guarantee of the security of people and things are reaffirmed.

Leonardo Fibonacci from Pisa.

Most of the biographical information available about Leonardo Fibonacci can be found in his own works, in particular the *Liber Abaci*. His birth date is unknown and has been the subject of much conjecture; today, it is usually considered to have been a little after 1170. As a young boy, his father brought him to Bugia, a town near today's Alger, where he was a notary of the Municipality of Pisa. Here, Leonardo learned the first elements of mathematics, on which he would improve later during several journeys all over the Mediterranean Sea. Because of his many travels, he was nicknamed 'Bigollo'.



Once back in his native country, in Fibonacci wrote the *Liber Abaci*, a work that brought him widespread fame. It is not known whether Fibonacci remained in Pisa afterwards or took up travelling again, since no information about his life before 1220 -- the date of publication of the *Practica Geometriae* -- is available. In 1226, he met Emperor Frederic the second in Pisa, and was to remain on very good terms with his court. The revised version of the *Liber Abaci*, published in 1228, is dedicated to Michele Scoto, a philosopher of the imperial court. During these years, he also wrote three smaller but not less important works: the *Liber Quadratorum*, the *Flos* and the *Letter to magister Theodorus*. Only vague details are known of his two other works, the *Comment to the tenth book of Euclid's Elements* and a *Book of less manners,* probably a compendium of the *Liber Abaci.* Not even the date of composition is known of these.

A document issued by the Municipality of Pisa in 1241, granting the mathematician a pension, proves that he was still alive in that year. Nothing more is known of Fibonacci thereafter.

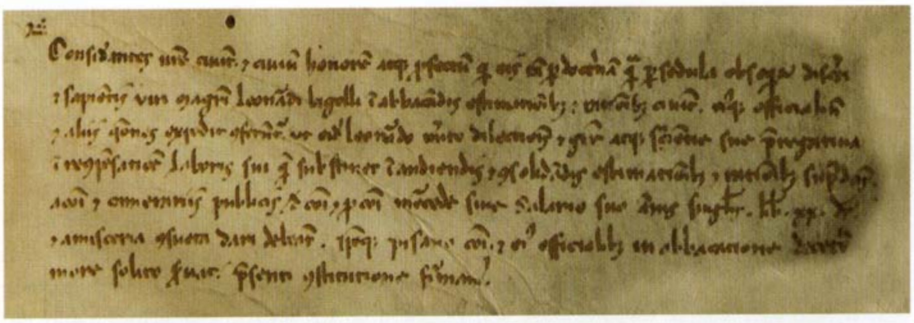
The *Liber Abaci.* 10

The *Liber Abaci* was written in 1202. In it, all the knowledge acquired by Fibonacci during his travels across the Arabian countries and the Mediterranean Sea can be found, together with reflections and elaborations of his own. The result is a work that can compete with its models in doctrine and that exceeds them so far as sheer scope is concerned. The *Liber Abaci* were indeed to remain unequalled in Western mathematics for a long time.

There is no aspect of commercial mathematics that is not given its own space in the *Liber Abaci*; from companies to loans, from the exchange to the melting of the coins, from purchases to barter, everything is systematically explained with a series of examples taken from contemporary commercial operations. For European mathematical culture, still modeled on the late Latin authors, such as Boethius and Cassiodorus, the *Liber Abaci* represented a disruption to the established norm. It provided the base for precise, reliable bookkeeping for commerce, that had gone beyond the limits of familiar management to assume European dimensions.

A pension for Leonardo Pisano. 11

Considering the honour and profit of our town and its citizens, deriving from the doctrine and diligent services of the wise and learned master in abacus problems and estimates, useful to the town and its officials and in other things when necessary, with this act we deliberate that Leonardo is to be given 20 Liras with the title of pay or yearly salary, together with the usual benefits, by the Municipality and the Public Treasure for his dedication and knowledge and as remuneration for the work he sustains in order to study and determine the above-mentioned estimates and problems. We also ask him to serve the Pisan Municipality and its officials in the practice of abacus.



The positional notation. 12

One of the most important contributions of the *Liber Abaci* is the diffusion of the positional notation and Indo-Arabic figures. Ancient Mediterranean civilisations had different methods to write numbers. Egyptians and Romans had different signs to signal units, tens, hundreds etc… For example, Romans indicated units with I, tens with X and hundreds with C. To indicate two hundred and three, they would write CCIII. Greeks and Hebrews used alphabetical letters: Greeks wrote one with α, two with β, three with γ, and so on. To indicate ten they wrote κ ; thirty was μ, ρ was one hundred and σ two hundred; so, two hundred and three was written as σγ. Those who came closer to the positional system were the Babylonians, who used a mixed sexagesimal system: numbers from one to fifty-nine were written the way Egyptians and Romans wrote them, while for greater numbers a positional system was used-- two hundred and three was indicated with 3 followed by 23, that is 3 sixties and twenty-three units. Apart from the latter, all the other systems had many difficulties in expressing large numbers.

In modern notation, which was invented by the Indians and arrived in the West thanks to the Arabs, every number has avalue according to its position: that on the right is the place of units, then proceeding to the left come the tens, the hundreds and so on. From here comes the necessity of a sign, the zero, indicating that the corresponding place is empty: in 203 there are two hundreds, no tens and three units.

Problems from the *Liber Abaci*: the rule of three. 13

If a cantare is sold for 40 lire, what is the value of 5 rolls?

In order to find the unknown number, one writes the first number, that is the quantity of the goods, on the right and its price besides it on the left. In case the second quantity of the goods is known, one has to write it under the goods, while if the number to be spent is known, this is to be written under the price, so as to always write a type under the same type: goods under goods and money under money. Once one has done this, the opposite numbers can be multiplied; the product divided by the left number will give the fourth desired number.

In our case, one can write one cantare, that is 100 rolls, on the right, and its price, that is 40 liras, on the left. Then, under 100 rolls one will write 5 rolls, since they are of the same type. Afterwards, the opposite numbers can be multiplied -- that is 5 by 40 gives 200 -- and this result is to be divided by 100, giving 2 liras as the price of 5 rolls.

The measures of weight in Pisa in the Thirteenth Century

4 grains of wheat make a carob

6 carobs make a denaro of cantare

25 denari of cantare make an ounce of libra	39 and half denari of cantare make an ounce
12 ounces of libra make a libra	12 ounces make a roll
158 thin libras make a Pisan cantare	100 rolls make a Pisan cantare

Problems from the *Liber Abaci*: the false position. 14

There is a tree that has $\frac{1}{3}$ and $\frac{1}{4}$ underground, while the remaining part of 21 palms is above the ground. We ask the length of the tree.

Assume that the tree is 12 palms tall and that, taken out $\frac{1}{3}$ and $\frac{1}{4}$ – that is 7 – 5 palms remain above the ground. One will then say: if when I have 12 I obtain 5, what do I need to do to obtain 21? Multiply the extremes, that is 12 by 21, and the division of the result by the medium 5 will give 50 and $\frac{2}{5}$.

The method is called "of false position" because from an initial assumption, usually false, one can get the solution by applying the rule of three. In the *Liber Abaci* the method of false position and its generalisations are used with great skill and virtuosity.

Problems from the *Liber Abaci*: if 3 were 4.

15

If 3 were 4, how much would 5 be? If this problem (or a similar one) were not in the *Liber Abaci* it would seem to be the ravings of a madman. But since it is Fibonacci who ponders, it is worth trying to find a solution. Actually, Leonardo himself provides the answer to the question:

If one asks of 5, to which number is it in proportion to as 3 is to 4, do this: multiply 4 by 5 and it gives 20, that divided by 3 gives 6 and $\frac{2}{3}$ and this is the desired number. So:

If 3 were 4, 5 would be 6 and $\frac{2}{3}$.

The same problem can also be seen from another point of view. Indeed, Fibonacci also states:

If 3 were 4, how much would 5 be? That is equivalent to saying: if 3 rolls cost 4 bezants, how much will 5 rolls cost? This question is to be treated as purchases are, by following the rules concerning similar matters.

So, behind an apparently extravagant problem hides an abstract formulation of the rule of three, and a general method of calculation.

Rabbits and the numbers of Fibonacci. 16

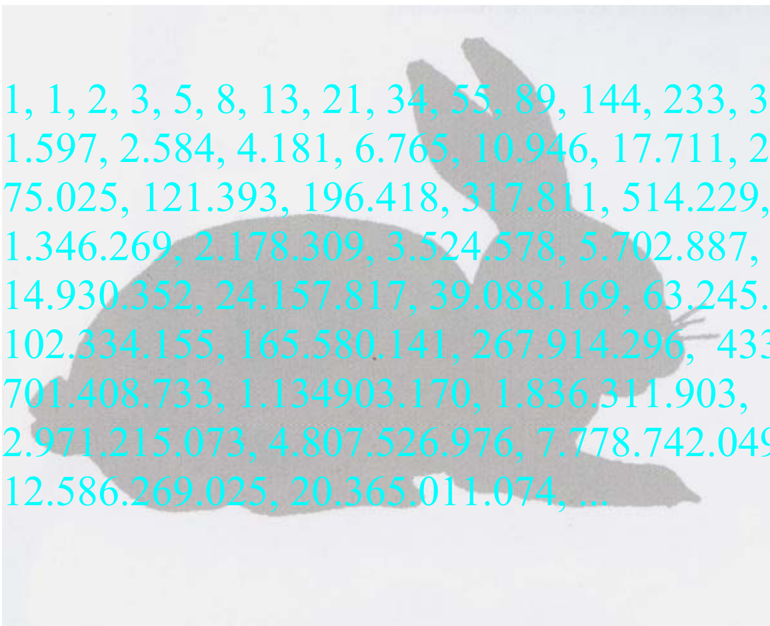
*How many couples of rabbits descend from a couple a year***.**

A lad put a couple of rabbits in a place completely surrounded by walls, in order to find out how many couples of rabbits descend from this one a year. By nature, each couple of rabbits begets another couple every month, but they start to procreate only in the second month of life.

In order to solve this problem, one can assume for instance that in November there is a certain number of rabbit couples, say 21, and that in October there were 13. Of the couples of November, 8 (that is 21-13) are the newborn that do not procreate. So, in December there will be the 21 couples of November plus 13 couples born from the rabbits that were already there in October. This is always true and therefore – Fibonacci observes – to find the number of rabbits one only needs to sum:

the first number to the second, that is 1 to 1; then the second to the third, the third to the fourth, the fourth to the fifth, and so on, until the tenth is summed to the eleventh, that is 89 to 144, to obtain the final quantity of 233 couples of rabbits; one can continue in an orderly way for endless months to come.

The sequence 1,1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610… is nowadays called "the series of Fibonacci" and the numbers it is composed of are called "the numbers of Fibonacci". Later, it was noted that this series can be found in nature and art. Today, the name of Fibonacci is known to a vast public thanks to a sequence that he probably considered a mere curiosity.



1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1.597, 2.584, 4.181, 6.765, 10.946, 17.711, 28.657, 46.368, 75.025, 121.393, 196.418, 317.811, 514.229, 832.040, 1.346.269, 2.178.309, 3.524.578, 5.702.887, 9.227.465, 14.930.352, 24.157.817, 39.088.169, 63.245.986, 102.334.155, 165.580.141, 267.914.296, 433.494.437, 701.408.733, 1.134903.170, 1.836.311.903, 2.971.215.073, 4.807.526.976, 7.778.742.049, 12.586.269.025, 20.365.011.074, …

Shells and other curiosities. 17

A geometrical problem which leads to Fibonacci's numbers is the building of adjacent squares. On the side of a square of side one, a second adjacent square with the side one is built. The two squares will form a rectangle 2×1 and the next adjacent square will have side 2. Together with the previous ones, this square will form a rectangle 3×2, on which a square of 3 will be based. If one continues, one will form a series of squares whose sides are the subsequent numbers of Fibonacci 1,1,2,3,5,8,13, etc.

By tracing the quarter of a circle in each square, one obtains the so-called "Fibonacci spiral", a form that can also be found in shells.

Shells are only an example of a recurring phenomenon: the presence of Fibonacci's numbers in nature. According to mechanisms as yet unclear, Fibonacci's numbers can be found in many natural phenomena: the arrangement of flowers' petals, the branching of some plants, the disposition of seeds in sunflowers and of squamas in cones. The latter are disposed so as to form two series of opposed spirals that converge in the centre; in the same cone or in the same sunflower, the number of the spirals that reel in two directions are consecutive Fibonacci's numbers.

18

Fibonacci's numbers and the golden section.

An unexpected property of Fibonacci's numbers is that, as far as one proceeds, the ratio between one number and the one that precedes it grows closer and closer to the irrational number $\gamma = \frac{\sqrt{5}+1}{2} = 1{,}6180339887498948482045 86....$ This ratio can already be found in Euclid's *Elements* as the solution to the problem of "the division of the segment in mean and extreme ratio". It was called "*Divina proportione*" by Luca Pacioli -- who dedicated a whole volume with this title to the subject -- and, later on, "golden section" or "golden number".

The golden ratio has peculiar properties of symmetry and has had an important role in the visual arts: Leonardo da Vinci built the proportions of the human body on the basis of the golden section and, more recently, this latter has been of greatest interest to both Mondrian and Severini. The Modulor by Le Corbusier is also connected to Fibonacci's numbers and the golden section, while the axis of the tower of the Palazzo Vecchio in Florence divides the width of the building in two according to the mean and extreme ratio.

Money and interests. 19

Loans and interests have always had a particular role in commercial arithmetic. Usually, the monetary unit is the lira, made up of 20 soldi that have the individual value of 12 denari; therefore, the lira has the value of 240 denari.

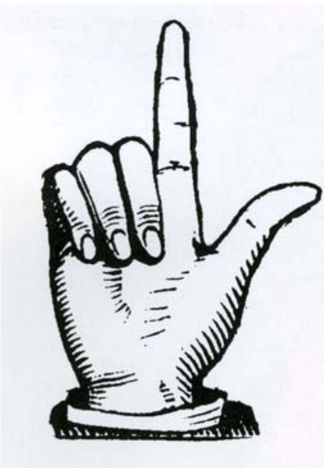
The interest is accrued only after one year (this is called: “new year's merit”), while as far as the fractions of the year are concerned, simple interest is calculated. This latter is expressed in denari per lira a month; a denaro per lira a month is equivalent to 12 denari per lira a year. Since 12 denari are one soldo, that is a twentieth of lira, it corresponds to an interest rate of 5%. Therefore, 4 denari per lira a month give an annual interest rate of 20%.

Hand memory.

During the Middle Ages, paper was a precious commodity and consequently many operations that are nowadays carried out by writing on paper were accomplished mentally or by writing precariously in the dust or in the sand. It was very important to have the oppportunity to memorise partial results, so as to recall and use them later.

The most common way to remember a number was “to hold it in one's hands” by means of an elaborate system of the position of fingers. Units and tenths (that is a number from 1 to 99) were held in the left hand, while the right hand was used symmetrically to register hundreds and thousands. So, the position that indicated a number in one's left hand, for instance 35, indicated as many hundreds -- that is 3500 -- in one's right one.

The art of holding numbers in one’s hands represented a very important part of the learning of arithmetic and therefore the treaties of the abacus always contained two pages with the figures for the position of one’s fingers at the beginning.



Problems from the *Liber Abaci*: old women and cats. 21

Some of the problems in the *Liber Abaci* have very ancient origins and were transmitted for millennia before reaching Leonardo Pisano and the present day. One of the most ancient problems was already in the Rhind Papyrus and consists of adding up a geometric sequence of ratio 7:

Seven houses; in each are seven cats; each cat kills 7 mice; each mouse would have eaten 7 grains of spelt; each grain of spelt will produce 7 hekat. What is the total of all these?

This problem has come down to the present day.

Along the road that led to Saint Ives
I met a man with seven wives.
Each wife had seven sacks
Each sack had seven cats,
Each cat had seven kits
Sacks, cats, kits and wives,
How many were going to Saint Ives?

In the *Liber Abaci* the statement is:

Seven old ladies go to Rome; each of them has seven mules, each mule has seven sacks, in each sack there are seven pieces of bread, each piece of bread has seven knives, each knife has seven cases. The question is what is the total amount of them all.

In the Rhind papyrus there are five terms, in the rigmarole of Saint Ives four, while in the Fibonacci's problem six.

Problems from the *Liber Abaci*: the chessboard. 22

Another very ancient problem that has remained unsolved until the present day concerns the game of chess. Over the years, it has recounted that its creator asked for a grain as reward for the creation of the first square, two for the second one, four for the third one, eight for the fourth one and so on, doubling until the last square of the chessboard, the sixty-fourth one.

Fibonacci did not mention the legend, but he calculated the number of grains as 18.446.744.073.709.551.615.

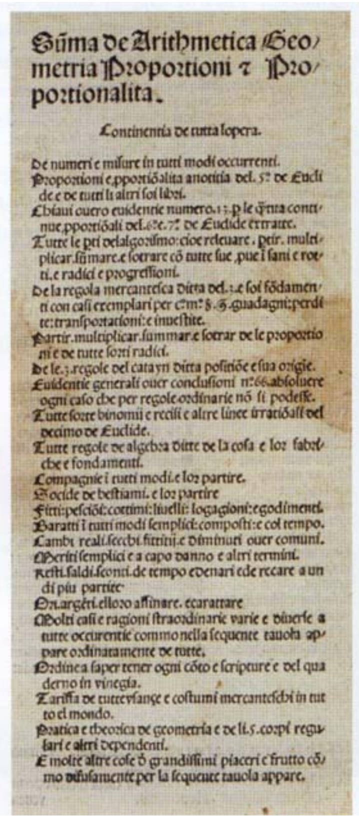
It is difficult to imagine what such a long number might mean and to have an idea of its enormity. But after all, when written, it does not seem to be so dreadfully great. In order to make the reader have a more definite idea about this number, Leonardo ponders: how many ships could be loaded if each of them can carry 500 Pisan bushels weighing 24 sextaries each, with a sextary composed of 140 libras of 12 ounces each, that in turn have the individual value of 25 denari, each of which weighs 24 grains? The result is surprising: one would load up 1.525.028.455 ships -- more than one and half billion, a number "that is apparently innumerable and almost infinite".

The legacy of the *Liber Abaci*.

Falling in a mathematically undeveloped environment, the *Liber Abaci* required a considerable amount of time before coming to fruition. We must wait until the last part of the thirteenth century until we have some concrete proofs of the influence of Fibonacci on the development of mathematics in Italy. These were almost always strictly connected with the activities of the schools of abacus. Indeed, most of the abacus treaties drew inspiration from the work of Pisano, who is now universally recognized as the founder and major exponent of Mediaeval mathematics.

Towards the middle of the fifteenth century, the invention of the printing press changed the way culture was transmitted, causing the progressive disappearance from collective knowledge of those authors whose works were not to be printed. Not even Fibonacci escaped this fate and, by the sixteenth century, he was merely a name: Cardano placed Leonardo "a few years before" Luca Pacioli; Bernardino Baldi, the author of *Cronica de' matematici,* mentioned him as a mathematician living in the fifteenth century.

Only in the nineteenth century was Fibonacci situated in the correct historical perspective.



Sũma de Arithmetica Geometria Proportioni e Proportionalita.

Continentia de tutta lopera.

De numeri e misure in tutti modi occurrenti.
Proportioni e ꝓportiõalita anotitia del .5°. de Euclide e de tutti li altri soi libri.
Chiaui ouero euidentie numero .13. ꝑ le q̃ntita continue ꝓportiõali del .6°. e .7°. de Euclide extratte.
Tutte le pti de lalgorismo: cioe relevare, ptir, multiplicar, sũmare, e sotrare cõ tutte sue ꝓue i sani e rotti, e radici e progressioni.
De la regola mercantesca ditta del .3. e soi fõdamenti con casi exemplari per c̃m° §. guadagni: perdite: transportationi: e investite.
Partir, multiplicar, summar, e sotrar de le proportioni e de tutte sorti radici.
De le .3. regole del catayn ditta positiõe e sua origie.
Euidentie generali ouer conclusioni n° 66. absoluere ogni caso che per regole ordinarie nõ si podesse.
Tutte sorte binomii e recisi e altre linee irratiõali del decimo de Euclide.
Tutte regole de algebra ditte de la cosa e lor fabriche e fondamenti.
Compagnie i tutti modi, e lor partire.
Socide de bestiami, e lor partire
Fitti: pesciõi: cottimi: livelli: logagioni: e godimenti.
Baratti i tutti modi semplici: composti: e col tempo.
Cambi reali, secchi, fittitii, e diminuti ouer comuni.
Meriti semplici e a capo danno e altri termini.
Resti, saldi, sconti, de tempo e denari e de recare a un di piu partite.
Ori, argẽti, e lor affinare, e carattare
Molti casi e ragioni straordinarie varie e diverse a tutte occurentie commo nella sequente tauola appare ordinatamente de tutte.
Ordine a saper tener ogni cõto e scripture e del quaderno in vinegia.
Tariffa de tutte vsanze e costumi mercanteschi in tutto el mondo.
Pratica e theorica de geometria e de li .5. corpi regulari e altri dependenti.
E molte altre cose d̃ grandissimi piaceri e frutto cõmo difusamente per la sequente tauola appare.

Commerce and mathematics.

At the beginning of the fourteenth century, the intensifying of trade led to the development of companies with branches in different towns. Their unity was based, on the one hand, on an intense exchange of correspondence and, on the other, on a tested accounting system that had been improved through practice. Aside from the first registration *pro memoria*, the journal for the daily writing of operations in chronological order appeared. Then the ledger was created in order to have a profit and loss account reserved for each usual correspondence agent. In addition, there were other specific account books concerning instrumental and patrimonial properties, goods and members.

Elementary arithmetic was no longer sufficient for such complex commercial organisations. Its bookkeeping necessities required updated notions - in the first place those Arabic figures, that were considered obstacles rather than useful tools in small companies. The necessities of advanced companies, often acting at an international level, constituted the main reason for the diffusion of the techniques and innovative notations contained in the *Liber Abaci.*

The abacus schools.

The diffusion of Arabic figures and the corresponding methods of calculation occured largely thanks to institutions that have probably been unique in the history of Europe, the abacus schools. These developed from the late thirteenth century, especially in the most active commercial centres where trade activities expanded, creating a wealthy commercial middle class that would soon claim for itself the political control of the republics.

In smaller centres, abacus teachers were usually paid by the Municipalities that used them as advisors for measurements and evaluations. In big towns, such as Venice and Florence, a great number of private schools of abacus were founded. These would operate continuously until the sixteenth century, when institutes of religious education would replace them.

Although incomplete, the first proofs of the presence of masters of abacus in many Italian towns indicate a clear prevalence of Tuscan centres and masters.

Pisa	1241	Leonardo Fibonacci
Bologna	1265	Pietro da Bologna
San Geminano	1279	Michele
Perugia	IInd half of the XIIIth Century	
Verona	1277	Lotto da Firenze (1285)
Venezia	1305	Gentile dall'abaco
Siena	1312	Gherardo di Chiaro da Firenze
Savona	1345	Nello da Pisa
Lucca	1345	Iacopo da Firenze
Pistoia	1353	Ricco di Vanni da Prato
Genova	1373	Tommaso di Miniato da Pisa
Genova	c. 1375	Tommaso di Bonaccio da Pisa
Arezzo	1394	Benedetto di Domenico da Prato
Volterra	1409	Filippo de Follis da Pisa
Modena	1421	Bonifacio di Ferro
Brescia	1436	Benedetto da Firenze

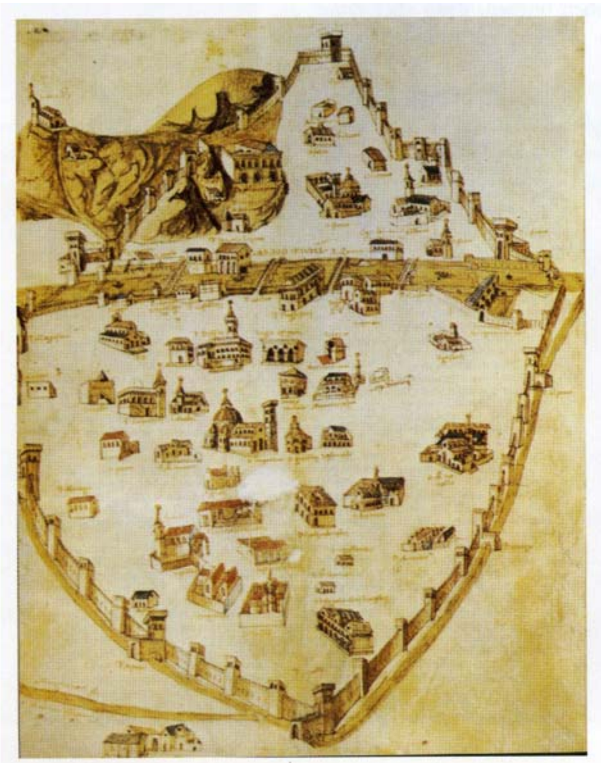
The abacus schools in Florence.

The diffusion of the abacus schools was particularly important in Florence, where the unique phenomenon of mass education occured. According to the *Cronica* by Giovanni Villani, in 1338:

We estimate the number of young boys and girls who learn to read from eight to ten thousand. The young boys who learn the abacus and the algorithm in six schools from one thousand to two thousand and two hundred. And those who are studying grammar and logic in four big schools from five hundred and fifty to six hundred.

From the middle of the fourteenth century to the first thirty years of the sixteenth century, there were twenty schools of abacus in Florence, a number that might grow as archival research continues.

Florence was then divided into the quarters of Santa Maria Novella, Santa Croce, San Giovanni and Santo Spirito. These in turn were subdivided into four Gonfalons.



The abacus schools in Florence: the quarter of Santa Maria Novella.

Gonfalon of the Unicorn	School of Santa Trinita (c. 1340–1450)	[Biagio il vecchio] [Paolo dell'abaco] [Michele di Gianni] Don Agostino di Vanni Antonio di Giusto Mazzinghi, Giovanni di Bartolo Lorenzo di Biagio Mariano di M° Michele Taddeo di Salvestro dei Micceri
	School of Lungarno Corsini (1367–1445)	Biagio di Giovanni [Antonio Mazzinghi] Michele di Gianni Luca di Matteo Giovanni di Luca Calandro di Piero Calandri
	School of Via dell'Inferno (1514)	Marco di Iacopo Grassini
	School of S. Maria della Scala (1458–1469)	Benedetto da Firenze
Gonfalon of the red Lion	School of the Corticina dell'abaco (c. 1460–1506)	Calandro di Piero Calandri Benedetto da Firenze Pier Maria Calandri Filippo Maria Calandri
	School of Via Ferravecchi (1493–1500)	Giovanni del Sodo
Gonfalon of the Viper	School of Santi Apostoli (1375–1527)	Michele di Gianni Orlando di Piero Mariano di Michele Benedetto di Antonio Banco di Piero Banchi Niccolò di Lorenzo Taddeo di Salvestro dei Micceri Niccolò di Taddeo Micceri Piero di Zanobi Giuliano di Buonaguida della Valle
	School of Piazza dei Pilli (c. 1447–1458)	Calandro di Piero Calandri

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The abacus schools in Florence: the quarter of Santa Croce.

Gonfalon of the Wheels	School of Via dei Neri (1475)	Niccolò di Taddeo Micceri
	School of Badia Fiorentina (1452–1453)	Bettino di Ser Antonio da Romena [Lorenzo di Biagio da Campi]
	[School of Borgo Pinti] (1519–1522)	Francesco di Leonardo Galigai Giuliano di Buonaguida della Valle
Gonfalon of the black Lion	[School of Piazza Peruzzi] (1283–1334)	Iacopo dell'abaco
	School of Via dei Rustici (c. 1530)	Antonio di Taddeo dei Micceri

The abacus schools in Florence: the quarter of San Giovanni. 29

Gonfalon of the Fur	School of Santa Margherita de' Ricci (1370–1376)	Tommaso di Davizzo dei Corbizzi Bernardo di Tommaso [Cristofano di Tommaso] Antonio Mazzinghi
	School of Canto di Croce Rossa (c. 1493–1495)	Iacopo di Antonio Grassini [Marco di Iacopo Grassini] [Raffaello di Giovanni Canacci]
Gonfalon of the Dragon	School of Via Teatina (1452–1464)	[Benedetto da Firenze]

The abacus schools in Florence: the quarter of Santo Spirito. 30

Gonfalon of the Dragon	[School of Via della Chiesa (1458–1469)]	[Lorenzo di Biagio da Campi]
Gonfalon of the Shell	School of Borgo S. Iacopo (c. 1495)	Raffaello di Giovanni Canacci
Gonfalon of the Ladder	School of Via dei Bardi (1495–1499)	Ser Filippo

An abacus school in Pisa 31

Among the documents that describe the teaching in the schools of abacus, the most detailed concerns the school of Cristofano di Gherardo di Dino, who taught abacus in Pisa in 1442:

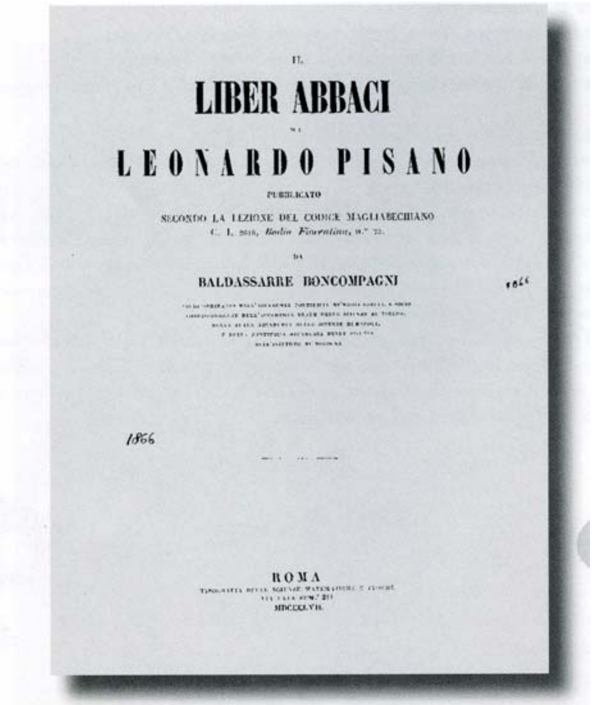
> *This is the way of teaching the abacus in Pisa, from the beginning to the end of the students' learning period, as we will say.*

- *At first, when the boy begins school, he is taught how to make figures, that is 9, 8, 7, 6, 5, 4, 3, 2, 1;*
- *Then he is taught how to keep numbers in his hands, that is his left hand units and tens and in his right hand hundreds and thousands;*
- *Then to draw numbers on tables: that is of two figures what it means, and then three figures, four figures and so on. Then how to keep them in one's hand.*
- *Then one explains the tables of multiplication. One draws it on the table, starting from one times one until ten times ten one hundred, and students learn it very well by heart.*
- *Then one teaches how to make divisions;*
- *Then how to multiply fractions;*
- *Then how to sum fractions;*
- *Then how to divide [fractions];*
- *Then how to accrue simple interests and the "new year's merit";*
- *Then how to measure lands or how to square a number;*
- *Then how to make simple discounts and new year's discounts;*
- *Then how to calculate the ounces of silver;*
- *Then the melting of silver;*
- *Then one makes the comparison between the two amounts;*
- *And note that to make the above-mentioned calculations, students are to use pencils according to their level. And sometimes have them sum with their hands, or else on the blackboard; occasionally give them some extraordinary homework, according to the teacher's will.*
- *Please, note also this general rule: every evening give them homework for the following day according to their level. And, in case of days of rest, homework is to be doubled.*

The rediscovery of Fibonacci in the Nineteenth Century.

At the end of the eighteenth century, with the awakening the historic-mathematical research in Italy, the work of Fibonacci acquired it proper historical context. The forerunners of the Fibonaccian revival were Pietro Cossali, who wrote the *Origine, trasporto in Italia, primi progressi in essa dell'algebra* (1798-99), and Gianbattista Guglielmini, author of the *Elogio di Leonardo Pisano* (1812). Some decades later, Guglielmo Libri and Michel Chasles were involved in a controversy concerning the evaluation of the role of Leonardo in the history of algebra and Diophantine analysis.

But the most important restorer of the name and work of Fibonacci was Baldassarre Boncompagni. After a profound study of the life and time of the Pisano, he published the *Opuscoli* (*Liber Quadratorum, Flos* and *Epistola*) in two subsequent editions (1854 and 1856) and later in a monumental edition of all the works of Fibonacci including, together with the already mentioned *Opuscoli*, the *Liber Abaci* (1857) and the *Practica Geometriae* (1862). Even today, Boncompagni's is the only edition of the works by Leonardo.



IL
LIBER ABBACI
DI
LEONARDO PISANO
PUBBLICATO
SECONDO LA LEZIONE DEL CODICE MAGLIABECHIANO
C. I, 2616, *Badia Fiorentina*, n.° 73.
DA
BALDASSARRE BONCOMPAGNI

ROMA
TIPOGRAFIA DELLE SCIENZE MATEMATICHE E FISICHE
MDCCCLVII.